Sums of hermitian squares as an approach to the BMV conjecture

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BMV conjecture

Conjecture (Bessis, Moussa, Villani, 1975)

For all symmetric $A, B \in \mathbb{R}^{s\times s}$, $B \geq 0$, $s \in \mathbb{N}$, exists a measure $\mu \geq 0$ s.t.

$$\mathrm{tr}[\exp(A - tB)] = \int e^{-tx} \mathrm{d}\mu(x).$$

Using Bernstein's Theorem leads to

Conjecture

For all symmetric $A, B \in \mathbb{R}^{s\times s}$, $B \geq 0$, $s \in \mathbb{N}$, the function

$$f : t \mapsto \mathrm{tr}[\exp(A - tB)]$$

is completely monotone.

BMV conjecture – algebraic version

Conjecture (Lieb, Seiringer, 2004)

For all symmetric $A, B \in S\mathbb{R}^{s \times s}$, $A, B \geq 0$, $s \in \mathbb{N}$, and all $m \in \mathbb{N}$, the polynomial

$$\mathrm{tr}[(A + tB)^m] = \sum_k \mathrm{tr}[S_{m,k}(A, B)]t^k \in \mathbb{R}[t]$$

has only nonnegative coefficients.

$S_{m,k}(A, B)$ is the sum of all products with $m - k$ times A and k times B.

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$S_{m,k}(A, B)$ is the sum of all products with $m-k$ times A and k times B.

Example

$S_{4,2}(A, B) = AABB + ABBA + BBAA + BAAB + ABAB + BABA$

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$S_{m,k}(A, B)$ is the sum of all products with $m - k$ times A and k times B.

Question 1

Let $m, k \in \mathbb{N}$ be fixed.
Is $\mathrm{tr}[S_{m,k}(A, B)] \geq 0$ for all psd $A, B \in S\mathbb{R}^{s\times s}$, all $s \in \mathbb{N}$?

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- Matrix product
- Matrix polynomial $f(A, B)$
- Matrix conjugation *
  - A, B symmetric,
  - $(ABAAB)^* = BAABA$

- word $w \in \langle X, Y\rangle$
- polynomial $f(X, Y) \in \mathbb{R}\langle X, Y\rangle$
- involution * on $\mathbb{R}\langle X, Y\rangle$
  - * fixes $\mathbb{R} \cup \{X, Y\}$,
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- Matrix product
- Matrix polynomial $f(A, B)$
- Matrix conjugation *
  - A, B symmetric,
  - $(ABAAB)^* = BAABA$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
  - $\mathrm{tr}([A, B]) = 0$
  - $\mathrm{tr}[w(A, B)] = \mathrm{tr}[\sigma(w(A, B))]$
  - tr linear

- word $w \in \langle X, Y \rangle$
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  - * fixes $\mathbb{R} \cup \{X, Y\}$,
  - $(XYXXY)^* = YXXYX$
- equivalence relation $\overset{\text{cyc}}{\sim}$
  - $v \overset{\text{cyc}}{\sim} w \Leftrightarrow v - w = [u, u']$
  - $v \overset{\text{cyc}}{\sim} w \Leftrightarrow v = \sigma(w)$
  - $f \overset{\text{cyc}}{\sim} g \Leftrightarrow f - g = \sum_i a_i [u_i, u_i']$

where $v, w, u, u' \in \langle X, Y \rangle$; $f, g \in \mathbb{R}\langle X, Y \rangle$; $a_i \in \mathbb{R}$; σ cyclic permutation; commutator $[a, b] := ab - ba$

Definition

For $g \in \mathbb{R}\langle X, Y\rangle$ the product g^*g is called a hermitian square

- A^*A is psd, in particular $\operatorname{tr}(A^*A) \geq 0$
- the same holds for hermitian squares $g^*g(A, B)$, **independent** of A and B
- $f \overset{\text{cyc}}{\sim} g \implies \operatorname{tr}(f(A, B)) = \operatorname{tr}(g(A, B))$ for all $A, B \in S\mathbb{R}^{s\times s}$

Useful fact

$$f \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i \implies \operatorname{tr}(f(A, B)) = \operatorname{tr}(\sum_i g_i^* g_i(A, B)) \geq 0$$

for all $A, B \in S\mathbb{R}^{s\times s}$, all $s \in \mathbb{N}$.

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Let $m, k \in \mathbb{N}$ be fixed.
Is $\mathrm{tr}(S_{m,k}(A, B)) \geq 0$ for all psd $A, B \in S\mathbb{R}^{s \times s}$, all $s \in \mathbb{N}$?

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Let $m, k \in \mathbb{N}$ be fixed.
Is $S_{m,k}(X, Y) \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i$ for some $g_i \in \mathbb{R}\langle X, Y\rangle$?

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Question 2 is quite too strong since A, B psd not provided.

Example

$\text{tr}(S_{3,0}(A, B)) = \text{tr}(AAA) \geq 0$ for all psd $A \in S\mathbb{R}^{s \times s}$. But $S_{3,0}(X, Y) = X^3$ is not (equivalent to) a square in $\mathbb{R}\langle X, Y \rangle$.

- If $A \in S\mathbb{R}^{s \times s}$ is psd, then $A = C^*C = C^2$ for some $C \in S\mathbb{R}^{s \times s}$
- consider $S_{m,k}(C^2, D^2)$ instead of $S_{m,k}(A, B)$

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Let $m, k \in \mathbb{N}$ be fixed.
Is $S_{m,k}(X, Y) \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i$ for some $g_i \in \mathbb{R}\langle X, Y\rangle$?

Question 3

Let $m, k \in \mathbb{N}$ be fixed.
Is $S_{m,k}(X^2, Y^2) \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i$ for some $g_i \in \mathbb{R}\langle X, Y\rangle$?

Question 3 is weaker than Question 2.

Example

$S_{3,0}(X^2, Y^2) = X^6 = (X^3)^*(X^3)$ is a square in $\mathbb{R}\langle X, Y\rangle$.

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But not equivalent to Question 1.

Example (Hägele,Klep, Schweighofer)

$\mathrm{tr}(S_{6,3}(A, B)) \geq 0$ for all psd $A, B \in S\mathbb{R}^{s\times s}$ but

$$S_{6,3}(X^2, Y^2) \overset{\text{cyc}}{\nsim} \sum g_i^* g_i$$

Theorem (Hillar)

If the BMV conjecture is true for some $M, K \in \mathbb{N}$ then it is true for all $m, k \in \mathbb{N}$ with $m \leq M, k \leq K, m - k \leq M - K$.

Question 3 is weaker than Question 2.

Example

$S_{3,0}(X^2, Y^2) = X^6 = (X^3)^*(X^3)$ is a square in $\mathbb{R}\langle X, Y\rangle$.

But not equivalent to Question 1.

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Theorem (Hillar)

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Plan: Find an infinite sequence (m, k) with $m, k \to \infty, m - k \to \infty$ for which the answer of Question 3 is 'yes'.

Gram matrix method

Let $f \in \mathbb{R}\langle X, Y\rangle$ symmetric, $\deg f = 2d$ and $\bar{v} = (v_i)$ be a vector containing all words $v_i \in \langle X, Y\rangle$ with $\deg v_i \leq d$. Then there exists a real symmetric matrix G such that

$$f = \bar{v}^* G \bar{v}.$$

- G is called a Gram matrix of f.
- Using Cholesky decomposition of real psd matrices:

$$G \text{ is psd} \Rightarrow \exists L \,\text{symmetric} : G = L^* L$$
$$\Rightarrow f = \sum g_i^* g_i \text{ with } L\bar{v} = (g_1, \ldots, g_s)^T$$

- Semidefinte programming to obtain psd Gram matrices G possible

Want

$$f \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i$$

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Example

$S_{2,1}(X^2, Y^2) = X^2Y^2 + Y^2X^2$ $\quad \bar{v} = (X^2, XY, YX, Y^2)$

$$\bar{v}^* \bar{v} = \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$

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$S_{2,1}(X^2, Y^2) = X^2Y^2 + Y^2X^2 \qquad \bar{v} = (X^2, XY, YX, Y^2)$

$$G_{nc} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \not\succeq 0 \qquad \bar{v}^* \bar{v} = \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$

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$$G_\sim = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0 \qquad \bar{v}^*\bar{v} = \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$

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Example

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$\Rightarrow f \overset{\text{cyc}}{\sim} (XY)^*(XY) + (YX)^*(YX)$

In general G is not unique.

Results I

Question 3

For which (m, k) exists a psd Gram matrix of $S_{m,k}(X^2, Y^2)$?

- $k = 0, 1; m \in \mathbb{N}$ trivial
- $k = 2; m \in \mathbb{N}$ easy
- $k = 4; m \in \mathbb{N}$ [ind.: $k = 4; m$ odd by Landweber, Speer]
- $(14, 4); (14, 6)$ by Klep, Schweighofer
- by symmetry: If (m, k) is 'yes' then $(m, m - k)$ is.

Results II

Question 2

For which (m, k) exists a psd Gram matrix of $S_{m,k}(X, Y)$?
Then $\text{tr}(S_{m,k}(A, B)) \geq 0$ for all $A, B \in S\mathbb{R}^{s \times s}$ not even psd.

- $k = 0, m \in 2\mathbb{N}$ trivial
- $k = 2, m \in 2\mathbb{N}$
- $k = 4, m = 4l + 2, l \in \mathbb{N}$
- by symmetry: If (m, k) is 'yes' then $(m, m - k)$ is.

$k = 2$

Question 3

- Each word in $S_{m,2}(X^2, Y^2)$ is $\overset{\text{cyc}}{\sim}$ to $X^{k_1}Y^2X^{k_2}Y^2X^{k_1}$, with even k_2. This is a palindrome and (obvious) a square.

Example

$$S_{4,2}(X^2, Y^2)$$
$$= \underbrace{X^4Y^4 + Y^4X^4}_{\overset{\text{cyc}}{\sim} 2X^2Y^4X^2} + X^2Y^4X^2 + Y^2X^4Y^2 + \underbrace{X^2Y^2X^2Y^2 + Y^2X^2Y^2X^2}_{\overset{\text{cyc}}{\sim} 2XY^2X^2Y^2X}$$

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- Using the Gram matrix:
  - Palindromes of even length are of the form $v_i^* v_i$ for some v_i
  - they correspond to diagonal entries of G
  - All (diagonal) entries are positive, thus G is psd.

- Does not work for $S_{m,2}(X, Y)$

Question 2

- 'Yes' only possible for even m, since $S_{m,k}(X, Y)$ is homogenous.

Example

$$S_{3,2}(X, Y) = XY^2 + YXY + Y^2X \implies \operatorname{tr}(S_{3,2}(-\mathbf{1}_s, \mathbf{1}_s)) = -3s < 0$$

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- $\bar{v} := (YX, XY)$ $\qquad \bar{v}^*\bar{v} = \begin{pmatrix} YX^2Y & YXYX \\ XYXY & XY^2X \end{pmatrix}$

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- $G = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

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Example

$$S_{4,2}(X,Y) = X^2Y^2 + Y^2X^2 + XY^2X + YX^2Y + XYXY + YXYX$$

- $\bar{v} := (YX, XY)$ $\qquad \bar{v}^*\bar{v} = \begin{pmatrix} YX^2Y & YXYX \\ XYXY & XY^2X \end{pmatrix}$
- $G = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}$
- $S_{4,2}(X,Y) \overset{\mathrm{cyc}}{\sim} 2(YX + \frac{1}{2}XY)^*(YX + \frac{1}{2}XY) + \frac{3}{2}(XY)^*(XY)$

$$\begin{aligned} S_{6,2}(X,Y) &= X^4Y^2 + X^3Y^2X + X^2Y^2X^2 + XY^2X^3 + Y^2X^4 + YX^4Y + \\ &+ YX^3YX + X^3YXY + X^2YXYX + XYXYX^2 + YXYX^3 + \\ &+ XYX^3Y + X^2YX^2Y + XYX^2YX + YX^2YX^2 \\ &\overset{\text{cyc}}{\sim} 6X^4Y^2 + 6X^3YXY + 3X^2YX^2Y \end{aligned}$$

$$S_{6,2}(X, Y) \overset{\text{cyc}}{\sim} 6X^4Y^2 + 6X^3YXY + 3X^2YX^2Y$$

$$\bar{v} := \left(YX^2, XYX, X^2Y\right) \qquad \bar{v}^*\bar{v} = \begin{pmatrix} X^2Y^2X^2 & X^2YXYX & X^2YX^2Y \\ XYXYX^2 & XYX^2YX & XYX^3Y \\ YX^2YX^2 & YX^3YX & YX^4Y \end{pmatrix}$$

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$$G = \tfrac{3}{2}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \tfrac{3}{2}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{3}{2}} & 0 \\ 0 & \sqrt{\tfrac{2}{3}} & \sqrt{\tfrac{4}{3}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{\tfrac{1}{2}} & 0 \\ 0 & \sqrt{\tfrac{3}{2}} & \sqrt{\tfrac{2}{3}} \\ 0 & 0 & \sqrt{\tfrac{4}{3}} \end{pmatrix}$$

$$\begin{aligned} S_{6,2}(X,Y) \overset{\text{cyc}}{\sim} & 3(YX^2 + \frac{1}{2}XYX)^*(YX^2 + \frac{1}{2}XYX) + \\ & + (XYX + \frac{3}{2}X^2Y)^*(XYX + \frac{3}{2}X^2Y) + 2(X^2Y)^*(X^2Y) \end{aligned}$$

$k = 4$

Question 3, m odd

$$S_{9,4}(X^2,Y^2) \overset{\text{cyc}}{\sim} 9X^{10}Y^8 + 9X^6Y^2X^4Y^6 + 9X^6Y^4X^4Y^4 + 9X^6Y^6X^4Y^2 +$$
$$+ 9X^8Y^2X^2Y^6 + 9X^8Y^4X^2Y^4 + 9X^8Y^6X^2Y^2 + 9X^4Y^2X^2Y^2X^4Y^4 +$$
$$+ 9X^4Y^2X^4Y^2X^2Y^4 + 9X^4Y^2X^4Y^4X^2Y^2 + 9X^6Y^2X^2Y^2X^2Y^4 +$$
$$+ 9X^6Y^2X^2Y^4X^2Y^2 + +9X^6Y^4X^2Y^2X^2Y^2 + 9X^4Y^2X^2Y^2X^2Y^2X^2Y^2$$

Question 3, m odd

$S_{9,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} \dots$

$\bar{v} := (Y^2X^2Y^2X^3, Y^4X^5, Y^2X^4Y^2X, X^2Y^4X^3, X^4Y^4X, X^2Y^2X^2Y^2X)$

$$G = 9 \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$S_{9,4} \overset{\text{cyc}}{\sim} 9 \sum_{i=1}^{3} g_i^* g_i$

- $g_1 = Y^2X^2Y^2X^3 + Y^4X^5 + Y^2X^4Y^2X$
- $g_2 = X^2Y^4X^3$
- $g_3 = X^2Y^2X^2Y^2X + X^4Y^4X$

$S_{17,4} \overset{\text{cyc}}{\sim} \ldots$

$$\bar{v} := (Y^2X^{10}Y^2X^3, Y^2X^8Y^2X^5, Y^2X^6Y^2X^7, Y^2X^4Y^2X^9, Y^2X^2Y^2X^{11}, Y^4X^{13}, Y^2X^{12}Y^2X, X^2Y^2X^4Y^2X^7, X^2Y^2X^6Y^2X^5, X^2Y^2X^2Y^2X^9, X^2Y^4X^{11}, X^2Y^2X^8Y^2X^3, X^4Y^2X^2Y^2X^7, X^4Y^4X^9, X^4Y^2X^4Y^2X^5, X^6Y^4X^7, X^8Y^4X^5, X^6Y^2X^2Y^2X^5, X^8Y^2X^2Y^2X^3, X^6Y^2X^4Y^2X^3, X^4Y^2X^6Y^2X^3, X^{12}Y^4X, X^{10}Y^2X^2Y^2X, X^{10}Y^4X^3, X^8Y^2X^4Y^2X, X^6Y^2X^6Y^2X, X^4Y^2X^8Y^2X, X^2Y^2X^{10}Y^2X)$$

$$G = 17 \begin{pmatrix} \boxed{1}_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1}_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1}_6 \end{pmatrix}$$

where

$$\boxed{1}_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}_{n \times n}$$

$$\bar{v} := (Y^2X^{10}Y^2X^3, Y^2X^8Y^2X^5, Y^2X^6Y^2X^7, Y^2X^4Y^2X^9, Y^2X^2Y^2X^{11}, Y^4X^{13}, Y^2X^{12}Y^2X, X^2Y^2X^4Y^2X^7, X^2Y^2X^6Y^2X^5, X^2Y^2X^2Y^2X^9, X^2Y^4X^{11}, X^2Y^2X^8Y^2X^3, X^4Y^2X^2Y^2X^7, X^4Y^4X^9, X^4Y^2X^4Y^2X^5, X^6Y^4X^7, X^8Y^4X^5, X^6Y^2X^2Y^2X^5, X^8Y^2X^2Y^2X^3, X^6Y^2X^4Y^2X^3, X^4Y^2X^6Y^2X^3, X^{10}Y^4X^3, X^{12}Y^4X, X^{10}Y^2X^2Y^2X, X^8Y^2X^4Y^2X, X^6Y^2X^6Y^2X, X^4Y^2X^8Y^2X, X^2Y^2X^{10}Y^2X)$$

$$G = 17\begin{pmatrix} \boxed{1}_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1}_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1}_6 \end{pmatrix}$$

Question 3, *m* even

$$S_{10,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} 10X^{12}Y^8 + 10X^6Y^2X^6Y^6 + 10X^8Y^2X^4Y^6 +$$
$$+ 10X^8Y^4X^4Y^4 + 10X^8Y^6X^4Y^2 + 10X^{10}Y^2X^2Y^6 + 10X^{10}Y^4X^2Y^4 +$$
$$+ 10X^{10}Y^6X^2Y^2 + 10X^4Y^2X^4Y^2X^4Y^4 + 10X^6Y^2X^2Y^2X^2Y^2X^2Y^2 +$$
$$+ 10X^6Y^2X^2Y^2X^4Y^4 + 10X^6Y^2X^2Y^4X^4Y^2 + 10X^6Y^2X^4Y^2X^2Y^4 +$$
$$+ 10X^6Y^2X^4Y^4X^2Y^2 + 10X^6Y^4X^2Y^2X^4Y^2 + 10X^6Y^4X^4Y^2X^2Y^2 +$$
$$+ 10X^8Y^2X^2Y^2X^2Y^4 + 10X^8Y^2X^2Y^4X^2Y^2 + 10X^8Y^4X^2Y^2X^2Y^2 +$$
$$+ 10X^4Y^2X^4Y^2X^2Y^2X^2Y^2 + 5X^4Y^2X^2Y^2X^4Y^2X^2Y^2 + 5X^6Y^4X^6Y^4$$

Question 3, m even

$S_{10,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} \dots$

$$\bar{v} := (Y^2X^4Y^2X^2, Y^2X^2Y^2X^4, Y^4X^6, X^2Y^4X^4, Y^2X^6Y^2, X^2Y^2X^2Y^2X^2, XY^2X^2Y^2X^3, XY^4X^5, XY^2X^4Y^2X, X^3Y^4X^3)$$

$$G = 5 \begin{pmatrix} 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & & & & \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 1 & 1 & \frac{1}{2} & 0 \\ & & & & & & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Question 3, m even

$S_{10,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} \dots$

$\bar{v} := (Y^2X^4Y^2X^2, Y^2X^2Y^2X^4, Y^4X^6, X^2Y^4X^4, Y^2X^6Y^2, X^2Y^2X^2Y^2X^2,$
$XY^2X^2Y^2X^3, XY^4X^5, XY^2X^4Y^2X, X^3Y^4X^3)$

$$G = 5\begin{pmatrix} 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & & & & \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 1 & 1 & \frac{1}{2} & 0 \\ & & & & & & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$S_{10,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} \dots$
$\bar{v}' = (Y^2X^4Y^2X^2, Y^2X^2Y^2X^4, Y^4X^6, X^2Y^4X^4, Y^2X^6Y^2, X^2Y^2X^2Y^2X^2)$

$$G = 5\begin{pmatrix} 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 2 & 2 & 2 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & & & & \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & & 1 & 1 & \frac{1}{2} & 0 \\ & & & & & & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \Rightarrow G' = 5\begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

- $S_{10,4}(X^2, Y^2) \overset{\text{cyc}}{\sim} \sum_i g_i^* g_i$ where $g_i \in \mathbb{R}\langle X^2, Y^2 \rangle$
- $S_{10,4}(X, Y) \overset{\text{cyc}}{\sim} \sum_i h_i^* h_i$

$$G' = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} &S_{10,4}(X^2,Y^2) \overset{\text{cyc}}{\sim} 5[\\ &\frac{1}{2}(X^2Y^2X^2Y^2X^2 + X^2Y^4X^4 + Y^2X^6Y^2 + 2Y^2X^4Y^2X^2 + 2Y^2X^2Y^2X^4 + 2Y^4X^6)^* \\ &\quad (X^2Y^2X^2Y^2X^2 + X^2Y^4X^4 + Y^2X^6Y^2 + 2Y^2X^4Y^2X^2 + 2Y^2X^2Y^2X^4 + 2Y^4X^6) + \\ &+ \frac{1}{6}(X^2Y^2X^2Y^2X^2 + 3X^2Y^4X^4 - Y^2X^6Y^2)^*(X^2Y^2X^2Y^2X^2 + 3X^2Y^4X^4 - Y^2X^6Y^2) + \\ &+ \frac{1}{3}(X^2Y^2X^2Y^2X^2 - Y^2X^6Y^2)^*(X^2Y^2X^2Y^2X^2 - Y^2X^6Y^2)] \end{aligned}$$

Question 2

$S_{10,4}(X,Y) \overset{\text{cyc}}{\sim} \sum_i h_i^* h_i$ for some $g_i \in \mathbb{R}\langle X, Y\rangle$
The same is true for all $m = 4l + 2$, $l \in \mathbb{N}$ but i.g. not for $m = 4l$.

Example

$S_{8,4}(X,Y) = 8X^4Y^4 + 8X^2YX^2Y^3 + 8X^3YXY^3 + 8X^3Y^2XY^2 + 8X^3Y^3XY + 8X^2YXYXY^2 + 8X^2YXY^2XY + 8X^2Y^2XYXY + 4X^2Y^2X^2Y^2 + 2XYXYXYXY$

$\bar{v} := (Y^2X^2Y^2X^2, Y^4X^4, Y^2X^4Y^2, X^2Y^4X^2, XY^4X^3, XY^2X^2Y^2X)$

$$G = 2\begin{pmatrix} 4 & 4 & 2 & 0 & & \\ 4 & 4 & 2 & 0 & & \\ 2 & 2 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ & & & & 4 & 2 \\ & & & & 2 & 1 \end{pmatrix}$$

Question 2

$S_{10,4}(X,Y) \overset{\text{cyc}}{\sim} \sum_i h_i^* h_i$ for some $g_i \in \mathbb{R}\langle X, Y\rangle$
The same is true for all $m = 4l + 2$, $l \in \mathbb{N}$ but i.g. not for $m = 4l$.

Example

$S_{8,4}(X,Y) = 8X^4Y^4 + 8X^2YX^2Y^3 + 8X^3YXY^3 + 8X^3Y^2XY^2 + 8X^3Y^3XY + 8X^2YXYXY^2 + 8X^2YXY^2XY + 8X^2Y^2XYXY + 4X^2Y^2X^2Y^2 + 2XYXYXYXY$

$\bar{v} := (Y^2X^2Y^2X^2, Y^4X^4, Y^2X^4Y^2, X^2Y^4X^2, XY^4X^3, XY^2X^2Y^2X)$

$$G = 2\begin{pmatrix} 4 & 4 & 2 & 0 & & \\ 4 & 4 & 2 & 0 & & \\ 2 & 2 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ & & & & 4 & 2 \\ & & & & 2 & 1 \end{pmatrix}$$