The conjecture of Geyer and Jarden about torsion of abelian varieties over large fields

(Joint work with Sara Arias-de-Reyne and Wojciech Gajda)

Let K be a finitely generated field, K_s a separable closure of K and $G_K = \mathrm{Gal}(K_s/K)$ its absolute Galois group. For a vector $\sigma = (\sigma_1, \cdots, \sigma_m) \in G_K^m$ we denote by $K_s(\sigma)$ the fixed field of the subgroup $\langle \sigma_1, \cdots, \sigma_m \rangle$ of G_K generated by the components of vector σ. If M is an abelian group, then let M_{tor} be its torsion subgroup. The following conjecture of Geyer and Jarden dates back to 1978.

Conjecture. *Let A/K be a non-zero abelian variety.*

a) *For almost all (in the sense of Haar measure on G_K) $\sigma \in G_K$ the group $A(K_s(\sigma))_{\mathrm{tor}}$ is infinite.*

b) *Let $m \geq 2$. Then for almost all $\sigma \in G_K^m$ the group $A(K_s(\sigma)_{tor})$ is finite.*

We discuss classical and recent results towards this conjecture. We explain how this conjecture can be translated into a question about the mod-ℓ Galois representations

$$\rho_{A,\ell} : G_K \to \mathrm{Aut}_{\mathbb{F}_\ell}(A[\ell]) \quad (\ell \text{ a prime number})$$

attached to A and how this question is related to other conjectures about these representations. Our main contribution is the following

Theorem. *Assume that $\mathrm{End}(A) = \mathbb{Z}$. Assume that A has semistable bad reduction of toric dimension 1 at some discrete valuation v of K. Then the conjecture of Geyer and Jarden holds true for A.*

The key ingredient in the proof of this theorem is an explicit computation of the images of the representations $\rho_{A,\ell}$ for this special class of abelian varieties. This computation extends (and heavily uses) recent work of Chris Hall. It has several other applications.