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Übungsblatt 6 zur Einführung in die Algebra: Solutions

Aufgabe 1. Sei A ein kommutativer Ring und $S \subseteq A$ eine multiplikative Menge ohne Nullteiler.

(a) Zeige, dass auf $A \times S$ durch

$$(a,s) \sim (b,t) \iff at = bs \qquad (a,b \in A,\ s,t \in S)$$

eine Äquivalenzrelation $\sim$ definiert wird.

(b) Durch

$$\widetilde{(a,s)} + \widetilde{(b,t)} := \widetilde{(at + bs, st)} \quad \text{und} \quad \widetilde{(a,s)}\widetilde{(b,t)} := \widetilde{(ab,st)} \qquad (a,b \in A,\ s,t \in S)$$

erhält man wohldefinierte Operationen $+$ und $\cdot$ auf $(A \times S)/\sim$.

(c) Mit den Operationen aus (b) wird $(A \times S)/\sim$ zu einem kommutativen Ring mit $0 = \widetilde{(0,1)}$ und $1 = \widetilde{(1,1)}$.

(d) Es ist $\iota\colon A \to (A \times S)/\sim,\ a \mapsto \widetilde{(a,1)}$ eine Einbettung.

Solution

(a) That $\sim$ is reflexive and symmetric is clear.

Transitive: Take $a,b,c \in R$ and $s,t,u \in S$ with $(a,s) \sim (b,t)$ and $(b,t) \sim (c,u)$. So $at = bs$ and $bu = ct$, and hence $atu = btu = bus = cts$, since R is commutative. Hence $(au - cs)t = 0$, but T has no zero divisors, so $au = cs$ so $(a,s) \sim (c,u)$.

(b) Take $a,a',b,b' \in R$ and $s,s',t,t' \in S$ with $\widetilde{(a,s)} \sim \widetilde{(a',s')}$ and $\widetilde{(b,t)} \sim \widetilde{(b',t')}$. We must show that $\widetilde{(at + bs, st)} = \widetilde{(a't' + b's', s't')}$.

So we have $as' = a's$ and $bt' = b't$, hence

$$\begin{aligned}(at + bs)s't' &= ats't' + bss't' = (as')tt' + (bt')ss' = a'stt' + b'tss' \\ &= (a't' + b's')st\end{aligned}$$

Hence $\widetilde{(at + bs, st)} = \widetilde{(a't' + b's', s't')}$. Therefore $+$ is well defined.

We now want $\widetilde{(ab,st)} = \widetilde{(a'b',s't')}$. This is clear from

$$abs't' = a'b'st.$$

Hence $\cdot$ is well defined.

(c) Take $a,b,c \in R$ and $s,t,u \in S$.

$$(A) \qquad \left(\widetilde{(a,s)} + \widetilde{(b,t)}\right) + \widetilde{(c,u)} = \widetilde{(at + bs, st)} + \widetilde{(c,v)} = \widetilde{(atu + bsu + cst, stu)}$$
$$= \widetilde{(a,s)} + \widetilde{(bu + ct, tu)} = \widetilde{(a,s)} + \left(\widetilde{(b,t)} + \widetilde{(c,u)}\right)$$

$$(K) \qquad \widetilde{(a,s)} + \widetilde{(b,t)} = \widetilde{(at + bs, st)} = \widetilde{(bs + at, ts)} = \widetilde{(b,t)} + \widetilde{(a,s)}$$

$$(N) \qquad \widetilde{(a,s)} + \widetilde{(0,1)} = \widetilde{(a \cdot 1 + 0 \cdot s,s)} = \widetilde{(a,s)}$$

$$(I) \qquad \widetilde{((a,s)} + \widetilde{(-a,s)} = \widetilde{(as + (-a)s,s^2)} = \widetilde{(as - as,s^2)} = \widetilde{(0,1)} = 0$$

$$(\dot{A}) \qquad \left(\widetilde{(a,s)} \cdot \widetilde{(b,t)}\right) \widetilde{(c,u)} = \widetilde{(ab,st)} \cdot \widetilde{(c,u)} = \widetilde{(abc,stu)} = \widetilde{(a,s)} \cdot \widetilde{(bc,tu)} = \widetilde{(a,s)} \left(\widetilde{(b,t)} \cdot \widetilde{(c,u)}\right)$$

$$(\dot{K}) \qquad \widetilde{(a,s)} \cdot \widetilde{(b,t)} = \widetilde{(ab,st)} = \widetilde{(ba,ts)} = \widetilde{(b,t)} \cdot \widetilde{(a,s)}$$

$$(\dot{N}) \qquad \widetilde{(a,s)} \cdot \widetilde{(1,1)} = \widetilde{(a \cdot 1,s \cdot 1)} = \widetilde{(a,s)}$$

$$(D) \qquad \left(\widetilde{(a,s)} + \widetilde{(b,t)}\right) \widetilde{(c,u)} = \widetilde{(at + bs,st)} \cdot \widetilde{(c,v)} = \widetilde{(act + bcs,stu)}$$
$$= \widetilde{(act,sut)} + \widetilde{(bcs,tus)} = \widetilde{(ac,su)} + \widetilde{(bc,tu)}$$

where we have used the rule $\widetilde{(at,st)} = \widetilde{(a,s)}$, which follows from $(at)s = a(st)$.

(d) For $a,b \in R$ we have

$$\iota(a+b) = \widetilde{(a+b,1)} = \widetilde{(a,1)} + \widetilde{(b,1)} = \iota(a) + \iota(b)$$

$$\iota(ab) = \widetilde{(ab,1)} = \widetilde{(a,1)} \cdot \widetilde{(b,1)} = \iota(a) \cdot \iota(b)$$

and

$$\iota(1) = \widetilde{(1,1)}.$$

Hence the map is a homomorphism. If $\widetilde{(a,1)} \sim \widetilde{(0,1)}$ then $a \cdot 1 = 0 \cdot 1$, and hence $a = 0$, and therefore this map is injective.

Aufgabe 2. Sei A ein Unterring des kommutativen Ringes B, $S \subseteq A \cap B^\times$ multiplikativ und $B = S^{-1}A$. Sei C ein weiterer Ring und $\varphi\colon A \to C$ ein Homomorphismus.

(a) Zeige, dass es genau dann einen Homomorphismus $\psi\colon S^{-1}A \to C$ mit $\varphi = \psi|_A$ gibt, wenn $\varphi(S) \subseteq C^\times$.

(b) Zeige, dass ein Homomorphismus ψ wie in (a) eindeutig bestimmt. Genauer: Zeige, dass für dieses ψ gilt
$$\psi\left(\frac{a}{s}\right) = \frac{\varphi(a)}{\varphi(s)} \qquad \text{für alle } a \in A \text{ und } s \in S.$$

Solution

(a) Assume that such a homomorphism ψ exists. Then for all $s \in S$, $\psi(s) = \varphi(s)$, and, since ψ is a ring homomorphism, $1 = \psi(s \cdot \frac{1}{s}) = \psi(s)\psi(\frac{1}{s}) = \varphi(s) \cdot \psi(\frac{1}{s})$. Hence $\varphi(S) \subseteq C^\times$.

Now assume $\varphi(S) \subseteq C^\times$. Then we can define the map

$$\psi\left(\frac{a}{s}\right) = \frac{\varphi(a)}{\varphi(s)}.$$

We must show this is well defined. Suppose we have $\frac{a}{s} = \frac{a'}{s'}$ for some $a,a' \in A, s,s' \in S$, that is there exists some $t \in S$ such that $t(as - a's') = 0$. Then

$$\varphi(t(as - a's')) = \varphi(t)(\varphi(a)\varphi(s) - \varphi(a')\varphi(s')).$$

But $\varphi(S) \subseteq C^\times$, hence this implies that $\varphi(a)\varphi(s) - \varphi(a')\varphi(s') = 0$, and, for the same reason

$$\frac{\varphi(a')}{\varphi(s')} = \frac{\varphi(a)}{\varphi(s)},$$

therefore the map is well-defined.

This is a ring homomorphism, $\psi(1) = 1$,

$$\psi\left(\frac{a}{s} \cdot \frac{b}{t}\right) = \frac{\varphi(ab)}{\varphi(st)} = \frac{\varphi(a)\varphi(b)}{\varphi(s)\varphi(t)} = \psi\left(\frac{a}{s}\right)\psi\left(\frac{b}{t}\right),$$

$$\begin{aligned} \psi\left(\frac{a}{s} + \frac{b}{t}\right) &= \psi\left(\frac{at+bs}{st}\right) = \frac{\varphi(at+bs)}{\varphi(st)} = \frac{\varphi(at)}{\varphi(st)} + \frac{\varphi(bs)}{\varphi(st)} \\ &= \frac{\varphi(a)\varphi(t)}{\varphi(t)\varphi(s)} + \frac{\varphi(b)\varphi(s)}{\varphi(s)\varphi(t)} = \frac{\varphi(a)}{\varphi(s)} + \frac{\varphi(b)}{\varphi(t)} \\ &= \psi\left(\frac{a}{s}\right) + \psi\left(\frac{b}{t}\right) \end{aligned}$$

and clearly $\varphi = \psi|_A$.

(b) We have already shown that φ is a homomorphism. We must now show that it is unique. Suppose that ρ is another homomorphism with the same properties. Then $\rho(a) = a$ for all $a \in A$. For all $s \in S$ we have

$$1 = \rho\left(\frac{s}{s}\right) = \rho\left(s \cdot \frac{1}{s}\right) = \rho(s)\rho\left(\frac{1}{s}\right) = \varphi(s)\rho\left(\frac{1}{s}\right).$$

Hence $\rho\left(\frac{1}{s}\right) = \varphi(s)^{-1}$, and hence

$$\rho\left(\frac{a}{s}\right) = \rho(a) \cdot \rho\left(\frac{1}{s}\right) = \frac{\varphi(a)}{\varphi(s)} \qquad \text{for all } a \in A \text{ and } s \in S,$$

that is $\rho = \psi$.

Aufgabe 3. Sei R ein kommutativer Ring, $n \in \mathbb{N}$ und $A := R[X_1, \dots, X_n]$

(a) Zeige $A^\times = R^\times$, wenn R ein Integritätsring ist.

(b) Gebe ein Beispiel für R mit $A^\times \neq R^\times$.

Solution

(a) From the lectures, we have that A is an integral domain as R is. Then the leading form of fg will be the leading form of f times the leading form of g, and hence the result, and hence $\deg(f) + \deg(g) = \deg(fg)$.

Let $f \in A$ be a unit. Then $fg = 1$ for some $g \in A$. Since A is an integral domain, this gives $\deg(f) + \deg(g) = 0$. But , since $\deg(f)$,$\deg(g) \geqslant 0$ (as $f \neq 0$), this implies that $\deg(f) = 0$ and $\deg(g) = 0$, i.e. f,$g \in R$. Hence, f,$g \in R^\times$. So $A^\times \subseteq R^\times$. That $R^\times \subseteq A^\times$ is clear. Hence $R^\times = A^\times$.

(b) Let $R = \mathbb{Z}/(4)$ and $f = 2X + 1 \in R[X] = A$. Then $f \in A^\times$ as $f^2 = 1$, so $A^\times \neq R^\times$.

Aufgabe 4. Seien S und T multiplikative Mengen eines kommutativen Ringes R mit $S \subseteq T$.

(a) Zeige, dass $\overline{T} := \iota_S(T)$ eine multiplikative Menge in der Lokalisierung R_S ist.

(b) Zeige $(R_S)_{\overline{T}} \cong R_T$.

Solution

(a) Since $1 \in T$ then $1 = \iota_S(1) \in \overline{T}$. Similarly, if $\iota_S(a), \iota_S b \in \overline{T}$ then $a, b \in T$, and hence $ab \in T$ as T is mutiplicative. Hence $\iota_S(ab) = \iota_S(a)\iota_S(b) \in \overline{T}$ and hence $\overline{T}$ is a multiplicative set in A_S.

(b) Since $\iota_T(S) \subseteq R_T^\times$ there exists a unique homomorphism $\varphi : R_S \to R_T$ (by problem 2 (a)), such that $\varphi \circ \iota_S = \iota_T$.

Similarly, as $\iota_T(T) = \varphi(\iota_S(T)) \subseteq R_T^\times$ there exists a unique homomorphism $\rho : (R_S)_{\overline{T}} \to R_T$ such that $\rho \circ \iota_{\overline{T}} = \varphi$, which gives $\rho \circ \iota_{\overline{T}} \circ \iota_S = \iota_T$.

Lastly, as $\iota_{\overline{T}} \circ \iota_S(T) \subseteq (R_S)_{\overline{T}}$, there exists a unique homomorphism $\alpha : R_T \to (R_S)_{\overline{T}}$ such that $\alpha \circ \iota_T = \iota_{\overline{T}} \circ \iota_S$.

We want to show that α and ρ are inverse to each other, and hence $(R_S)_{\overline{T}} \cong R_T$.

Take the map $(\rho \circ \alpha) \circ \iota_T : R \to R_T$. Then $(\rho \circ \alpha) \circ \iota_T = \rho \circ \iota_{\overline{T}} \circ \iota_S = \iota_T$. We also have $\mathrm{id}_{R_T} \circ \iota_T : R \to R_T$ and $\mathrm{id}_{R_T} \circ \iota_T = \iota_T$. But, since we clearly have $\iota_T(T) \subseteq R_T^\times$, any map $\gamma : R_T \to R_T$ such that $\gamma \circ \iota_T = \iota_T$ is unique. Hence $\gamma = \rho \circ \alpha = \mathrm{id}_{R_T}$.

We claim now that $(\alpha \circ \rho) \circ \iota_{\overline{T}} = \iota_{\overline{T}}$. If this is true, then we can argue as above to show that $\alpha \circ \rho = \mathrm{id}_{(R_S)_{\overline{T}}}$ and the proof is complete.

Since $(\iota_{\overline{T}} \circ \iota_S)(S) \subseteq (R_S)_{\overline{T}}^\times$ there is a unique map $\delta : R_S \to (R_S)_{\overline{T}}$, such that $\delta \circ \iota_S = \iota_{\overline{T}} \circ \iota_S$. Clearly $\iota_{\overline{T}}$ fulfills this condition, hence $\delta = \iota_{\overline{T}}$. We also have that $(\alpha \circ \rho) \circ \iota_{\overline{T}} \circ \iota_S = \iota_{\overline{T}} \circ \iota_S$, and hence $\delta = \iota_{\overline{T}} = (\alpha \circ \rho) \circ \iota_{\overline{T}}$ and we are done.