



Trace positive binary quartics

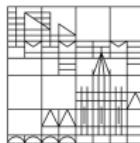
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Advanced Course on Optimization:
Theory, Methods and Applications

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Motivation

psd = sos ?



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Theorem (Hilbert)

Let $f \in \mathbb{R}[X, Y]$, $\deg f \leq 4$ and $f \geq 0$ on \mathbb{R}^2 . Then $f = \sum_i g_i^2$.



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- ▶ What does hold in the non-commutative case?



Matrix-positive polynomials

- ▶ $\mathbb{R}\langle X, Y \rangle$ polynomial ring in non-commuting variables X, Y

Definition

$f \in \mathbb{R}\langle X, Y \rangle$ is called **matrix-positive** ($f \succeq 0$) if

$$f(A, B) \succeq 0 \text{ for all } A, B \in \mathbb{S}\mathbb{R}^{t \times t}, t \in \mathbb{N}.$$



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 - ▶ A^2 is not psd for all $A \in \mathbb{S}\mathbb{R}^{t \times t}$
 - ▶ but A^*A



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 - ▶ A^2 is not psd for all $A \in \mathbb{S}\mathbb{R}^{t \times t}$
 - ▶ but A^*A
 - ▶ $\sum g_i^2$ replaced by $\sum g_i^*g_i$
 - ▶ Involution $*$: fixes $\mathbb{R} \cup \{X, Y\}$ pointwise, $(XYX^2)^* = X^2YX$



Theorem (Hilbert)

Let $f \in \mathbb{R}[X, Y]$, $\deg f \leq 4$ and $f \geq 0$ on \mathbb{R}^2 . Then $f = \sum_i g_i^2$.

Theorem (Helton)

Let $f \in \mathbb{R}\langle X, Y \rangle$, $f \succeq 0$. Then $f = \sum_i g_i^* g_i$.



Trace-positive polynomials

Definition

$f \in \mathbb{R}\langle X, Y \rangle$ is called **trace-positive** ($\text{tr}(f) \geq 0$) if

$$\text{tr}(f(A, B)) \geq 0 \text{ for all } A, B \in \mathbb{S}\mathbb{R}^{t \times t}, t \in \mathbb{N}.$$

Further structure

- ▶ $\text{tr}(AB) = \text{tr}(BA)$ for all $A, B \in \mathbb{S}\mathbb{R}^{t \times t}$
 - ▶ $v \stackrel{\text{cyc}}{\approx} w \Leftrightarrow \exists u_1, u_2 : v = u_1 u_2, w = u_2 u_1$
- ▶ $\text{tr}([A, B]) = 0$
 - ▶ $v \stackrel{\text{cyc}}{\approx} w \Leftrightarrow v - w = [u_1, u_2] = u_1 u_2 - u_2 u_1$
- ▶ trace is linear
 - ▶ $f \stackrel{\text{cyc}}{\approx} g \Leftrightarrow f - g = \sum_i [u_i, u'_i]$

$$v, w \in \langle X, Y \rangle, f, g \in \mathbb{R}\langle X, Y \rangle$$



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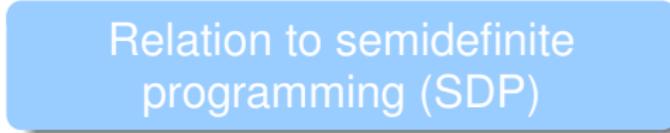
Theorem

Let $f \in \mathbb{R}\langle X, Y \rangle$ **homogenous** of $\deg f \leq 4$, $\text{tr}(f) \geq 0$. Then

$$f \stackrel{\text{cyc}}{\sim} \sum_i g_i^* g_i.$$



Relation to semidefinite
programming (SDP)





Gram matrix

Theorem (commutative case)

Let $f \in \mathbb{R}[X, Y]$, $\deg f \leq 2d$ and v a monomial vector of $\deg \leq d$.

Then

$$\exists G \in \mathbb{S} \mathbb{R}^{s \times s} : f = v^* G v.$$



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Theorem (non-commutative case)

Let $f \in \mathbb{R}\langle X, Y \rangle$, $f = f^*$, $\deg f \leq 2d$ and v a monomial vector of $\deg \leq d$. Then

$$\exists G \in \mathbb{S} \mathbb{R}^{s \times s} : f = v^* G v.$$



Gram matrix

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Theorem (trace case)

Let $f \in \mathbb{R}\langle X, Y \rangle$, $f \stackrel{\text{cyc}}{\sim} f^*$, $\deg f \leq 2d$ and v a monomial vector of $\deg \leq d$. Then

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Useful fact

$$f \text{ is sos} \Leftrightarrow \exists \text{ Gram matrix } G \succeq 0.$$



Example

$$f = X^2Y^2 + Y^2X^2$$

$$v = \begin{pmatrix} X^2 \\ XY \\ YX \\ Y^2 \end{pmatrix} \implies \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$



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► $G_{nc} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \not\succeq 0 \implies f \text{ is not sos}$



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► $G_{\sim} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0 \implies f \stackrel{\text{cyc}}{\sim} (XY)^*(XY) + (YX)^*(YX)$



Relation to SDP

Theorem

Let $f \in \mathbb{R}\langle X, Y \rangle$, $\deg f = 2d$, v monomial vector of $\deg \leq d$ and G_0 a fixed Gram matrix of f . Then the following is equivalent:

- 1 $f \stackrel{\text{cyc}}{\approx} \sum g_i^* g_i$
- 2 $\min \operatorname{tr} G \quad s.t. \quad f \stackrel{\text{cyc}}{\approx} v^* G v; G \succeq 0$ is feasible



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- 1 $f \stackrel{\text{cyc}}{\approx} \sum g_i^* g_i$
- 2 $\min \operatorname{tr} G \quad s.t. \quad f \stackrel{\text{cyc}}{\approx} v^* G v; G \succeq 0 \quad \text{is feasible}$
- 3 $G = G_0 + M \succeq 0 \text{ for some } M \in \mathcal{M}$

$$\mathcal{M} := \{A \in \mathbb{S} \mathbb{R}^{s \times s} \mid v^* A v \stackrel{\text{cyc}}{\approx} 0\}.$$



Dual problem

- ▶ $\langle A, B \rangle := \text{tr}(B^* A)$ on $S\mathbb{R}^{s \times s}$

$$\begin{aligned}\implies \mathcal{M}^\perp &= \{A \in S\mathbb{R}^{s \times s} \mid \text{tr}(AM) = 0 \ \forall M \in \mathcal{M}\} \\ &= \{A \in S\mathbb{R}^{s \times s} \mid A_{ij} = A_{kl} \text{ if } v_i^* v_j \stackrel{\text{cyc}}{\sim} v_k^* v_l\}.\end{aligned}$$



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- ▶ Linear functional $\ell_f : \mathcal{M}^\perp \longrightarrow \mathbb{R}$, $\ell_f(M) = \text{tr } G_0 M$.

Theorem

Assuming $(\mathcal{M}^\perp)^+$ contains a positive definite matrix. Then

$$f \stackrel{\text{cyc}}{\sim} \sum g_i^* g_i \Leftrightarrow \ell_f \geq 0 \text{ on } (\mathcal{M}^\perp)^+$$

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Assuming $(\mathcal{M}^\perp)^+$ contains a positive definite matrix. Then

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primal problem

strict solution
 \longleftrightarrow

dual problem

primal proof:

Find $G \succeq 0$ with $f \stackrel{\text{cyc}}{\sim} v^* G v$

dual proof:

Show $\ell_f \geq 0$ on $(\mathcal{M}^\perp)^+$



Primal proof

$$f = X^4 + aY^4 + 2bX^2Y^2 + 2cXYXY + 2dXY^3$$



Primal proof

$$f = X^4 + aY^4 + 2bX^2Y^2 + 2cXYXY + 2dXY^3$$

$$v = (X^2, XY, YX, Y^2)$$

$$G_\lambda = \begin{pmatrix} 1 & 0 & 0 & \lambda \\ 0 & b - \lambda & c & d/2 \\ 0 & c & b - \lambda & d/2 \\ \lambda & d/2 & d/2 & a \end{pmatrix}$$

- ▶ $c \leq 0$: $\lambda = \lambda_{\text{comm}}$
- ▶ $c \geq 0, b \leq c, a \leq b^2 + 5c^2 - 6bc$: $\lambda = b - c$
- ▶ $c \geq 0, c \leq b$ or $c \geq 0, b \leq c, a \geq b^2 + 5c^2 - 6bc$:

$$\lambda = \frac{b+c}{3} - \frac{1}{3}\sqrt{3a + (b+c)^2}$$



Dual proof

$$\begin{aligned}(\mathcal{M}^\perp)^+ &= \{M \in \mathbb{S}\mathbb{R}^{4 \times 4} \mid M_{ij} = M_{kl} \text{ for } v_i^* v_j \stackrel{\text{cyc}}{\sim} v_k^* v_l, M \succeq 0\} \\ &= \{T \mid T = (\text{tr}(v_i^*(A, B)v_j(A, B)))_{i,j} \text{ for some } A, B \in \mathbb{S}\mathbb{R}^{14 \times 14}\}\end{aligned}$$



Dual proof

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$$\begin{aligned}\ell_f(M) &= \text{tr}(G_0 M) \\ &= \text{tr}(G_0 (\text{tr}(v_i^*(A, B)v_j(A, B)))_{i,j}) \\ &= \text{tr}(v^*(A, B) G_0 v(A, B)) = \text{tr}(f(A, B)) \geq 0.\end{aligned}$$



Conclusion

primal proof

dual proof

- + constructive
- difficult to generalize
- + non constructive
- + more general