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Übungsblatt 3 zur Einführung in die Algebra

Aufgabe 1. Sei G eine Gruppe und $H \triangleleft G$ und $I \triangleleft G$, mit $H \subseteq I$. Zeige $I/H \triangleleft G/H$ und

$$(G/H)/(I/H) \cong G/I.$$

Solution

As H is a normal subgroup in G, for all $g \in G$, $gH = Hg$, and hence for all $i \in I$ we have $(gH)^{-1}iH(gH) = (g^{-1}ig)H$. This is an element of I/H, as I is normal, so $I/H \triangleleft G/H$.

Define p,q to be the natural homomorphisms from G to G/I, G/H respectively:

$$p(g) = gI \qquad q(g) = gH \qquad \forall g \in G$$

H is a subset of $\ker(p)$, so there exists a unique homomorphism $\varphi : G/H \to G/I$ so that $\varphi \circ q = p$ by the homorphism theorem.

p is surjective, so φ is surjective as well; hence $\mathrm{im}\varphi = G/I$. The kernel of φ is $\ker(p)/H = I/H$. So by the first isomorphism theorem we have

$$(G/H)/\ker(\varphi) = (G/H)/(I/H) \cong \mathrm{im}(\varphi) = G/I.$$

Aufgabe 2. Sei G eine Gruppe, $H \leq G$ und $N \triangleleft G$. Zeige $(H \cap N) \triangleleft H$, $N \triangleleft HN = NH \leq G$ und

$$H/(H \cap N) \cong (HN)/N.$$

Solution

First, we shall prove that HN is a subgroup of G: Since $e \in H$ and $e \in N$, clearly $e = e^2 \in HN$. Take h_1,$h_2 \in H, n_1, n_2 \in N$. Clearly $h_1n_1, h_2n_2 \in HN$. Further,

$$h_1n_1h_2n_2 = h_1(h_2h_2^{-1})n_1h_2n_2 = h_1h_2(h_2^{-1}n_1h_2)n_2$$

Since N is a normal subgroup of G and $h_2 \in G$, then $h_2^{-1}n_1h_2 \in N$. Therefore $h_1h_2(h_2^{-1}n_1h_2)n_2 \in HN$, so HN is closed under multiplication.

Also, $(hn)^{-1} \in HN$ for $h \in H$, $n \in N$, since

$$(hn)^{-1} = n^{-1}h^{-1} = h^{-1}hn^{-1}h^{-1}$$

and $hn^{-1}h^{-1} \in N$ since N is a normal subgroup of G. So HN is closed under inverses, and is thus a subgroup of G.

Similarly, for $h \in H$, $n \in N$ we have

$$hn = (nn^{-1})hn = n(n^{-1}hn) \in NH,$$

so $HN \subset NH$. That $NH \subset HN$ follows similarly, and hence $NH = HN$.

Since HN is a subgroup of G, the normality of N in HN follows immediately from the normality of N in G. That $H \cap N$ is a subgroup of H follows similarly.

Clearly $H \cap N$ is a subgroup of G, since it is the intersection of two subgroups of G.

Finally, define ϕ: $H \to HN/N$ by $\phi(h) = hN$. We claim that ϕ is a surjective homomorphism from H to HN/N. Let h_0n_0N be some element of HN/N; since $n_0 \in N$, then $h_0n_0N = h_0N$, and $\phi(h_0) = h_0N$. Now

$$\ker(\phi) = \{h \in H \mid \phi(h) = N\} = \{h \in H \mid hN = N\}$$

and if $hN = N$, then we must have $h \in N$. So

$$\ker(\phi) = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since $\phi(H) = HN/N$ and $\ker\phi = H \cap N$, by the Isomorphism Theorem we see that $H \cap N$ is normal in H and that there is a canonical isomorphism between $H/(H \cap N)$ and HN/N.

Aufgabe 3. Sei R ein kommutativer Ring und $n \in \mathbb{N}_0$. Zeige

$$Z(\mathrm{GL}_n(R)) = \{aI_n \mid a \in R^\times\} \qquad \text{und} \qquad Z(\mathrm{SL}_n(R)) = \{aI_n \mid a^n = 1\}.$$

Solution The result is clear for $n = 1$, assume $n > 1$.

Let E_{pq} be the matrix with 1 in the (p,q)th position, and 0 elsewhere. Let $B_{pq} = E_{pq} + I_n$. If a matrix commutes with B_{pq} then it commutes with E_{pq} by distributivity, and the fact that all matrices commute with the identity. B_{pq} is invertible when $p \neq q$, hence if a matrix is in $Z(\mathrm{GL}_n(R))$ it commutes with E_{pq} for $p \neq q$.

Now suppose A is a matrix with $a_{ji} \neq 0$ for some $i \neq j$. Consider the matrix E_{ij}. Then, the (j,j)th entry of AE_{ij} is nonzero, while the (j,j)th entry of $E_{ij}A$ is zero. Thus, any matrix that commutes with all the E_{pq} for $p \neq q$ cannot have any non-zero off-diagonal entries.

Now note that conjugation by the matrix (which, note, is its own inverse)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

swaps the first two diagonal entries of the matrix, hence they must be the same. This shows that $Z(\mathrm{GL}_n(R)) \subseteq \{aI_n \mid a \in R^\times\}$. That $\{aI_n \mid a \in R^\times\} \subseteq Z(\mathrm{GL}_n(R))$ is clear, hence the result for $\mathrm{GL}_n(R)$.

The proof follows similarly for $\mathrm{SL}_n(R)$. We can show that any element of the centre is a diagonal matrix exactly as above, as $B_{pq} \in \mathrm{SL}_n(R)$ for $p \neq q$. To show that all the diagonal entries are equal we cannot conjugate by the above matrix, as it is not in $\mathrm{SL}_n(R)$ (it has determinant -1). However, the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

is in $\mathrm{SL}_n(R)$, and conjugating by this matrix again gives that all the diagonal entries must be equal. Finally, that $a^n = 1$ for $aI_n \in Z(\mathrm{SL}_n(R))$ simply follows from the fact that the determinant is 1 for all elements in $Z(\mathrm{SL}_n(R))$.

Aufgabe 4. Sei K ein endlicher Körper mit q Elementen. Was ist die Gruppenordnung von $\mathrm{SL}_n(K)$?

Solution

Consider the homomorphism $\det : \mathrm{GL}_n(K) \to K^\times$. This map is surjective. Since $\mathrm{SL}_n(K)$ is the kernel of the homomorphism, it follows from the First Isomorphism Theorem that $\mathrm{GL}_n(K)/\mathrm{SL}_n(K) \cong K^\times$. We know $|\mathrm{GL}_n(K)| = \prod_{k=0}^{n-1}(q^n - q^k)$, therefore

$$|\mathrm{SL}_n(K)| = \frac{|\mathrm{GL}_n(K)|}{|\mathrm{SL}_n(K)|} = \frac{\prod_{k=0}^{n-1}(q^n - q^k)}{q-1}.$$