WORKSHOP ON CENTRAL SIMPLE ALGEBRAS OVER FUNCTION FIELDS OF SURFACES KONSTANZ - AUGUST, 2007 INTRODUCTION

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Karim Becher asked me to give an introduction at the workshop on *Central Simple Algebras over Function Fields of Surfaces* that took place in Konstanz from August 27 till August 31. These notes are based on the slides I used for that talk. To prepare the talk I used especially the papers of Colliot-Thélène, [2, 3, 4].

1. The Brauer group of a field

It is known that the *Brauer group* of a field plays an important role in the arithmetic of a field. We mention only two applications, but many more could be given.

- (1932) For instance the calculation of the Brauer group of a global field k, expressed in the exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

  where v runs over "all" valuations, non-Archimedean and Archimedean, of the global field k, (cf. [1]),
  - contains implicitly the higher reciprocity laws, and
  - implies the Hasse Norm theorem for cyclic extensions
- (2006) T. Scanlon proved a conjecture of Florian Pop, (cf. [7]),
  - *Fields finitely generated over the prime field that are* elementary equivalent *are isomorphic.*
    Cyclic division algebras play an essential role in Scanlon's proof.

The aim of this introduction is to state the theorem of de Jong, on the period-index problem for central simple algebras over function fields in two variables over an algebraically closed field. To do this we need to give a survey of the basic definitions and facts from the theory of central simple algebras, (P.Gille explained all these facts extensively in his talks).

2. Central simple algebras over a field

Throughout k is a field.

1. Definition A central simple k-algebra is a finite dimensional k-algebra, with centre k and no non-trivial two sided ideals.

Date: Konstanz - August, 2007.

A central simple k-algebra of dimension n is a "twisted form" of the endomorphism algebra $\mathrm{End}(V) \cong M_n(k)$ of a finite dimensional k-vector space V. This is expressed more formally by the statement that the following properties are equivalent,

(1) A is a c.s. k-algebra,
(2) for all field extensions K, $A \otimes_k K$ is a c.s. K-algebra,
(3) there exists a (finite separable) field extension K/k, such that $A \otimes_k K \cong M_n(K)$. Such a field extension K/k is called a *splitting field for A*.

From this one obtains the following facts:

(a) $\dim_k A = n^2$, the square root of the dimension of A is called its *degree*, $\deg A := n$,
(b) If A, B c.s. k-alg., then $A \otimes_k B$ is a c.s. k-alg.
(c) The opposite algebra A^0, obtained by defining the multiplication on the k-vector space A as; $a \cdot b := ba$ is a c.s. k-alg, and $A \otimes A^0 \cong M_n(k)$. This can be seen as follows,

$$\begin{array}{rcl} A \otimes_k A^0 & \rightarrow & \mathrm{End}_{k-vec}(A) \\ (a \otimes b) & \mapsto & axb \end{array}$$

defines an injective morphism of algebras of the same dimension, so it defines an isomorphism.

3. Classifying central division algebras over k

2. Definition Consider the following equivalence relation on the set of isomorphism classes of c.s. k=alg.

$$A \sim B \Leftrightarrow M_r(A) \cong M_s(B) \text{ for some } r, s \in \mathbb{N}_{>0},$$

The tensor product induces the structure of an *abelian group* on the set of equivalence classes,

$$[A] \cdot [B] := [A \otimes B].$$

The Brauer group $Br(k)$ of the field k is this group of equivalence classes of c.s.

It follows from Wedderburn's theorem,

3. Theorem (Wedderburn) *let A be a c.s. k-algebra, then $A = M_n(D)$, with D a (finite dimensional) division algebra with centre k.*
$M_r(D) \cong M_s(D')$ iff $r = s$ and $D \cong D'$.

that the Brauer group *classifies division algebras over k.*

4. Definition The *index* of A, is the degree of the "division algebra part" of A, $i(A) :=$ $(\mathrm{ind}(A) :=) \deg D$

Brauer proved the following basic facts

5. Proposition *Let $A \cong M_n(D)$, be a c.s. k-algebra,*

- *if K is any maximal commutative subfield of D, $i(A) = [K : k]$,*
- *$i(A) = gcd\{[L : K] | D \otimes_k L \cong M_m(L)\}$,*
- *for any (tensor) power A^l of A, one has $i(A^l) | i(A)$,*

- $[A^{i(A)}] = [k]$.

It follows from the last property that the Brauer group of a field is a torsion group and the *period* (or *exponent*) of A, i.e., this is the period of the class $[A]$ in the abelian torsion group $\mathrm{Br}(k)$, divides the index of A. The relation between the period and the index of A is stronger,

6. Proposition *Let D be a c.d.a. over k. Let $i(D) = p_1^{n_1} \cdots p_r^{n_r}$, be the prime factorization of the index of D Then the period of D is of the form $p(D) = p_1^{m_1} \cdots p_r^{m_r}$, and for all i, $1 \leq m_i \leq n_i$.*

As a first application, of the fact that division algebras over a field k are classified by an abelian torsion group, Brauer proved,

7. Theorem *Let D be a c.d.a. over k. Then*

$$D \cong D_1 \otimes \cdots \otimes D_r,$$

with D_j central division algebras over k, and $i(D_j) = p_j^{n_j}$.

Proof: (Sketch)
The structure theorem of abelian groups implies that the element $[D]$ of period $p(D) = p_1^{m_1} \cdots p_r^{m_r}$, is a product of elements of period $p_i^{m_i}$, so $D \sim D_1 \otimes \cdots \otimes D_r$, with D_j k-division algebras and $p(D_j) = p_j^{m_j}$. Moreover it follows that $D_j \sim D^{\otimes l_j}$, $(l_j = \prod_{i \neq j} p_i^{m_i}))$.
Since $i(D_j) | i(D)$ we obtain, using proposition 6, $i(D_j) = p_j^{a_j}$, with $a_j < n_j$ Finally $p_1^{n_1} \cdots p_r^{n_r} = i(D) = i(D_1 \otimes \cdots \otimes D_r) = p_1^{a_1} \cdots p_r^{a_r} \leq p_1^{n_1} \cdots p_r^{n_r}$, implies, for all $j = 1, \ldots, r$, that $i(D_j) = p_j^{n_j}$. Comparing the degree of D with the degree of $D_1 \otimes \cdots \otimes D_r$ yields $D \cong D_1 \otimes \cdots \otimes D_r$. □

4. Examples - Cyclic algebras

We assume now (for the sake of simplicity) that the characteristic of k does not divide n.

8. Definition Let K/k be a cyclic Galois extension, $[K : k] = n$, $\mathrm{Gal}(K/k) = \langle \sigma \rangle$. Define on the K-vector space $A := \bigoplus_{i=0}^{n-1} K v^i$, an algebra structure by

$$v^n = a \in k, \text{ and } \forall x \in K, vx = \sigma(x)v.$$

Then one can proof that A is a central simple k-algebra of degree n. Such a c.s. k-algebra is called a *cyclic algebra*, it is denoted by $(K/k, \sigma, a)$.
For $n = 2$ one obtains the quaternion algebras $(a, b)_k := (k(\sqrt{(}b))/k, \sigma, a)$ over k.

9. Proposition

- *$(K/k, \sigma, a)$ is trivial in $Br(k)$ iff $a \in N_{K/k}(K^*)$.*
- *$(K/k, \sigma, a) \otimes_k (K/k, \sigma, b) \sim (K/k, \sigma, ab)$, so $p((K/k, \sigma, a))$ is the smallest integer such that $a^t \in N_{K/k}(K^*)$.*

10. Remark

- Not all division algebras are cyclic algebras.

- Central simple algebras have a Galois splitting field but not all *division algebras* have a maximal subfield which is Galois splitting field. (This is Amitsur's non-crossed product theorem.)
- The main open question on the structure of division algebras is: *Are algebras of prime index cyclic*?
  (Division algebras of index 2 and 3 are known to be cyclic, the problem is open for algebras of prime index ≥ 5. We refer to the talk by P. Gille.)

5. Period - index problem

The research on central simple algebras shows that the relation between the *period* and the *index* is very important. Some facts,

- let k be a global field then $p(A) = i(A)$, and any central simple algebra A over k is a cyclic algebra.
- Brauer gave the first example of a c.s.a. for which $p(A) \neq i(A)$:

$$k = \mathbb{C}(x, y, z, t) \text{ and } A = (x, y)_k \otimes (z, t)_k.$$

- the period-index problem is related to the problem of decomposing division algebras as tensor products,
  - there exist division algebras of period p, p prime, that are not isomorphic in a tensor product of division algebras of index p. (e.g., examples by Amitsur and Tignol for $p = 2$).

Up to Brauer equivalence Merkurjev and Suslin proved the following deep theorem,

11. Theorem (Merkurjev-Suslin)
Let $\zeta_n \in k$, then every c.s. k-algebra of period n is equivalent to a tensor product of cyclic algebras of index n.

Examples of algebras A with $p(A) \neq i(A)$. . The following propositions can be used to construct examples of c.s.a. with period not equal to the index.

12. Proposition (Jacobson) *The biquaternion algebra $A = (a, b)_k \otimes_k (c, d)_k$ is a division algebra iff the quadratic form, (called the Albert form of A), $\langle a, b, -ab, -c, -d, cd\rangle$ is anisotropic.*

The following is due to different authors, see for instance [2, proposition 2].

13. Proposition *Let K/k be a cyclic extension, $Gal(K/k) = \langle\sigma\rangle$, t a variable. A a c.s. k-algebra.*

$$i(A_{k(t)} \otimes (K(t)/k(t), \sigma, t)) = i(A_K) \cdot [K : k].$$

14. Corollary *Let $a, b, c \in k^*$, $c \notin k^{*2}$. Let $K = k(\sqrt{c})$. Then $(a, b)_K$ is a division algebra iff $(a, b)_k \otimes_k (c, d)_k$ is a division algebra.*

Examples

- The example given above $k = \mathbb{C}(x,y,z,t)$ and $A = (x,y)_k \otimes (z,t)_k$, $p(A) = 2$ and $i(A) = 4$. This since A is a division algebra because the quadratic form $\langle x, y, -xy, -z, -t, zt\rangle$ is anisotropic over k.
- Let $k = \mathbb{C}(x)(y,z)$ then one can use corollary 14 to prove that $(f(x),y)_k \otimes (g(x),z)_k$ is a division algebra iff $f(x) \not\equiv g(x) \bmod k^{*2}$.
  It follows $(x,y)_k \otimes_k (x+1,y)_k$ is a division algebra of index 4 and period 2.
- Let $a \in k^*$, $a \notin k^{*2}$, $K = k(x,y)$ then $(x,a)_K \otimes_K (x+1,y)_K$ is a division algebra of index 4 and period 2. (This example also follows from corollary 14.)

6. Intermezzo - Brauer group of a scheme - cohomology

We only mentioned definitions and basic facts concerning the Brauer group of a field. Azumaya (1951) extended the definition of the Brauer group to the case of local rings, Auslander-Goldman (1960) extended it further to the case of any commutative ring.
This is not sufficient to develop the theory. The deeper theorems, also the theorem of de Jong are based on other aspects of the Brauer group: cohomological interpretations and the algebraic geometric definition of "algebras" and of the Brauer group. This aspect will be addressed to extensively in the other lecture. We Grothendieck from:
Alexander GROTHENDIECK, *Le Groupe de Brauer*, in "Dix exposés sur la cohomologie des schémas"

- *Les développements de Azumaya-Auslander-Goldman peuvent se généraliser d'ailleur de façon essentiellement triviale au cas des préschémas de base généraux, ce qui est un des buts du présent exposé.*
- *Cependant, l'interprétation cohomologique du groupe de Brauer, qui a joué un rôle important dans le cas classique, ne pouvait être développée dans [3] et [2], faute de disposer d'une théorie de la cohomologie étale, développée dans [1] et [11]. C'est elle qui donne tout son charme à la variante "globale" du group de Brauer.*

7. Brauer group and cohomological dimension

From now on we assume, for the sake of simplicity, that k is a field of characteristic zero! Before we state de Jong's theorem we indicate some links between the Brauer group and the *cohomological dimension* of a field. We will not give a formal definition of cohomological dimension, we refer to [8] for it, instead we will use characterizations of fields of "low" cohomological dimension, characterizations that are strongly related to the Brauer group.

15. Definition The *reduced norm* of a c.s. k-algebra A, of degree n, is the map

$$nr : A \to A \otimes_k \overline{k} \to M_n(\overline{k}) \stackrel{\det}{\to} \overline{k}.$$

If $A = \bigoplus_{i=1}^{n^2} ke_i$, and $x = \sum_{i=1}^{n^2} x_i e_i \in A$ then $nr(x)$ is a homogeneous polynomial, of degree n, in the n^2 variables x_i. The reduced norm takes it values in k.
A is a division algebra iff the homogeneous polynomial $nr(x)$ has no non-trivial zero in k. In the same way one can define the *reduced characteristic polynomial* $\chi_{red}(t)$ for the elements $x \in A$. It is a polynomial of degree n. The coefficient of t^{n-1} is called the *reduced trace*, it is

a homogenous linear polynomial in the x_i. In general the coefficient of t^d is a homogeneous polynomial of degree $n-d$ in the x_i.

16. Remark Alternatively one can replace the algebraic closure $\overline{k}$ by any splitting field of A. One can show that the map one obtains is independent of the chosen splitting field. A priori the determinant takes values in the algebraic closure $\overline{k}$, however one can prove that the determinant of elements in the image of A does take values in k.

Fields of (cohomological) dimension ≤ 1. The following properties are equivalent

- For all (finite) algebraic extensions K/k, $Br(K) = 0$
- For all L/K finite Galois extensions, K/k (finite) algebraic, the norm $N_{L/K} : L^* \to k^*$ is surjective.

(Note that (1) implies (2), follows since the hypothesis implies that all cyclic algebras are trivial, use proposition 9. If we allow us to use the theorem of Merkurjev-Suslin it is also clear that (2) implies (1). However this implication can be obtained with cohomological methods without using the deep theorem of Merkurjev-Suslin.

17. Definition Fields k which satisfy these equivalent properties are said to be of *dimension* ≤ 1, (since we assume the characteristic to be zero it are the fields of cohomological dimension ≤ 1).

18. Definition A field k is called a C_i-field if all systems of homogeneous n-dimensional forms $f_j, j = 1, \ldots, r$ over k, with $\deg f_j = d_j$ and $n > \sum_j d_j^i$ have a non-trivial solution in k.

19. Lemma *Let k be a field for which there exist finite field extensions of arbitrarily large degree. Finite extensions of C_i fields are C_i fields.*
Extensions of transcendence degree d over a C_i field are C_{i+d} fields.

PROOF: See [6, page 310-312]. (See also the appendix at the end of this note.) □

Examples of fields of (cohomological) dimension ≤ 1.

- Algebraically closed fields.
- Let k be a C_1-field. The reduced norm form of a c.s. k-algebra is of degree $\deg A$ and has $\deg A^2$ variables. It follows that there are no non-trivial division algebras over k.
  (Note also that it is easy to see that the C_1 condition implies that the norm forms of finite field extension are surjective!)
- Chevalley and Warning proved that finite fields are C_1 fields.
- Tsen's theorem: Let k be an algebraically closed field then $k(x)$ is a C_1-field.

Fields of (cohomological) dimension ≤ 2. Merkurjev-Suslin's results imply:

20. Theorem *A field k is of (cohomological) dimension ≤ 2 iff for all finite extensions K/k and all K-division algebras D, the reduced norm map $nr_D : D^* \to K^*$ is surjective.*

PROOF: See [8] □

Examples of fields of (cohomological) dimension ≤ 2.

(1) Totally imaginary number fields are of dimension ≤ 2.
(2) C_2-fields, (e.g. using lemma 19, $\mathbb{C}(x, y)$, $\mathbb{F}_q(x)$), are of cohomological dimension ≤ 2.
(3) Merkurjev constructed for all N, fields of dimension ≤ 2 over which there exists quadratic forms of dimension N that are anisotropic. (The so called u-invariant of fields of cohomological dimension ≤ 2, can be arbitrarily high). It follows that there are fields of dimension ≤ 2 which are not C_i for any i.

8. de Jong's Theorem

21. Theorem (de Jong, cf. [5])
Let k be an algebraically closed field and let K/k be a finitely generated field extension of transcendence degree 2. Let A be a c.s. K-algebra. Then $i(A) = p(A)$.

We discuss a special case of this theorem, namely the case $p(A) = 2^m 3^n$. That the theorem holds in this case was noticed by several authors: Artin-Harris, Artin-Tate, Merkurjev-Suslin, Yanchevskii, Platonov, cf. [2]. First we show that it suffices to prove the theorem for algebras of prime period p.

Proof: Brauer's theorem, see theorem 7, implies that it suffices to consider algebras A with $p(A) = p^r$, p prime.
Assume one can prove that algebras of period p, p prime, are of index p. Let A be a c.s.a. of period p^r. We argue by induction on r. The algebra $A^{\otimes p}$ is of period p^{r-1}, so by induction it is split by an extension K'/K with $[K' : K] = p^{r-1}$. Then $A' = A \otimes_K K'$ has period p. We assume the theorem holds for algebras of prime period, so A' is split by an extension K''/K', with $[K'' : K'] = p$. It follows that K'' is a splitting field for A and $[K'' : K] = p^r$, it follows that $i(A) = p^r$. $\square$

The case $p(A) = 2$.
Proof: Let K/k be as in de Jong's theorem, let A be a c.s.a. over K of period 2. By Merkurjev-Suslin, A is equivalent a tensor product of quaternion algebra. But K is a C_2-field so $(a, b) \otimes (c, d)$ is never a division algebra since the quadratic form $\langle a, b, -ab, -c, -d, cd\rangle$ is isotropic. It follows that $i(A) = 2$. $\square$

The case $p(A) = 3$.
Proof: Again, invoking Merkurjev-Suslin, it suffices to show that any two cyclic division algebras A, B of deg 3 have a common splitting field of the form $K(\gamma)$, $[K(\gamma) : K] = 3$.
Write $A = K \oplus V_A$, with V_A a vector space of dimension 8 over K, similarly $B = K \oplus V_B$, $\dim_K V_N = 8$. We like to find $\alpha \in V_A$ and $\beta \in V_B$ such that α and β have the same reduced characteristic polynomial. Let $x_1 v_1 + \cdots + x_8 v_8$ represent a general element in V_A, and $y_1 w_1 + \cdots + y_8 w_8$ a general element in V_B. Comparing the coefficients of the characteristic polynomials of x, y, leads to 3 homogeneous equation in the 16 variables $x_1, \ldots, x_8, y_1, \ldots, y_8$, one of degree one (comparing the reduced traces), one of degree 2 (comparing the coefficients in degree 1), and one of degree 3 (comparing the reduced norms). The field K is a C_2-field it follows, since $16 > 1 + 2^2 + 3^2 = 14$, that this system

has a non-trivial solution. This solution defines a polynomial $\chi(t)$ which is the reduced characteristic polynomial of an element in $V_A \subset A$ and of an element in $V_B \subset B$. So $\chi(t)$ has to be irreducible. Let $L = K[t]/\chi(t)$, then $A \otimes_K B \otimes L \cong M_{n^2}(L)$. □

22. Remark The proof of these special cases only uses the fact that K is a C_2-field. (To use the Merkurjev-Suslin theorem we need, in the second case, the fact that a primitive third root of unity ζ_3 is in K. However this can be avoided, since adjoining ζ_3 to K yields an extension of degree 2, and neither the index nor the period of algebras of degree prime to 2 changes after quadratic extensions.

M. Artin formulates the following question (cf. citeart):

> *Let K be a C_2-field and A a c.s. K-algebra. Is the period of A equal to the index of A?*

The result of de Jong gives evidence for the answer to this question to be positive.

We recall the examples of algebras for which the index is not equal to the period mentioned above.

- $k = \mathbb{C}(x, y, z, t), A = (x, y)_k \otimes (z, t)_k.$, $p(A) = 2$, $i(A) = 4$
- $k = \mathbb{C}(x, y, z), A = (y, x)_k \otimes_k (y + 1, z)_k$, $p(A) = 2$, $i(A) = 4$
- k a field which is not quadratically closed, $a \in k^* \setminus k^{*2}$, $K = k(x, y)$, $A = (x, a)_K \otimes_K (x + 1, y)_K$ then $p(A) = 2$ and $i(A) = 4$.
- Merkurjev's fields of u-invariant $2n$ and cohomological dimension 2. The Clifford invariants associated to the anisotropic forms of maximal dimension yield c.s.a. of exponent 2 and arbitrarily high index.

The first two examples show that the hypothesis "*transcendence degree* 2" in de Jong's theorem is necessary. The third example shows that some (strong) condition on the base field is necessary, in de Jong's theorem the condition is: k is algebraically closed. A field of transcendence degree 2 over an algebraically closed field is a field of cohomolgical dimension ≤ 2, but the last example shows that cohomological dimension ≤ 2 alone does not suffice to have period equal to index.

We mentioned above that one can define the Brauer group not only for fields and commutative rings but also for schemes in general. The other lectures will spend ample time to explain this. We mention only that if k is an algebraically closed field and Y is a smooth projective geometrically connected variety over k then there is an injection of the Brauer group of Y into the Brauer group of the function field of Y, $\mathrm{Br}(Y) \hookrightarrow \mathrm{Br}(k(Y))$. The image of the Brauer of Y in $\mathrm{Br}(k(Y)$ defines an important subgroup, the so called unramified part of the Brauer group of $k(Y)$. In general it is not so easy to describe the elements of the unramified part of the Brauer group. The following result of Colliot-Thélène uses de Jong's theorem, and proves the existence of "unramified" division algebras for which the index is an arbitrarily high power of the period. (Previous to Colliot-Thélène's result, A.Kresh gave an example of an unramified biquaternion division algebra, cf. [2].)

23. Theorem (Colliot-Thélène, cf. [2]) *Soit l premier. Soit k un corps algébriquement clos de caractéristique différent de l. Pour $1 \leq n < m$, il existe une variété projective et lisse Y sur k, de dimension $l^{m-n}+1$, et une algèbre à division sur le corps des fonction $k(Y)$, non ramifiée sur Y, d'exposant l^n et d'indice l^m.*

9. Appendix: Remarks on C_i-fields

I add a few words on C_i fields, thereby replying on questions and comments that I obtained during the workshop.

The definition of C_i-field one mostly find in the literature is different form the one I gave in these notes.

24. Definition A field k is called a $\tilde{C}_i$-field if every homogeneous form of degree d in n variables, with $n > d^i$, has a non-trivial zero in k.

It is possible to prove that the $\tilde{C}_i$ property, in the sense of definition 24, is equivalent with the following statement,

> *Every system of r homogeneous forms of degree d in n variables, with $n > rd^i$ has a non-trivial zero in k.*

We refer to [6, page 310-312] for a proof of this equivalence. The proof is based on the fact that a field that is not algebraically closed has a *normic form* of arbitrarily large degree. A normic form over a field k, is a form of degree n in n-variables that has only the trivial zero in k. For instance the norm of a finite field extension K/k, $[K : k]$, gives rise to a normic form of degree n over k.
If a field the $\tilde{C}_i$ property, in the sense of definition 24, holds for a field k that allows a normic form of degree n for any n, then it is possible to prove that the field is also C_i in the stronger sense, i.e., in the sense of definition 18,

> *Every system of homogeneous forms, $f_1, \ldots, f_r$, in n variables,* $\deg f_i = d_i$*, with $n > \sum_{i=1}^r d^i$, has a non-trivial zero in k.*

As far as I know the question whether or not a field that is not algebraically closed admits a normic form for every degree n is still open.

During the lectures several examples and questions concerning C_i and $\tilde{C}_i$ fields came up.

- A field of transcendence degree d over an algebraically closed field is a C_d field.
- A field of transcendence degree d over a finite field is a C_{d+1} field.
- Certain properties of imaginary-global and local fields show that these fields "behave as C_2 or $\tilde{C}_2$-fields", e.g., the period of a division algebra over a global or a local field is equal to its index, quadratic forms over imaginary-global, or over local fields, of dimension > 4 have a non-trivial zero in their field of definition. However imaginary-global and local fields are not $\tilde{C}_i$ for any i.

Some comments concerning local fields. Terjanian (1966) gave an example of an form of degree 4 in 20 variables over $\mathbb{Q}_2$ that has only the trivial zero in $\mathbb{Q}_2$.
Work of russian mathematicians (and not of Ax, as Tamás Szamuely rightly pointed out during the last talk), Arkhipov, Karatsuba (1981), of Lewis, Montgomery (1983) and of Alemu (1985) proves that $\mathfrak{p}$-adic fields are not $\tilde{C}_i$ for any i. This work also showed that there exists forms of degree d over $\mathbb{Q}_p$, *d not a multiple of p*, in more that d^2 variables that only have trivial zeros in $\mathbb{Q}_p$. More recently David Leep and Nicolas Bartholdi (independently) found explicit examples of such forms.

Ax and-Kochen proved the following statement: For a fixed degree d there exists a finite (but non-constructible) set $S(d)$ of prime numbers such that any form f of degree d in $n > d^2$ variables over $\mathbb{Q}_p$ has a non-trivial zero provided $p \notin S$.

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