Trace positive polynomials, sums of hermitian squares and the tracial moment problem

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February 3, 2011

Content

Introduction

- $\mathbb{R}\langle \underline{X} \rangle$ free algebra on $\underline{X} = (X_1, \dots, X_n)$
- Polynomials in the non-commuting variables $X_1, \dots, X_n$
  - $f = \sum_w f_w w$, $w \in \langle \underline{X} \rangle, f_w \in \mathbb{R}$
- $\mathbb{R}\langle \underline{X} \rangle_k = \{f \in \mathbb{R}\langle \underline{X} \rangle \mid \deg f \leq k\}$

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- Evaluation in symmetric matrices
  - $\underline{A} = (A_1, \ldots, A_n) \in (\mathcal{S}\mathbb{R}^{s \times s})^n$:
  - $f(\underline{A}) = f_1 \mathbf{1}_s + f_{X_1} A_1 + f_{X_2} A_2 + \ldots + f_{X_1^2 X_3 X_2^3} A_1^2 A_3 A_2^3 + \ldots$

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- $\mathcal{S}^n := \bigcup_{s \in \mathbb{N}} (\mathcal{S}\mathbb{R}^{s \times s})^n$
- canonical projection $\check{}: \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}[x_1, \ldots, x_n]$, $X_i \mapsto x_i$

Trace-positivity

Definition

$f \in \mathbb{R}\langle \underline{X} \rangle$ is trace-positive if

$$\operatorname{Tr}(f(\underline{A})) \geq 0 \text{ for all tuples } \underline{A} \in \mathcal{S}^n.$$

If $\operatorname{Tr}(f(\underline{A})) \geq 0$ for all $\underline{A} \in K \subseteq \mathcal{S}^n$, f is trace-positive on K.

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Example

$f = X^2Y^2 - XYXY \in \mathbb{R}\langle X, Y \rangle$ is trace-positive: Let $A, B \in \mathcal{S}\mathbb{R}^{s\times s}$,

$$\begin{aligned}
\operatorname{Tr}(f(A,B)) &= Tr(A^2B^2 - ABAB) \\
&= \frac{1}{2}\operatorname{Tr}\left(AB^2A + BA^2B - ABAB - BABA\right) \\
&= \frac{1}{2}\operatorname{Tr}\left((AB-BA)^T(AB-BA)\right).
\end{aligned}$$

Cyclic equivalence

Definition

- $[p, q] = pq - qp$ for $p, q \in \mathbb{R}\langle \underline{X} \rangle$ is a commutator.
- $f, g \in \mathbb{R}\langle \underline{X} \rangle$ are cyclically equivalent ($f \overset{\text{cyc}}{\sim} g$) if

$$f - g = \sum_{i=1}^{k} [p_i, q_i] \text{ for some } k \in \mathbb{N}, p_i, q_i \in \mathbb{R}\langle \underline{X} \rangle.$$

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Example

$f = 2XY^2X - Y^2X^2 + 2YXY \overset{\mathrm{cyc}}{\sim} g = X^2Y^2 + 2XY^2$:

$$f - g = [XY^2, X] + [X, Y^2X] + 2[Y, XY].$$

Fact

$f,g \in \mathbb{R}\langle \underline{X} \rangle, f \overset{\mathrm{cyc}}{\sim} g \implies \forall \underline{A} \in \mathcal{S}^n : \mathrm{Tr}(f(\underline{A})) = \mathrm{Tr}(g(\underline{A}))$

Conjecture (Bessis, Moussa, Villani, 1975)

For all hermitian $A, B \in \mathbb{C}^{s\times s}$, $s \in \mathbb{N}$, $B \succeq 0$, exists a measure $\mu \geq 0$ supported on $\mathbb{R}_{\geq 0}$ s.t. for all $t \in \mathbb{R}$:

$$\mathrm{Tr}(e^{A-tB}) = \int e^{-tx} \mathrm{d}\mu(x).$$

BMV conjecture

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Theorem (Lieb, Seiringer, 2004)

The BMV conjecture holds iff for all $A, B \in S\mathbb{R}^{s\times s}$, $s \in \mathbb{N}$, $A, B \succeq 0$ and all $m \in \mathbb{N}$, we have

$$\mathrm{Tr}[(A+tB)^m] \in \mathbb{R}_{\geq 0}[t].$$

BMV Conjecture

- Write
$$\mathrm{Tr}[(A+tB)^m] = \sum_{k=0}^{m} \mathrm{Tr}[S_{m,k}(A,B)]t^k$$
- $S_{m,k}(A,B)$: Sum of all products with $m-k$ $A's$ and k $B's$.

Example

$S_{4,2}(A,B) = AABB + ABBA + BBAA + BAAB + ABAB + BABA$

BMV Conjecture

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- Replace $A, B \succeq 0$ by variables X^2, Y^2:

$$S_{m,k}(X^2, Y^2) \in \mathbb{R}\langle X, Y\rangle$$

Question

Is $S_{m,k}(X^2, Y^2)$ trace-positive for all $m, k \in \mathbb{N}_0, k \leq m$?

Sums of hermitian squares

- Involution $^* : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}\langle \underline{X} \rangle$ compatible with transposition ^{T}:

$$f^*(\underline{A}) = f(\underline{A})^T \text{ for all } \underline{A} \in \mathcal{S}^n$$

Definition

- $\Sigma^2 := \{f \in \mathbb{R}\langle \underline{X} \rangle \mid f = \sum_{i=1}^r {g_i}^* g_i \text{ for some } g_i \in \mathbb{R}\langle \underline{X} \rangle, r \in \mathbb{N}_0\}$
- $\Theta^2 := \{f \in \mathbb{R}\langle \underline{X} \rangle \mid f \overset{\text{cyc}}{\sim} g \text{ for some } g \in \Sigma^2\}$.

Example

$f = X^2Y^2 - XYXY$:

$$f \overset{\text{cyc}}{\sim} \frac{1}{2}(XY^2X + YX^2Y - XYXY - YXYX)$$
$$= \frac{1}{2}(XY - YX)^*(XY - YX) \in \Sigma^2$$

Sums of hermitian squares

Fact

$$f \in \Theta^2 \implies f \text{ trace-positive.}$$

Question

$$f \text{ trace-positive} \implies f \in \Theta^2 \;?$$

- Answer: In general not

Cyclically sorted polynomials

Definition

$f \in \mathbb{R}\langle X, Y\rangle$ is cyclically sorted if $f \overset{\text{cyc}}{\sim} \sum_{i,j} a_{ij} X^i Y^j$ for some $i, j \in \mathbb{N}_0$, $a_{ij} \in \mathbb{R}$.

Theorem

Let $f \in \mathbb{R}\langle X, Y\rangle$ be cyclically sorted. Then:

1. f trace-positive $\Leftrightarrow \check{f} \in \mathbb{R}[x, y]$ positive on $\mathbb{R}^2$,
2. $f \in \Theta^2 \Leftrightarrow \check{f} \in \sum \mathbb{R}[x, y]^2$.

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Example

- univariate polynomials
- quadratic polynomials in $\mathbb{R}\langle X, Y\rangle$

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Example

- univariate polynomials
- quadratic polynomials in $\mathbb{R}\langle X, Y\rangle$
- $f = X^2Y^4 + X^4Y^2 - 3X^2Y^2 + 1 \notin \Theta^2$

Bivariate Quartics

Theorem (B., Klep)

Let $f \in \mathbb{R}\langle X, Y\rangle_4$. The following statements are equivalent:

1. f trace-positive;
2. $\mathrm{Tr}(f(A, B)) \geq 0$ for all $A, B \in (\mathcal{S}\mathbb{R}^{2\times 2})^2$;
3. $f \overset{\mathrm{cyc}}{\sim} \sum_{i=1}^{4} g_i{}^* g_i$ with $g_i \in \mathbb{R}\langle X, Y\rangle_2$;
4. $f \in \Theta^2$.

- The bound in 3 is sharp!

Example

$f = 1 + \frac{1}{2}X^2 + X^4 + Y^4 + 2XYXY$

Tracial Gram matrix method

Theorem

Let $f \in \mathbb{R}\langle \underline{X} \rangle$. Then $f \in \Theta^2$ iff there exists a $G \succeq 0$ such that

$$f \overset{\text{cyc}}{\sim} \mathbf{v}^* G \mathbf{v},$$

where $\mathbf{v}$ consists of all words $w \in \langle \underline{X} \rangle$ with

$$\operatorname{mindeg}(f) \leq 2 \deg(w) \leq \deg(f).$$

Given such a $G \succeq 0$ of rank r, one can construct polynomials $g_1, \ldots, g_r \in \mathbb{R}\langle \underline{X} \rangle$ such that

$$f \overset{\text{cyc}}{\sim} \sum_{i=1}^{r} {g_i}^* g_i$$

and vice versa.

Example

BMV polynomials

Theorem (B.)

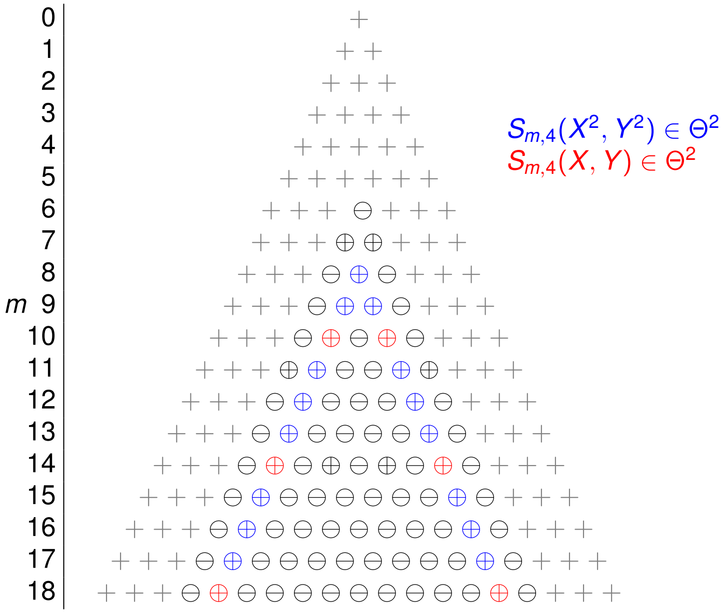
For all $m \in \mathbb{N}$, $m \geq 4$, we have $S_{m,4}(X^2, Y^2) \in \Theta^2$.

Theorem (B.)

For all $r \in \mathbb{N}$, we have $S_{4r+2,4}(X, Y) \in \Theta^2$.

Examples

BMV polynomials



Connes' embedding conjecture

Conjecture (Connes, 1976)

If ω is a free ultrafilter on $\mathbb{N}$ and $\mathcal{F}$ is a II_1 factor with separable predual, then $\mathcal{F}$ can be embedded into the ultrapower $\mathcal{R}^\omega$.

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If ω is a free ultrafilter on $\mathbb{N}$ and $\mathcal{F}$ is a II_1 factor with separable predual, then $\mathcal{F}$ can be embedded into the ultrapower $\mathcal{R}^\omega$.

- $\mathrm{QM} := \{f \in \mathbb{R}\langle \underline{X} \rangle \mid f \overset{\text{cyc}}{\sim} \sigma + \sum_{i,j} p_{ij}^*(1 - X_i^2)p_{ij}, p_{ij} \in \mathbb{R}\langle \underline{X} \rangle, \sigma \in \Sigma^2\}$
- $K := \{\underline{A} \in \mathcal{S}^n \mid \mathbf{1} - A_i^2 \succeq 0 \text{ for all } i = 1, \ldots, n\}.$

Theorem (Klep, Schweighofer, 2008)

Connes' conjecture holds iff the following is equivalent:

(i) f trace-positive on K,

(ii) $\forall \varepsilon \in \mathbb{R}_{>0} : f + \varepsilon \in \mathrm{QM}.$

The tracial moment problem

Definition

$L \in (\mathbb{R}\langle \underline{X} \rangle)^*$ is tracial if $L(pq - qp) = 0$ for all $p, q \in \mathbb{R}\langle \underline{X} \rangle$.

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Tracial moment problem

For which tracial linear functionals $L \in (\mathbb{R}\langle \underline{X} \rangle)^*$ exist some $s \in \mathbb{N}$ and a probability measure μ on $(\mathcal{S}\mathbb{R}^{s \times s})^n$, such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

$$L(f) = \int \operatorname{Tr}(f(\underline{A}))\, d\mu(\underline{A})?$$

- μ is a representing measure for L
- $s = 1$: Classical moment problem

The tracial moment problem

Truncated tracial moment problem

For which tracial $L \in (\mathbb{R}\langle \underline{X} \rangle_k)^*$ exist some $s \in \mathbb{N}$ and a probability measure μ on $(\mathcal{S}\mathbb{R}^{s \times s})^n$, such that for all $f \in \mathbb{R}\langle \underline{X} \rangle_k$:

$$L(f) = \int \mathrm{Tr}(f(\underline{A}))\, d\mu(\underline{A})?$$

Tracial K-moment problem

Let $K \subseteq \mathcal{S}^n$ be closed. For which tracial $L \in (\mathbb{R}\langle \underline{X} \rangle)^*$ exist some $s \in \mathbb{N}$ and a probability measure μ with $\operatorname{supp} \mu \subseteq K \cap (\mathcal{S}\mathbb{R}^{s \times s})^n$, such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

$$L(f) = \int \mathrm{Tr}(f(\underline{A}))\, d\mu(\underline{A})?$$

Duality

- $K \subseteq (\mathcal{S}\mathbb{R}^{s\times s})^n$ non-empty, closed set, $s \in \mathbb{N}$ fixed

- $\mathcal{P}(K) := \{p \in \mathbb{R}\langle \underline{X} \rangle \mid p$ trace-positive on $K\}$
- $\mathcal{M}(K) := \{L \in (\mathbb{R}\langle \underline{X} \rangle)^* \mid L$ has a K-representing measure$\}$

Duality

- $K \subseteq (\mathcal{S}\mathbb{R}^{s\times s})^n$ non-empty, closed set, $s \in \mathbb{N}$ fixed
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Proposition (B.)

The convex cones $\mathcal{M}(K)$ and $\mathcal{P}(K)$ are dual to each other.

Proof

- $\mathcal{P}(K)^* \subseteq \mathcal{M}(K)$:
  - Reduction to truncated case
  - Positive extensions of tracial linear functionals
  - Limits of sequences of measures

Tracial Hankel matrix

- Associate to a tracial $L \in (\mathbb{R}\langle \underline{X} \rangle)^*$ the bilinear form

$$B_L : \mathbb{R}\langle \underline{X} \rangle \times \mathbb{R}\langle \underline{X} \rangle, (f, g) \mapsto L(f^* g).$$

Definition

The tracial Hankel matrix $M(L)$, indexed by $u, v \in \langle \underline{X} \rangle$, is given by

$$M(L) := [L(u^* v)]_{u,v}.$$

The tracial Hankel matrix of order k is the submatrix $M_k(L)$ of $M(L)$ indexed by $u, v \in \mathbb{R}\langle \underline{X} \rangle_k$.

- For tracial $L \in (\mathbb{R}\langle \underline{X} \rangle_{2k})^*$ we can define $M_d(L)$ for all $d \leq k$.

Necessary Conditions

- Fix basis $(w_1, w_2, \dots)$; $\vec{p} = (p_{w_1}, p_{w_2}, \dots)^T$
- $I_M := \{p \in \mathbb{R}\langle \underline{X} \rangle \mid M\vec{p} = 0\}$, M tracial Hankel Matrix
- $V_s(I_M) = \{\underline{A} \in (\mathcal{S}\mathbb{R}^{s\times s})^n \mid p(\underline{A}) = 0 \text{ for all } p \in I_M\}$

Proposition (B.,Klep)

Let $L \in (\mathbb{R}\langle \underline{X} \rangle_{2k})^*$ be tracial with representing measure μ on $(\mathcal{S}\mathbb{R}^{s\times s})^n$ for some $s \in \mathbb{N}$. Then

1. $M_k(L) \succeq 0$,
2. $\operatorname{supp} \mu \subseteq V_s(I_{M_k(L)})$ and
3. $\operatorname{rank} M_k(L) \leq |\operatorname{supp} \mu| s^2$.

- Let $L := \operatorname{Tr} \circ \varepsilon_{\underline{A}}$ where $\varepsilon_{\underline{A}} : \mathbb{R}\langle \underline{X} \rangle_k \to \mathbb{R}^{s\times s}, p \mapsto p(\underline{A})$
- $\operatorname{rank} M \leq \dim(\mathbb{R}\langle \underline{X} \rangle_k / \ker \varepsilon_{\underline{A}}) \leq s^2$

Examples

Tracial Hankel matrices of finite rank

Theorem (B.,Klep)

Let $L \in (\mathbb{R}\langle \underline{X} \rangle)^*$ be tracial with

(i) $M(L) \succeq 0$

(ii) $\operatorname{rank} M(L) = s < \infty$.

Then L has a representing measure on $(\mathcal{S}\mathbb{R}^{s\times s})^n$.

- GNS construction
  - *-subalgebra $\mathcal{A}$ of $\mathbb{R}^{s\times s}$
  - Tracial state $\hat{L}$ on $\mathcal{A}$ with $L(p) = \hat{L}(p(\underline{A}))$ for all $p \in \mathbb{R}\langle \underline{X} \rangle$
- Artin-Wedderburn theorem
  - Decomposition into fin. dim. simple algebras $\mathcal{A}^{(i)}$ over $\mathbb{R}$
- Riesz representation theorem
  - Tracial state on $\mathcal{A}^{(i)}$ is Tr

Flat extensions

Theorem (B.,Klep)

Let $L \in (\mathbb{R}\langle \underline{X} \rangle_{2k})^*$ be tracial with

1. $M_k(L) \succeq 0$
2. $\operatorname{rank} M_k(L) = \operatorname{rank} M_{k-1}(L) =: s$.

Then L has a representing measure on $(\mathcal{S}\mathbb{R}^{s \times s})^n$.

Theorem (B.,Klep)

Let $L \in (\mathbb{R}\langle \underline{X} \rangle_{2k})^*$ be tracial. If $\operatorname{rank} M_k(L) = \operatorname{rank} M_{k-1}(L)$ then there exist a unique tracial extension $\tilde{L} \in (\mathbb{R}\langle \underline{X} \rangle_{2k+2})^*$ of L such that $\operatorname{rank} M_{k+1}(\tilde{L}) = \operatorname{rank} M_k(L)$.

▸ Proof

▸ Outlook

Summary

Investigation of trace-positive polynomials

- Motivated by two conjectures (BMV, Connes)
- Trace-positivity on $\mathcal{S}^n$
  - Sums of hermitian squares and commutators
    - Bivariate Quartics
    - BMV polynomials with $k = 4$
- Trace-positivity on a closed semialgebraic set K
  - Tracial moment problem
    - Dual perspective
    - Tracial Hankel matrices of finite rank
    - Tracial Hankel matrices with a flat extension

Zusammenfassung

Untersuchung spurpositiver Polynome

- BMV-Vermutung, Einbettungsvermutung von Connes
- Spurpositivität auf $\mathcal{S}^n$
  - Summen hermitescher Quadrate und Kommutatoren
    - Bivariate Quartiken
    - BMV-Polynome mit $k = 4$
- Spurpositivität auf semialgebraischer Menge K
  - Spuriges Momentenproblem
    - Duale Perspektive
    - Spurige Hankelmatrizen von endlichem Rang
    - Spurige Hankelmatrizen mit flacher Erweiterung

Résumé

Recherche des polynômes à trace positive

- Motivation: Conjecture BMV et Conjecture de Connes
- Positivité sur $\mathcal{S}^n$
  - Sommes de carrés hermitiens et de commutateurs
    - Quartiques binaires
    - Polynômes de BMV avec $k = 4$
- Positivité sur un ensemble K semi-algébrique
  - Problème des moments traciaux
    - Dualité
    - Matrices traciales de Hankel de rang fini
    - Matrices traciales de Hankel avec une extension plate

Example

- $S_{2,1}(X^2, Y^2) = X^2Y^2 + Y^2X^2$
- $\mathbf{v} = (X^2, XY, YX, Y^2)^T \qquad \mathbf{v}^* = (X^2, YX, XY, Y^2)$

$$\mathbf{v}^*\mathbf{v} = \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$

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- $f \overset{\text{cyc}}{\sim} \mathbf{v}^* G \mathbf{v}$

Example

- $S_{2,1}(X^2, Y^2) = X^2Y^2 + Y^2X^2$
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- $f \overset{\text{cyc}}{\sim} \mathbf{v}^* G \mathbf{v}$

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0$$

- $f \overset{\text{cyc}}{\sim} (XY)^*(XY) + (YX)^*(YX)$

Gram matrix method

$S_{9,4}(X^2, Y^2)$

$$S_{9,4}(X^2,Y^2) \overset{\text{cyc}}{\sim} 9X^{10}Y^8 + 9X^6Y^2X^4Y^6 + 9X^6Y^4X^4Y^4 + 9X^6Y^6X^4Y^2 + \\ + 9X^8Y^2X^2Y^6 + 9X^8Y^4X^2Y^4 + 9X^8Y^6X^2Y^2 + 9X^4Y^2X^2Y^2X^4Y^4 + \\ + 9X^4Y^2X^4Y^2X^2Y^4 + 9X^4Y^2X^4Y^4X^2Y^2 + 9X^6Y^2X^2Y^2X^2Y^4 + \\ + 9X^6Y^2X^2Y^4X^2Y^2 + 9X^6Y^4X^2Y^2X^2Y^2 + 9X^4Y^2X^2Y^2X^2Y^2X^2Y^2$$

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$$S_{9,4}(X^2,Y^2) \overset{\text{cyc}}{\sim} 9X^{10}Y^8 + 9X^6Y^2X^4Y^6 + 9X^6Y^4X^4Y^4 + 9X^6Y^6X^4Y^2 + 9X^8Y^2X^2Y^6 + 9X^8Y^4X^2Y^4 +$$
$$+ 9X^8Y^6X^2Y^2 + 9X^4Y^2X^2Y^2X^4Y^4 + +9X^4Y^2X^4Y^2X^2Y^4 + 9X^4Y^2X^4Y^4X^2Y^2 + 9X^6Y^2X^2Y^2X^2Y^4 +$$
$$+ 9X^6Y^2X^2Y^4X^2Y^2 + 9X^6Y^4X^2Y^2X^2Y^2 + 9X^4Y^2X^2Y^2X^2Y^2X^2Y^2$$

$$\mathbf{v} = (Y^2X^2Y^2X^3, Y^4X^5, Y^2X^4Y^2X, X^2Y^4X^3, X^4Y^4X, X^2Y^2X^2Y^2X)^T$$

$$G = 9\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- $S_{9,4} \overset{\text{cyc}}{\sim} 9\sum_{i=1}^{3} g_i{}^* g_i$ with
  - $g_1 = Y^2X^2Y^2X^3 + Y^4X^5 + Y^2X^4Y^2X$
  - $g_2 = X^2Y^4X^3$
  - $g_3 = X^2Y^2X^2Y^2X + X^4Y^4X$

$S_{17,4}(X^2, Y^2)$

$$\mathbf{v} = (Y^2X^{10}Y^2X^3, Y^2X^8Y^2X^5, Y^2X^6Y^2X^7, Y^2X^4Y^2X^9, Y^2X^2Y^2X^{11},$$
$$Y^4X^{13}, Y^2X^{12}Y^2X, X^2Y^2X^4Y^2X^7, X^2Y^2X^6Y^2X^5, X^2Y^2X^2Y^2X^9,$$
$$X^2Y^4X^{11}, X^2Y^2X^8Y^2X^3, X^4Y^2X^2Y^2X^7, X^4Y^4X^9, X^4Y^2X^4Y^2X^5,$$
$$X^6Y^4X^7, X^8Y^4X^5, X^6Y^2X^2Y^2X^5, X^8Y^2X^2Y^2X^3, X^6Y^2X^4Y^2X^3,$$
$$X^4Y^2X^6Y^2X^3, X^{10}Y^4X^3, X^{12}Y^4X, X^{10}Y^2X^2Y^2X, X^8Y^2X^4Y^2X,$$
$$X^6Y^2X^6Y^2X, X^4Y^2X^8Y^2X, X^2Y^2X^{10}Y^2X)$$

$$G = 17\begin{pmatrix} \boxed{1}_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1}_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1}_6 \end{pmatrix}$$

$S_{10,4}(X^2, Y^2)$

$$\mathbf{v} = (Y^2X^4Y^2X^2, Y^2X^2Y^2X^4, Y^4X^6, X^2Y^4X^4, Y^2X^6Y^2, X^2Y^2X^2Y^2X^2, XY^2X^2Y^2X^3, XY^4X^5, XY^2X^4Y^2X, X^3Y^4X^3)$$

$$G = 5\begin{pmatrix} 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & & & & \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 1 & 1 & \frac{1}{2} & 0 \\ & & & & & & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{v} = (Y^2X^4Y^2X^2, Y^2X^2Y^2X^4, Y^4X^6, X^2Y^4X^4, Y^2X^6Y^2, X^2Y^2X^2Y^2X^2, XY^2X^2Y^2X^3, XY^4X^5, XY^2X^4Y^2X, X^3Y^4X^3)$$

$$G = 5\begin{pmatrix} 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & & & & \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 1 & 1 & \frac{1}{2} & 0 \\ & & & & & & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$S_{10,4}(X^2, Y^2)$

$$\mathbf{v}' = (Y^2X^4Y^2X^2, Y^2X^2Y^2X^4, Y^4X^6, X^2Y^4X^4, Y^2X^6Y^2, X^2Y^2X^2Y^2X^2)$$

$$G = 5\begin{pmatrix} 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & & & & \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 1 & 1 & \frac{1}{2} & 0 \\ & & & & & & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \Rightarrow G' = 5\begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

- $S_{10,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} {g_1}^* g_1 + {g_2}^* g_2 + {g_3}^* g_3$ where $g_i \in \mathbb{R}\langle X^2, Y^2 \rangle$
- $S_{10,4}(X, Y) \in \Theta^2$

Theorems

Reduction to the truncated case

- $K \subseteq (\mathcal{S}\mathbb{R}^{s \times s})^n$ non-empty closed set, $s \in \mathbb{N}$ fixed

Theorem (B.)

Let $L \in (\mathbb{R}\langle \underline{X} \rangle)^*$ be tracial. L has a K-representing measure iff

$$L(f) \geq 0 \text{ for all } f \in \mathcal{P}(K).$$

Theorem (B.,Klep)

Let $L \in (\mathbb{R}\langle \underline{X} \rangle)^*$. If there is for all $k \in \mathbb{N}$ a measure μ_k supported in K such that

$$L(f) = \int \operatorname{Tr}(f(\underline{A}))\, d\mu_k(\underline{A})$$

for all $f \in \mathbb{R}\langle \underline{X} \rangle_k$, then L has a K-representing measure.

- Show: $L|_{\mathbb{R}\langle \underline{X} \rangle_{2k}}$ has a K-representing measure ($k \in \mathbb{N}$)

The truncated case

- $K \subseteq (\mathcal{S}\mathbb{R}^{s\times s})^n$ non-empty, closed set, $s \in \mathbb{N}$ fixed
- $\mathcal{P}_d(K) := \{p \in \mathbb{R}\langle \underline{X}\rangle_d \mid p \text{ trace-positive on } K\}$

Theorem (B.)

Let $L \in (\mathbb{R}\langle \underline{X}\rangle_{2k})^*$ be tracial. If $L(f) \geq 0$ for all $f \in \mathcal{P}_{2k}(K)$ then $L|_{\mathbb{R}\langle \underline{X}\rangle_{2k-1}}$ has a K-representing measure.

Corollary (B.)

Let $L \in (\mathbb{R}\langle \underline{X}\rangle_{2k})^*$ be tracial. L has a K-representing measure iff L admits a positive extension $\tilde{L} : \mathbb{R}\langle \underline{X}\rangle_{2k+2} \to \mathbb{R}$, i.e. $\tilde{L}(f) \geq 0$ for all $f \in \mathcal{P}_{2k+2}(K)$.

- $L|_{\mathbb{R}\langle \underline{X}\rangle_{2k+2}}$ is a positive extension of $L|_{\mathbb{R}\langle \underline{X}\rangle_{2k}}$

Duality

Example 1

Example

W.r.t. the basis $(1, X, X^2)$:

$$M := M_2(L) := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \succeq 0$$

$L \in (\mathbb{R}\langle X \rangle_4)^*$ does not have a representing measure.

- $I_M = \operatorname{span}\{1 - X\}$
- $\operatorname{supp} \mu \subseteq \{\mathbf{1}_s\}$

Example 2

W.r.t. the basis $(1, X, Y, X^2, XY, YX, Y^2)$:

$$M := M_2(L) := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 4 \end{bmatrix} \succeq 0$$

$L \in (\mathbb{R}\langle X, Y\rangle_4)^*$ does not have a representing measure.

- $s = 1$: rank $M = 5 > 2 = |V_1(I_M)| \geq |\operatorname{supp} \mu|$
- $s = 2$: $|V_2(I_M)| = 4 \implies \operatorname{rank} M = 5 \leq 4|V_2(I_M)| = 16$
- Cauchy-Schwarz inequality implies $M_{55} = M_{56}$

Proposition

Proof (flat extension)

- Construct a tracial Hankel matrix

$$M_{k+1} := \begin{bmatrix} M_k & B \\ B^T & C \end{bmatrix}$$

- Fix $V \subseteq \langle \underline{X} \rangle_{k-1}$ minimal s.t. $\operatorname{span}\{[M_k]_v \mid v \in V\} = \operatorname{ran} M_k$
- For $p \in \mathbb{R}\langle \underline{X} \rangle_k$ exists $r_p \in \operatorname{span}(V)$ such that $M_k \vec{p} = M_k \vec{r_p}$.
- $M_k \overrightarrow{vX_i} = M_k \overrightarrow{r_v X_i}$ for $v \in \langle \underline{X} \rangle_{k-1}$
- Define B such that for all $v \in \langle \underline{X} \rangle_k$

$$[M_k \;\; B]\overrightarrow{vX_i} = [M_k \;\; B]\overrightarrow{r_v X_i}$$

- $B = M_k W$, $C := W^T M_k W \implies \operatorname{rank} M_{k+1} = \operatorname{rank} M_k$
- M_{k+1} is a tracial Hankel matrix, i.e.

$$[M_{k+1}]_{u,v} = [M_{k+1}]_{u_1,v_1} \text{ whenever } u^* v \overset{\text{cyc}}{\sim} u_1^* v_1$$

▸ Theorem

Outlook

1. Trace-positive polynomials & sums of hermitian squares
   - Consider $n \geq 3$ variables
   - Replace Θ^2 by tracial quadratic module
   - Bounds on matrix size?
2. Tracial moment problem
   - Dual cones of $\mathcal{P}(K)$, $\mathcal{M}(K)$ for arbitrary closed $K \subseteq \mathcal{S}^n$
   - Tracial version of Putinar's Theorem
   - Extend finite rank/flat extensions theorems to K-moment problem
3. Applications

Summary

Investigation of trace-positive polynomials

- Motivated by two conjectures (BMV, Connes)
- Trace-positivity on $\mathcal{S}^n$
  - Sums of hermitian squares and commutators
    - Bivariate Quartics
    - BMV polynomials with $k = 4$
- Trace-positivity on a closed semialgebraic set K
  - Tracial moment problem
    - Dual perspective
    - Tracial Hankel matrices of finite rank
    - Tracial Hankel matrices with a flat extension

Zusammenfassung

Untersuchung spurpositiver Polynome

- BMV-Vermutung, Einbettungsvermutung von Connes
- Spurpositivität auf $\mathcal{S}^n$
  - Summen hermitescher Quadrate und Kommutatoren
    - Bivariate Quartiken
    - BMV-Polynome mit $k = 4$
- Spurpositivität auf semialgebraischer Menge K
  - Spuriges Momentenproblem
    - Duale Perspektive
    - Spurige Hankelmatrizen von endlichem Rang
    - Spurige Hankelmatrizen mit flacher Erweiterung

Résumé

Recherche des polynômes à trace positive

- Motivation: Conjecture BMV et Conjecture de Connes
- Positivité sur $\mathcal{S}^n$
  - Sommes de carrés hermitiens et de commutateurs
    - Quartiques binaires
    - Polynômes de BMV avec $k = 4$
- Positivité sur un ensemble K semi-algébrique
  - problème des moments traciaux
    - Dualité
    - Matrices traciales de Hankel de rang fini
    - Matrices traciales de Hankel avec une extension plate