Trace positive binary quartics

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Advanced Course on Optimization:
Theory, Methods and Applications

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Motivation

psd = sos ?

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Theorem (Hilbert)

Let $f \in \mathbb{R}[X, Y]$, $\deg f \leq 4$ and $f \geq 0$ on $\mathbb{R}^2$. Then $f = \sum_i g_i^2$.

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- What does hold in the non-commutative case?

Matrix-positive polynomials

- $\mathbb{R}\langle X, Y\rangle$ polynomial ring in non-commuting variables X, Y

Definition

$f \in \mathbb{R}\langle X, Y\rangle$ is called matrix-positive ($f \succeq 0$) if

$$f(A, B) \succeq 0 \text{ for all } A, B \in \mathbb{S}\mathbb{R}^{t \times t}, t \in \mathbb{N}.$$

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  - $\sum g_i^2$ replaced by $\sum g_i^* g_i$
  - Involution $*$: fixes $\mathbb{R} \cup \{X, Y\}$ pointwise, $(XYX^2)^* = X^2YX$

psd=sos?

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Let $f \in \mathbb{R}\langle X, Y\rangle$, $f \succeq 0$. Then $f = \sum_i g_i^* g_i$.

Definition

$f \in \mathbb{R}\langle X, Y\rangle$ is called trace-positive ($\mathrm{tr}(f) \geq 0$) if

$$\mathrm{tr}(f(A, B)) \geq 0 \text{ for all } A, B \in \mathrm{S}\,\mathbb{R}^{t\times t}, t \in \mathbb{N}.$$

Further structure

- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in \mathrm{S}\,\mathbb{R}^{t\times t}$
  - $v \overset{\text{cyc}}{\sim} w \Leftrightarrow \exists u_1, u_2 : v = u_1 u_2, w = u_2 u_1$
- $\mathrm{tr}([A, B]) = 0$
  - $v \overset{\text{cyc}}{\sim} w \Leftrightarrow v - w = [u_1, u_2] = u_1 u_2 - u_2 u_1$
- trace is linear
  - $f \overset{\text{cyc}}{\sim} g \Leftrightarrow f - g = \sum_i [u_i, u_i']$

$v, w \in \langle X, Y\rangle, f, g \in \mathbb{R}\langle X, Y\rangle$

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Let $f \in \mathbb{R}\langle X, Y\rangle$, $f \succeq 0$. Then $f = \sum_i g_i^* g_i$.

Theorem

Let $f \in \mathbb{R}\langle X, Y\rangle$ homogenous of $\deg f \leq 4$, $\operatorname{tr}(f) \geq 0$. Then

$$f \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i.$$

Relation to semidefinite programming (SDP)

Gram matrix

Theorem (commutative case)

Let $f \in \mathbb{R}[X, Y]$, $\deg f \leq 2d$ and v a monomial vector of deg $\leq d$. Then

$$\exists G \in \mathrm{S}\,\mathbb{R}^{s \times s} : \ f = v^* G v.$$

Gram matrix

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Theorem (non-commutative case)

Let $f \in \mathbb{R}\langle X, Y\rangle$, $f = f^*$, $\deg f \leq 2d$ and v a monomial vector of $\deg \leq d$. Then
$$\exists G \in \mathrm{S}\,\mathbb{R}^{s\times s} : \; f = v^* G v.$$

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Useful fact

$$f \text{ is sos } \Leftrightarrow \exists \text{ Gram matrix } G \succeq 0.$$

Example

$$f = X^2Y^2 + Y^2X^2$$

$$v = \begin{pmatrix} X^2 \\ XY \\ YX \\ Y^2 \end{pmatrix} \implies \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$

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- $G_{nc} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \not\succeq 0 \implies f$ is not sos

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- $G_{nc} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \not\succeq 0 \implies f$ is not sos
- $G_{\sim} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0 \implies f \overset{\text{cyc}}{\sim} (XY)^*(XY) + (YX)^*(YX)$

Theorem

Let $f \in \mathbb{R}\langle X, Y\rangle$, $\deg f = 2d$, v monomial vector of deg $\leq d$ and G_0 a fixed Gram matrix of f. Then the following is equivalent:

1. $f \overset{\text{cyc}}{\sim} \sum g_i^* g_i$
2. $\min \operatorname{tr} G \quad s.t.\ f \overset{\text{cyc}}{\sim} v^* G v;\ G \succeq 0$ is feasible

Relation to SDP

Theorem

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1. $f \overset{\text{cyc}}{\sim} \sum g_i^* g_i$
2. $\min \operatorname{tr} G \quad s.t.\ f \overset{\text{cyc}}{\sim} v^* G v;\ G \succeq 0$ is feasible
3. $G = G_0 + M \succeq 0$ for some $M \in \mathcal{M}$

$$\mathcal{M} := \{A \in \mathrm{S}\,\mathbb{R}^{s \times s} \mid v^* A v \overset{\text{cyc}}{\sim} 0\}.$$

Dual problem

- $\langle A, B\rangle := \operatorname{tr}(B^* A)$ on $\mathrm{S}\,\mathbb{R}^{s\times s}$

$$\implies \mathcal{M}^{\perp} = \{A \in \mathrm{S}\,\mathbb{R}^{s\times s} \mid \operatorname{tr}(AM) = 0\ \forall M \in \mathcal{M}\}$$
$$= \{A \in \mathrm{S}\,\mathbb{R}^{s\times s} \mid A_{ij} = A_{kl} \text{ if } v_i^* v_j \overset{\text{cyc}}{\sim} v_k^* v_l\}.$$

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- Linear functional $\ell_f : \mathcal{M}^\perp \longrightarrow \mathbb{R}$, $\ell_f(M) = \operatorname{tr} G_0 M$.

Theorem

Assuming $(\mathcal{M}^\perp)^+$ contains a positive definite matrix. Then

$$f \overset{\text{cyc}}{\sim} \sum g_i^* g_i \iff \ell_f \geq 0 \text{ on } (\mathcal{M}^\perp)^+$$

Theorem

Assuming $(\mathcal{M}^\perp)^+$ contains a positive definite matrix. Then

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primal problem $\xleftrightarrow{\text{strict solution}}$ dual problem

primal proof:
Find $G \succeq 0$ with $f \overset{\text{cyc}}{\sim} v^* G v$

dual proof:
Show $\ell_f \geq 0$ on $(\mathcal{M}^\perp)^+$

Primal proof

$$f = X^4 + aY^4 + 2bX^2Y^2 + 2cXYXY + 2dXY^3$$

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$$f = X^4 + aY^4 + 2bX^2Y^2 + 2cXYXY + 2dXY^3$$

$$v = (X^2, XY, YX, Y^2)$$

$$G_\lambda = \begin{pmatrix} 1 & 0 & 0 & \lambda \\ 0 & b-\lambda & c & d/2 \\ 0 & c & b-\lambda & d/2 \\ \lambda & d/2 & d/2 & a \end{pmatrix}$$

- $c \le 0$: $\lambda = \lambda_{comm}$
- $c \ge 0, b \le c, a \le b^2 + 5c^2 - 6bc$: $\lambda = b - c$
- $c \ge 0, c \le b$ or $c \ge 0, b \le c, a \ge b^2 + 5c^2 - 6bc$:

$$\lambda = \frac{b+c}{3} - \frac{1}{3}\sqrt{3a + (b+c)^2}$$

Dual proof

$$
\begin{aligned}
(\mathcal{M}^{\perp})^{+} &= \{M \in \mathrm{S}\,\mathbb{R}^{4\times 4} \mid M_{ij} = M_{kl} \text{ for } v_i^* v_j \overset{\text{cyc}}{\sim} v_k^* v_l, M \succeq 0\} \\
&= \{T \mid T = (\mathrm{tr}(v_i^*(A,B) v_j(A,B)))_{i,j} \text{ for some } A, B \in \mathrm{S}\,\mathbb{R}^{14\times 14}\}
\end{aligned}
$$

Dual proof

$$(\mathcal{M}^{\perp})^{+} = \{M \in \mathrm{S}\,\mathbb{R}^{4\times 4} \mid M_{ij} = M_{kl} \text{ for } v_i^* v_j \overset{\text{cyc}}{\sim} v_k^* v_l, M \succeq 0\}$$
$$= \{T \mid T = (\mathrm{tr}(v_i^*(A,B)v_j(A,B)))_{i,j} \text{ for some } A, B \in \mathrm{S}\,\mathbb{R}^{14\times 14}\}$$

$$\begin{aligned}\ell_f(M) &= \mathrm{tr}(G_0 M)\\ &= \mathrm{tr}(G_0(\mathrm{tr}(v_i^*(A,B)v_j(A,B)))_{i,j})\\ &= \mathrm{tr}(v^*(A,B)G_0 v(A,B)) = \mathrm{tr}(f(A,B)) \geq 0.\end{aligned}$$

Conclusion

primal proof	dual proof
+ constructive	– non constructive
– difficult to generalize	+ more general