Valuations on central simple algebras

Jean-Pierre Tignol

Louvain University
Louvain-la-Neuve, Belgium

Positivity, Valuations, and Quadratic Forms

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Outline

Valuations on division rings

Definition (Schilling)

$v\colon D \to \Gamma \cup \{\infty\}$ *abelian* totally ordered group

- $v(x) = \infty \iff x = 0$,
- $v(x+y) \geq \min\big(v(x), v(y)\big)$,
- $v(xy) = v(x) + v(y)$.

Useful to obtain information on the fine structure of D:

- construction of noncrossed products (Amitsur 1972)
- counterexamples to the Kneser–Tits conjecture (Platonov 1977)

Examples

Twisted Laurent series

$D = E((x;\theta)) = \{\sum_{i\geq k} e_i x^i \mid e_i \in E\}, \qquad xe = \theta(e)x$

$v\colon D \to \mathbb{Z} \cup \{\infty\}, \qquad \sum e_i x^i \mapsto \min\{k \mid e_k \neq 0\}$

Composite valuations

If $u\colon E \to \Gamma \cup \{\infty\}$ is a valuation, define

$w\colon D \to (\Gamma \times \mathbb{Z}) \cup \{\infty\}$ by $d = \sum e_i x^i \mapsto \big(u(e_{v(d)}), v(d)\big)$.

valuation for the right-to-left lexicographic ordering

Specific example

$$D = \mathbb{C}((x_1))((x_2;\theta))((y_1))((y_2;\rho)),$$
$$\theta(x_1) = -x_1,\ \rho(x_i) = x_i,\ \rho(y_1) = -y_1$$

center $Z = \mathbb{C}((x_1^2))((x_2^2))((y_1^2))((y_2^2))$ and
D is biquaternion: $D = (x_1^2, x_2^2)_Z \otimes_Z (y_1^2, y_2^2)_Z$

$v\colon D \to \mathbb{Z}^4 \cup \{\infty\}$ $\quad v(D^\times)/v(Z^\times) = (\mathbb{Z}/2)^4$
$\overline{D} = \overline{Z} = \mathbb{C}$

$$D^\times \times D^\times \longrightarrow [D^\times, D^\times] \xrightarrow{\text{residue}} \mathbb{C}$$
$$D^\times \times D^\times \xrightarrow{v} (\mathbb{Z}/2)^4 \times (\mathbb{Z}/2)^4 \longrightarrow \mathbb{C}$$

Property

$i\ (=\sqrt{-1}) \notin [D^\times, D^\times]$ because $\overline{[D^\times, D^\times]} \subset \{\pm 1\}$
$\mathrm{Nrd}(i) = i^4 = 1$, so $\ker \mathrm{Nrd} \neq [D^\times, D^\times]$
(counterexample to Kneser–Tits)

Existence of valuations

D finite-dimensional division algebra over its center F

Each valuation v on D is an extension of $v|_F$ on F.

Theorem (Ershov 1982 – Wadsworth 1986)

Every Henselian valuation on F extends to D. The extension is unique.

Theorem (Ershov 1988 – Morandi 1989)

A valuation on F extends to D iff $D \otimes F^h$ is division (F^h = Henselization). The extension is unique.

Example

The local invariants of $(-1,-1)_{\mathbb{Q}}$ are trivial except at 2, ∞: the p-adic valuation on $\mathbb{Q}$ extends to $(-1,-1)_{\mathbb{Q}}$ only for $p = 2$.

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From division rings to quadratic forms

Common feature of central simple algebras and quadratic forms: the group of automorphisms is a simple linear algebraic group:

$\mathrm{Aut}(M_n) = \mathrm{PGL}_n$ type A_{n-1}

$\mathrm{Aut}(x_1^2 + \cdots + x_n^2) = \mathrm{O}_n$ type $D_{n/2}$ or $B_{(n-1)/2}$

Corresponding adjoint group:

$$\begin{aligned} \mathrm{PGO}_n &= \{g \in \mathrm{GL}_n \mid g^t \cdot g \text{ scalar}\}/\text{scalars} \\ &= \{\mathrm{Int}(g) \mid \mathrm{Int}(g) \circ t = t \circ \mathrm{Int}(g)\} \\ &= \mathrm{Aut}(M_n, t) \end{aligned}$$

From division rings to quadratic forms

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Theorem (Weil, 1960)

Adjoint classical groups of type A_{n-1}

$= $ *twisted forms of* PGL_n

$= \mathrm{Aut}(A)$ *for* A *central simple of degree* n

Adjoint classical groups of type D_m

$=$ *twisted forms of* PGO_{2m}

$= \mathrm{Aut}(A,\sigma)$ *for* (A,σ) *such that*

$$(A \otimes F_{alg}, \sigma \otimes \mathrm{Id}) \simeq (M_{2m}(F_{alg}), t)$$

i.e. A *is central simple of degree* $2m$ *and* $\sigma\colon A \to A$ *is an orthogonal involution*
in particular, σ *is linear,* $\sigma(xy) = \sigma(y)\sigma(x)$, $\sigma^2 = \mathrm{Id}$.

Anisotropy

For A central simple,

$$\begin{array}{rll} \mathrm{Aut}(A) \text{ is anisotropic} & \text{iff} & A \text{ is division} \\ \mathrm{Aut}(A,\sigma) \text{ is anisotropic} & \text{iff} & \sigma \text{ is anisotropic, i.e.} \\ & & \sigma(x)\cdot x = 0 \Rightarrow x = 0 \end{array}$$

Example

q quadratic form in n variables with Gram matrix b

σ = adjoint involution on M_n: $\quad \sigma(x) = b^{-1} \cdot x^t \cdot b$

$$\begin{array}{lll} \sigma \text{ anisotropic} & \text{iff} & x^t \cdot b \cdot x = 0 \Rightarrow x = 0 \\ & \text{iff} & q \text{ anisotropic.} \end{array}$$

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Question

Is there an analogue of a valuation for central simple algebras with anisotropic involution?

Note:

$$\left.\begin{array}{l} v(x) = \infty \iff x = 0 \\ v(xy) = v(x) + v(y) \end{array}\right\} \Longrightarrow \text{no zero-divisor}$$

Special value functions

$v\colon F \to \Gamma \cup \{\infty\}$ valuation with Γ divisible, caract $\overline{F} \neq 2$
$\sigma\colon A \to A$ involution on a central simple F-algebra

Theorem (Tignol–Wadsworth)

If v is Henselian and σ is anisotropic, there is a unique map $g\colon A \to \Gamma \cup \{\infty\}$ such that

- *g is a vector space valuation:*
  - $g(a) = \infty$ *iff* $a = 0$
  - $g(a+b) \geq \min\big(g(a), g(b)\big)$
  - $g(a\lambda) = g(a) + v(\lambda)$ *for* $\lambda \in F$
- *g is surmultiplicative:*
  - $g(1) = 0$ *and* $g(ab) \geq g(a) + g(b)$
- *g is σ-special:*
  - $g(\sigma(a)a) = 2g(a)$ *for* $a \in A$

Existence of special value functions

$A = \operatorname{End}_D V$, $\quad \sigma$ adjoint to $h\colon V \times V \to D$

σ anisotropic $\Rightarrow$ h anisotropic

v extends to $v_D\colon D \to \Gamma \cup \{\infty\}$

Define $\alpha\colon V \to \Gamma \cup \{\infty\}$ by
$$\alpha(x) = \tfrac{1}{2} v_D\big(h(x,x)\big)$$

α is a vector space valuation.

Every orthogonal base of V splits α:
$$\alpha\big(\textstyle\sum e_i\lambda_i\big) = \min\big(\alpha(e_i) + v_D(\lambda_i)\big)$$

Define $g\colon A \to \Gamma \cup \{\infty\}$ by

$$\begin{aligned} g(a) &= \min\big(\alpha\big(a(x)\big) - \alpha(x) \mid x \in V,\ x \neq 0\big) \\ &= \min\big(\alpha\big(a(e_i)\big) - \alpha(e_i) \mid i = 1, \ldots, n\big). \end{aligned}$$

Uniqueness: induction on $\dim V$. If $A = D$, then $g = v_D$.

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Examples

Hensel is needed:

$V = \mathbb{Q}^2$, h polar form of $q = x_1^2 + x_2^2$, $v =$ 5-adic valuation

$$(1,0) = (3,-1) + (-2,1)$$

$$\tfrac{1}{2} v\big(q(1,0)\big) = \tfrac{1}{2} v(1) = 0 \not\geq \min\big(\tfrac{1}{2} v\big(q(3,-1)\big), \tfrac{1}{2} v\big(q(-2,1)\big)\big) = \tfrac{1}{2}$$

Example

$A = M_n(F)$, v discrete valuation on F with uniformizer π,

σ adjoint to $\langle u_1, \ldots, u_r\rangle \oplus \langle \pi u_1', \ldots, \pi u_s'\rangle$

(u_1, ..., u_s' units, $r + s = n$)

$$g\left(\begin{array}{c|c} a_{ij} & b_{ij} \\ \hline c_{ij} & d_{ij} \end{array}\right) = \min\big(v(a_{ij}),\ v(b_{ij}) - \tfrac{1}{2},\ v(c_{ij}) + \tfrac{1}{2},\ v(d_{ij})\big)$$

Residue involutions

For g surmultiplicative and σ-special,

$$2g(a) = g(\sigma(a)a) \geq g\big(\sigma(a)\big) + g(a)$$

so $g(a) \geq g\big(\sigma(a)\big)$ for all $a \in A$

hence $g(a) = g\big(\sigma(a)\big)$ for all $a \in A$.

Define σ_0 on $A_0 = \{g \geq 0\}/\{g > 0\}$ by

$$\sigma_0(\bar{a}) = \overline{\sigma(a)}.$$

Example

$A = M_n(F)$, v discrete,

σ adjoint to $\langle u_1, \ldots, u_r\rangle \oplus \langle \pi u_1', \ldots, \pi u_s'\rangle$:

$A_0 = M_r(\overline{F}) \times M_s(\overline{F})$,

$$\sigma_0 = \mathrm{ad}\langle \overline{u_1}, \ldots, \overline{u_r}\rangle \times \mathrm{ad}\langle \overline{u_1'}, \ldots, \overline{u_s'}\rangle$$

…but the residue is only part of the story

For valuations on division rings, Γ_D acts on $\overline{D}$.

Example: $D = E((x;\theta))$:
$\Gamma_D = \mathbb{Z}$ acts through θ on $\overline{D} = E$.

Consider the graded ring associated to the filtration by g:

$$\mathbf{gr}(A) = \bigoplus_{\gamma\in\Gamma}\{g \geq \gamma\}/\{g > \gamma\}.$$

Let $\quad \widetilde{a} = a + \{g > g(a)\} \in \mathbf{gr}(A) \quad$ for $a \in A$.

g surmultiplicative $\Rightarrow$ $\mathbf{gr}(A)$ is a graded ring

$$\widetilde{a}\cdot\widetilde{b} = \begin{cases} \widetilde{ab} & \text{if } g(ab) = g(a) + g(b), \\ 0 & \text{if } g(ab) > g(a) + g(b). \end{cases}$$

Example

F with discrete valuation, uniformizer π:

$$\mathbf{gr}(F) = \overline{F}[\widetilde{\pi}^{\pm 1}] \quad \text{(Laurent polynomials)}$$

$A = M_n(F)$,

$$g\left(\begin{array}{c|c} a_{ij} & b_{ij} \\ \hline c_{ij} & d_{ij} \end{array}\right) = \min\bigl(v(a_{ij}),\ v(b_{ij}) - \tfrac{1}{2},\ v(c_{ij}) + \tfrac{1}{2},\ v(d_{ij})\bigr),$$

$$\mathbf{gr}(A) = M_n(\mathbf{gr}(F)), \qquad A_0 = \begin{pmatrix} M_r(\overline{F}) & 0 \\ 0 & M_s(\overline{F}) \end{pmatrix},$$

$$A_{1/2} = \begin{pmatrix} 0 & \widetilde{\pi} M_{r\times s}(\overline{F}) \\ M_{s\times r}(\overline{F}) & 0 \end{pmatrix}, \ \ldots$$

Proposition

A central simple over F Henselian, σ anisotropic, g the unique σ-special value function:
$\mathbf{gr}(A)$ is graded central simple over $\mathbf{gr}(F)$, and

$$[\mathbf{gr}(A) : \mathbf{gr}(F)] = [A : F].$$

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Gauges

A central simple over F, arbitrary valuation $v\colon F \to \Gamma \cup \{\infty\}$.

Definition

A gauge on A is a surmultiplicative vector space valuation $g\colon A \to \Gamma \cup \{\infty\}$ such that $\mathbf{gr}(A)$ is graded central simple over $\mathbf{gr}(F)$ and $[\mathbf{gr}(A) : \mathbf{gr}(F)] = [A : F]$.

Examples

$A =$ division algebra: a valuation is a gauge iff it is defectless.

$A = (-1,-1)_{\mathbb{Q}}, \quad v =$ 3-adic valuation:
$g(a_0 + a_1 i + a_2 j + a_3 k) = \min\big(v(a_0),\ v(a_1),\ v(a_2),\ v(a_3)\big)$
is a gauge with $\mathbf{gr}(A) \simeq M_2\big(\mathbb{F}_3[t^{\pm 1}]\big) \qquad (t = \widetilde{3})$.

Other gauges: $g_u(a) = g(uau^{-1})$
$g_u = g$ iff $\widetilde{u}$ is invertible in $\mathbf{gr}(A)$.

Gauges on algebras with involution

$\sigma\colon A \to A$ involution on a central simple F-algebra,
$v\colon F \to \Gamma \cup \{\infty\}$ valuation with char $\overline{F} \neq 2$

Theorem (Tignol–Wadsworth)

There is a σ-special gauge on A iff $(A,\sigma) \otimes F^h$ is anisotropic (F^h = Henselization). When it exists, the σ-special gauge is unique.

Note:

$$g(\sigma(a)a) = 2g(a) \iff \widetilde{\sigma(a)} \cdot \widetilde{a} \neq 0$$

so g is σ-special $\iff$ $\widetilde{\sigma}$ is anisotropic on $\mathbf{gr}(A)$.

Sketch of Proof:

If g is σ-special on A, then $g \otimes v^h$ is σ-special on $(A,\sigma) \otimes F^h$, hence $(A,\sigma) \otimes F^h$ is anisotropic.
If $(A,\sigma) \otimes F^h$ is anisotropic, it has a σ-special gauge g^h.
Then $g^h|_A$ is a σ-special gauge on A. □

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Toward a noncommutative Bröcker–Prestel theorem?

A quadratic form q is strongly anisotropic if $n \times q$ is anisotropic for all $n \in \mathbb{N}$. ($q \neq 0 \Rightarrow$ formally real base field)

Theorem (Bröcker – Prestel 1974)

q is strongly anisotropic iff either

- *there exists an ordering on F for which q is definite, or*
- *there exists a valuation on F such that q has at least two residue forms, and each residue form is strongly anisotropic.*

An involution $\sigma\colon A \to A$ is strongly anisotropic if $t \otimes \sigma\colon M_n \otimes A \to M_n \otimes A$ is anisotropic for all $n \in \mathbb{N}$.

σ is definite for an ordering P on F if $(A, \sigma) \otimes F_P$ is anisotropic $\quad (F_P =$ real closure$)$.

Toward a noncommutative Bröcker–Prestel theorem?

Problem:

If σ is strongly anisotropic and indefinite at each ordering, find a gauge on A such that σ has at least two residues, each strongly anisotropic.

OK if index$(A) \leq 2$: Kulshrestha (to appear)

Variant: Astier–Unger (2008)

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