Valuations on central simple algebras

Jean-Pierre Tignol

Louvain University
Louvain-la-Neuve, Belgium

Positivity, Valuations, and Quadratic Forms
Konstanz, 3 October, 2009

Outline

Valuations on division rings

From division rings to quadratic forms

Special value functions

Gauges
Valuations on division rings

Definition (Schilling)
\[ v : D \to \Gamma \cup \{\infty\} \] is an abelian totally ordered group.

- \( v(x) = \infty \iff x = 0 \),
- \( v(x + y) \geq \min(v(x), v(y)) \),
- \( v(xy) = v(x) + v(y) \).

Useful to obtain information on the fine structure of \( D \):
- construction of noncrossed products (Amitsur 1972)
- counterexamples to the Kneser–Tits conjecture (Platonov 1977)

Examples

Twisted Laurent series
\[ D = E((x; \theta)) = \{\sum_{i \geq k} e_i x^i \mid e_i \in E\} \quad \text{x e} = \theta(e)x \]

\[ v : D \to \mathbb{Z} \cup \{\infty\}, \quad \sum e_i x^i \mapsto \min\{k \mid e_k \neq 0\} \]

Composite valuations
If \( u : E \to \Gamma \cup \{\infty\} \) is a valuation, define
\[ w : D \to (\Gamma \times \mathbb{Z}) \cup \{\infty\} \text{ by } d = \sum e_i x^i \mapsto (u(e_{v(d)}), v(d)). \]

valuation for the right-to-left lexicographic ordering
Specific example

\[ D = \mathbb{C}((x_1)((x_2; \theta))(y_1)(y_2; \rho)), \]
\[ \theta(x_1) = -x_1, \rho(x_i) = x_i, \rho(y_1) = -y_1 \]

center \( Z = \mathbb{C}((x_1^2)((x_2^2)((y_1^2)((y_2^2)))) \) and

\[ D \text{ is biquaternion: } D = (x_1^2, x_2^2)Z \otimes Z (y_1^2, y_2^2)Z \]

\[ \nu: D \to \mathbb{Z}^4 \cup \{ \infty \} \]
\[ \nu(D^\times)/\nu(Z^\times) = (\mathbb{Z}/2)^4 \]
\[ \overline{D} = \overline{Z} = \mathbb{C} \]

\[ D^\times \times D^\times \xrightarrow{\nu} [D^\times, D^\times] \xrightarrow{\text{residue}} \mathbb{C} \]
\[ \nu \downarrow \quad (\mathbb{Z}/2)^4 \times (\mathbb{Z}/2)^4 \]

Property

\( i = \sqrt{-1} \notin [D^\times, D^\times] \) because \([D^\times, D^\times] \subset \{ \pm 1 \}\]

\[ \text{Nrd}(i) = i^4 = 1, \text{ so ker Nrd} \neq [D^\times, D^\times] \]

(counterexample to Kneser–Tits)

Existence of valuations

\( D \) finite-dimensional division algebra over its center \( F \)

Each valuation \( \nu \) on \( D \) is an extension of \( \nu|_F \) on \( F \).

Theorem (Ershov 1982 – Wadsworth 1986)

Every Henselian valuation on \( F \) extends to \( D \). The extension is unique.

Theorem (Ershov 1988 – Morandi 1989)

A valuation on \( F \) extends to \( D \) iff \( D \otimes F^h \) is division (\( F^h = \text{Henselization} \)). The extension is unique.

Example

The local invariants of \((-1, -1)_\mathbb{Q}\) are trivial except at 2, \( \infty \): the \( p \)-adic valuation on \( \mathbb{Q} \) extends to \((-1, -1)_\mathbb{Q}\) only for \( p = 2 \).
From division rings to quadratic forms

Common feature of central simple algebras and quadratic forms: the group of automorphisms is a simple linear algebraic group:

\[ \text{Aut}(M_n) = \text{PGL}_n \quad \text{type } A_{n-1} \]
\[ \text{Aut}(x_1^2 + \cdots + x_n^2) = O_n \quad \text{type } D_{n/2} \text{ or } B_{(n-1)/2} \]

Corresponding adjoint group:

\[ \text{PGO}_n = \{ g \in \text{GL}_n \mid g^t \cdot g \text{ scalar} \} / \text{scalars} \]
\[ = \{ \text{Int}(g) \mid \text{Int}(g) \circ t = t \circ \text{Int}(g) \} \]
\[ = \text{Aut}(M_n, t) \]

From division rings to quadratic forms

Theorem (Weil, 1960)

*Adjoint classical groups of type $A_{n-1}$*

\[ = \text{twisted forms of } \text{PGL}_n \]
\[ = \text{Aut}(A) \text{ for } A \text{ central simple of degree } n \]

*Adjoint classical groups of type $D_m$*

\[ = \text{twisted forms of } \text{PGO}_{2m} \]
\[ = \text{Aut}(A, \sigma) \text{ for } (A, \sigma) \text{ such that } \]
\[ (A \otimes F_{\text{alg}}, \sigma \otimes \text{Id}) \simeq (M_{2m}(F_{\text{alg}}), t) \]

i.e. $A$ is central simple of degree $2m$ and $\sigma: A \to A$ is an orthogonal involution

in particular, $\sigma$ is linear, $\sigma(xy) = \sigma(y)\sigma(x)$, $\sigma^2 = \text{Id}$. 
Anisotropy

For a central simple algebra $A$,

\[ \text{Aut}(A) \text{ is anisotropic} \iff A \text{ is division} \]

\[ \text{Aut}(A, \sigma) \text{ is anisotropic} \iff \sigma \text{ is anisotropic, i.e.} \quad \sigma(x) \cdot x = 0 \Rightarrow x = 0 \]

Example

$q$ quadratic form in $n$ variables with Gram matrix $b$

$\sigma = \text{adjoint involution on } M_n$: \[ \sigma(x) = b^{-1} \cdot x^t \cdot b \]

$\sigma$ anisotropic \iff \[ x^t \cdot b \cdot x = 0 \Rightarrow x = 0 \]

\iff $q$ anisotropic.

Question

Is there an analogue of a valuation for central simple algebras with anisotropic involution?

Note:

\[ v(x) = \infty \iff x = 0 \]

\[ v(xy) = v(x) + v(y) \quad \Rightarrow \quad \text{no zero-divisor} \]
Special value functions

\[ v : F \to \Gamma \cup \{\infty\} \] valuation with \( \Gamma \) divisible, \( \text{caract } F \neq 2 \)

\[ \sigma : A \to A \] involution on a central simple \( F \)-algebra

**Theorem (Tignol–Wadsworth)**

If \( v \) is Henselian and \( \sigma \) is anisotropic, there is a unique map \( g : A \to \Gamma \cup \{\infty\} \) such that

- **\( g \) is a vector space valuation:**
  - \( g(a) = \infty \) if \( a = 0 \)
  - \( g(a + b) \geq \min(g(a), g(b)) \)
  - \( g(a\lambda) = g(a) + v(\lambda) \) for \( \lambda \in F \)

- **\( g \) is surmultiplicative:**
  - \( g(1) = 0 \) and \( g(ab) \geq g(a) + g(b) \)

- **\( g \) is \( \sigma \)-special:**
  - \( g(\sigma(a)a) = 2g(a) \) for \( a \in A \)

Existence of special value functions

\[ A = \text{End}_D V, \quad \sigma \text{ adjoint to } h : V \times V \to D \]

\( \sigma \) anisotropic \( \Rightarrow \) \( h \) anisotropic

\( v \) extends to \( v_D : D \to \Gamma \cup \{\infty\} \)

Define \( \alpha : V \to \Gamma \cup \{\infty\} \) by

\[ \alpha(x) = \frac{1}{2} v_D(h(x, x)) \]

\( \alpha \) is a vector space valuation.

Every orthogonal base of \( V \) splits \( \alpha \):

\[ \alpha(\sum e_i \lambda_i) = \min(\alpha(e_i) + v_D(\lambda_i)) \]

Define \( g : A \to \Gamma \cup \{\infty\} \) by

\[ g(a) = \min(\alpha(a(x)) - \alpha(x) | x \in V, x \neq 0) \]
\[ = \min(\alpha(a(e_i)) - \alpha(e_i) | i = 1, \ldots, n). \]

Uniqueness: induction on \( \text{dim } V \). If \( A = D \), then \( g = v_D \).
Examples

Hensel is needed:

\[ V = \mathbb{Q}^2, \ h \ \text{polar form of} \ q = x_1^2 + x_2^2, \ \nu = 5\text{-adic valuation} \]

\[ (1, 0) = (3, -1) + (-2, 1) \]

\[ \frac{1}{2} \nu(q(1, 0)) = \frac{1}{2} \nu(1) = 0 \geq \min\left( \frac{1}{2} \nu(q(3, -1)), \frac{1}{2} \nu(q(-2, 1)) \right) = \frac{1}{2} \]

Example

\[ A = M_n(F), \ \nu \ \text{discrete valuation on} \ F \ \text{with uniformizer} \ \pi, \]
\[ \sigma \ \text{adjoint to} \ \langle u_1, \ldots, u_r \rangle \oplus \langle \pi u'_1, \ldots, \pi u'_s \rangle \]
\[ \quad (u_1, \ldots, u'_s \ \text{units}, r + s = n) \]

\[ g \left( \begin{array}{c|c}
    a_{ij} & b_{ij} \\
    \hline
    c_{ij} & d_{ij}
  \end{array} \right) = \min(\nu(a_{ij}), \nu(b_{ij}) - \frac{1}{2}, \nu(c_{ij}) + \frac{1}{2}, \nu(d_{ij})) \]

Residue involutions

For \( g \) surmultiplicative and \( \sigma \)-special,

\[ 2g(a) = g(\sigma(a)) + g(a) \]

so \( g(a) \geq g(\sigma(a)) \) for all \( a \in A \).

hence \( g(a) = g(\sigma(a)) \) for all \( a \in A \).

Define \( \sigma_0 \) on \( A_0 = \{ g \geq 0 \}/\{ g > 0 \} \) by

\[ \sigma_0(\bar{a}) = \overline{\sigma(a)}. \]

Example

\[ A = M_n(F), \ \nu \ \text{discrete}, \]
\[ \sigma \ \text{adjoint to} \ \langle u_1, \ldots, u_r \rangle \oplus \langle \pi u'_1, \ldots, \pi u'_s \rangle: \]
\[ A_0 = M_r(F) \times M_s(F), \]
\[ \sigma_0 = \text{ad}\langle \overline{u_1}, \ldots, \overline{u_r} \rangle \times \text{ad}\langle \overline{u'_1}, \ldots, \overline{u'_s} \rangle \]
For valuations on division rings, $\Gamma_D$ acts on $\overline{D}$.

Example: $D = E((x; \theta))$: 
$\Gamma_D = \mathbb{Z}$ acts through $\theta$ on $\overline{D} = E$.

Consider the graded ring associated to the filtration by $g$: 
$$\text{gr}(A) = \bigoplus_{\gamma \in \Gamma} \{ g \geq \gamma \}/\{ g > \gamma \}.$$ 
Let $\tilde{a} = a + \{ g > g(a) \} \in \text{gr}(A)$ for $a \in A$.

$g$ surmultiplicative $\Rightarrow$ $\text{gr}(A)$ is a graded ring 
$$\tilde{a} \cdot \tilde{b} = \begin{cases} 
    ab & \text{if } g(ab) = g(a) + g(b), \\
    0 & \text{if } g(ab) > g(a) + g(b).
\end{cases}$$

**Example**

$F$ with discrete valuation, uniformizer $\pi$:
$$\text{gr}(F) = F[\pi^{\pm 1}] \quad \text{(Laurent polynomials)}$$

$A = M_n(F),$
$$g \left( \begin{array}{c|c}
    a_{ij} & b_{ij} \\
    \hline
    c_{ij} & d_{ij}
\end{array} \right) = \min \left( v(a_{ij}), v(b_{ij}) - \frac{1}{2}, v(c_{ij}) + \frac{1}{2}, v(d_{ij}) \right),$$

$$\text{gr}(A) = M_n(\text{gr}(F)), \quad A_0 = \begin{pmatrix}
    M_r(F) & 0 \\
    0 & M_s(F)
\end{pmatrix},$$

$$A_{1/2} = \begin{pmatrix}
    0 & \pi M_{r \times s}(F) \\
    M_{s \times r}(F) & 0
\end{pmatrix}, \ldots$$

**Proposition**

$A$ central simple over $F$ Henselian, $\sigma$ anisotropic, $g$ the unique $\sigma$-special value function:
$\text{gr}(A)$ is graded central simple over $\text{gr}(F)$, and 
$$[\text{gr}(A) : \text{gr}(F)] = [A : F].$$
Gauges

A central simple over \( F \), arbitrary valuation \( \nu : F \to \Gamma \cup \{\infty\} \).

Definition

A gauge on \( A \) is a surmultiplicative vector space valuation \( g : A \to \Gamma \cup \{\infty\} \) such that \( \text{gr}(A) \) is graded central simple over \( \text{gr}(F) \) and \( [\text{gr}(A) : \text{gr}(F)] = [A : F] \).

Examples

\( A = \) division algebra: a valuation is a gauge iff it is defectless.

\( A = (-1, -1)_{\mathbb{Q}}, \quad \nu = 3\text{-adic valuation}: \)

\[ g(a_0 + a_1i + a_2j + a_3k) = \min(\nu(a_0), \nu(a_1), \nu(a_2), \nu(a_3)) \]

is a gauge with \( \text{gr}(A) \cong M_2(\mathbb{F}_3[t^{\pm 1}]) \quad (t = \tilde{3}) \).

Other gauges: \( g_u(a) = g(ua^{-1}) \)

\[ g_u = g \text{ iff } \tilde{u} \text{ is invertible in } \text{gr}(A). \]

Gauges on algebras with involution

\( \sigma : A \to A \) involution on a central simple \( F \)-algebra, \( \nu : F \to \Gamma \cup \{\infty\} \) valuation with char \( F \neq 2 \)

Theorem (Tignol–Wadsworth)

There is a \( \sigma \)-special gauge on \( A \) iff \( (A, \sigma) \otimes F^h \) is anisotropic \( (F^h = \text{Henselization}) \). When it exists, the \( \sigma \)-special gauge is unique.

Note:

\[ g(\sigma(a)a) = 2g(a) \iff \overline{\sigma(a) \cdot a} \neq 0 \]

so \( g \) is \( \sigma \)-special \( \iff \overline{\sigma} \) is anisotropic on \( \text{gr}(A) \).

Sketch of Proof:

If \( g \) is \( \sigma \)-special on \( A \), then \( g \otimes \nu^h \) is \( \sigma \)-special on \( (A, \sigma) \otimes F^h \), hence \( (A, \sigma) \otimes F^h \) is anisotropic.

If \( (A, \sigma) \otimes F^h \) is anisotropic, it has a \( \sigma \)-special gauge \( g^h \).

Then \( g^h|_A \) is a \( \sigma \)-special gauge on \( A \). \( \square \)
A quadratic form $q$ is strongly anisotropic if $n \times q$ is anisotropic for all $n \in \mathbb{N}$. $(q \neq 0 \Rightarrow$ formally real base field)

**Theorem (Bröcker – Prestel 1974)**

$q$ is strongly anisotropic iff either

- there exists an ordering on $F$ for which $q$ is definite, or
- there exists a valuation on $F$ such that $q$ has at least two residue forms, and each residue form is strongly anisotropic.

An involution $\sigma : A \to A$ is strongly anisotropic if $t \otimes \sigma : M_n \otimes A \to M_n \otimes A$ is anisotropic for all $n \in \mathbb{N}$.

$\sigma$ is definite for an ordering $P$ on $F$ if $(A, \sigma) \otimes F_P$ is anisotropic ($F_P =$ real closure).

**Problem:**

If $\sigma$ is strongly anisotropic and indefinite at each ordering, find a gauge on $A$ such that $\sigma$ has at least two residues, each strongly anisotropic.

OK if index$(A) \leq 2$: Kulshrestha (to appear)