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Wintersemester 2010/2011

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Übungsblatt 9 zur Einführung in die Algebra: Solutions

Aufgabe 1. Bestimme die Menge $A := \{a \in \mathbb{C} \,|\, \mathbb{C}[X]/(X^2 + a) \cong \mathbb{C} \times \mathbb{C}\}$, also die Menge aller $a \in \mathbb{C}$, für die Ringe $\mathbb{C}[X]/(X^2 + a)$ und $\mathbb{C} \times \mathbb{C}$ isomorph sind.

Solution

We will show that $A = \mathbb{C} \setminus \{0\}$.

First we consider the ring $\mathbb{C}[X]/(X^2)$. This is not isomorphic to $\mathbb{C} \times \mathbb{C}$, as in $\mathbb{C}[X]/(X^2)$ there is an nonzero element that squares to zero (namely the residue class of X), but there is no such element in $\mathbb{C} \times \mathbb{C}$.

Now, take $a \in \mathbb{C} \setminus \{0\}$. We will show that the rings $\mathbb{C}[X]/(X^2 - a)$ and $\mathbb{C} \times \mathbb{C}$ are isomorphic. Choose $b \in \mathbb{C}$ with $b^2 = -a$. Then

$$X^2 + a = (X + b)(X - b)$$

and, since $b \neq 0$ we have $b \neq -b$. Hence we have that the ideals $I := (X + b)$ and $J := (X - b)$ are coprime as $\frac{1}{2b}(X + b) \in I$ and $\frac{-1}{2b}(X - b) \in J$ and therefore

$$1 = \frac{1}{2b}(X + b) - \frac{1}{2b}(X - b) \in I + J.$$

By the Chinese remainder theorem we get the isomorphism

$$\mathbb{C}[X]/(X^2 + a) \cong (\mathbb{C}[X]/(X + b)) \times (\mathbb{C}[X]/(X - b)).$$

We have a map $\mathbb{C}[X] \to \mathbb{C}$ given by $f \mapsto f(b)$ for $f \in \mathbb{C}[X]$. This has kernel $(X - b)$, and hence $\mathbb{C}[X]/(X - b) \cong \mathbb{C}$. Similarly $\mathbb{C}[X]/(X + b) \cong \mathbb{C}$. Hence

$$\mathbb{C}[X]/(X^2 - a) \cong \mathbb{C} \times \mathbb{C}.$$

Aufgabe 2. Sei $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$ wobei T die Gruppe von invertierbaren unteren 2×2–Dreiecksmatrizen bezeichnet.

Sei $X \in \{\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}, \{0\}\}$. Es wirke G auf X in natürlicher Weise (d.h. durch Multiplikation einer Matrix mit einem Vektor).

Gebe für jeden $5 \cdot 3 = 15$ Fälle für (G,X) an, ob die Wirkung transitiv ist, ob sie treu ist und ob sie frei ist.

Solution

Note first that all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$ are subgroups of $\mathrm{GL}_2(\mathbb{R})$.

Transitivity:

An action of G on X is transitive if for all $v,v' \in X$, there exists an $A \in G$ such that $Av = v'$.

The action does not act transitively when $X = \mathbb{R}^2$ for all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$ as, for all $A \in \mathrm{GL}_2(\mathbb{R})$ we have that $A \cdot 0 = 0$.

The action does act transitively when $X = \{0\}$ for all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$ as, for all $A \in \mathrm{GL}_2(\mathbb{R})$ we have that $A \cdot 0 = 0$.

The action does act transitively when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathrm{GL}_2(\mathbb{R})$, as we now show. Pick a non-zero vector $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$. We will find an $A \in \mathrm{GL}_2(\mathbb{R})$ such that $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v$, and hence every $v \neq 0$ is in the G-orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $a \neq 0$, let $A = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$. If $b \neq 0$, let $A = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$. These matrices are invertible in each case, and they send $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} a \\ b \end{pmatrix}$.

This shows that the action is transitive, as for every $v, v' \in \mathbb{R}^2 \setminus \{0\}$ there exist $A, B \in \mathrm{GL}_2(\mathbb{R}^2)$ such that $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v$ and $B \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v'$, and hence $BA^{-1}v = v'$.

The action does act transitively when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathrm{SL}_2(\mathbb{R})$. We pick a non-zero vector $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ and find an $A \in \mathrm{SL}_2(\mathbb{R})$ such that $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v$. Take the matrices $A = \begin{pmatrix} a & 0 \\ b & 1/a \end{pmatrix}$ if $a \neq 0$, and $A = \begin{pmatrix} a & -1/b \\ b & 0 \end{pmatrix}$ if $b \neq 0$. That the action is transitive now follows as above.

The action does not act transitively when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathrm{O}_2(\mathbb{R})$. Assume there exists an $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}_2(\mathbb{R})$ such that $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then by carrying out the multiplication, we see that $a = 1$ and $c = 1$. Since $A \in \mathrm{O}_2(\mathbb{R})$ we have $AA^t = I$, hence

$$\begin{pmatrix} 1 & b \\ 1 & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives $1 + b^2 = 1 + d^2 = 1$, and hence $b = d = 0$. But $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is not invertible, hence $A \notin \mathrm{O}_2(\mathbb{R})$, a contradiction. Hence the action is not transitive.

The action does not act transitively when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathrm{SO}_2(\mathbb{R})$. This follows from the previous case.

The action does not act transitively when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = T$. Let $w = \begin{pmatrix} d \\ e \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ and $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in T$. Then $Aw = \begin{pmatrix} ad \\ bd + ce \end{pmatrix}$. We must have that $a \neq 0$ as A is invertible, so $Aw = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ only if $w = \begin{pmatrix} 0 \\ e \end{pmatrix}$ for some $e \in \mathbb{R}$. More specifically, there exists no $A \in T$ such that $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and hence the action is not transitive.

Faithful (treu)

An action of G on X is faithful if, for any $A \in G$, we have that $Av = v$ for all $v \in X$ implies that $A = I$.

The action acts faithfully when $X \in \{\mathbb{R}^2, \mathbb{R}\setminus\{0\}\}$ for all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$ as we now show. Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R})$. Then if $Av = v$ for all $v \in X$, then in particular $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Solving this gives $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity in all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$.

The action does not act faithfully when $X = \{0\}$ for all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$, as $A \cdot 0 = 0$ for all $A \in G$ and $G \neq \{1\}$.

Free

An action of G on X is free if, for any $A \in G$, we have that $Av = v$ for any $v \in X$ implies that $A = I$.

The action is not free when $X \in \{\mathbb{R}^2, \{0\}\}$ for all $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}), T\}$, as $A \cdot 0 = 0$ for all $A \in G$ and $G \neq \{1\}$.

The action is not free when $X = \mathbb{R}^2 \setminus \{0\}$ for $G \in \{\mathrm{GL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}), T\}$. Take $I \neq \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in G$. Then

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence the action is not free.

The action is not free when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathrm{O}_2(\mathbb{R})$. Take $I \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G$. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence the action is not free.

The action is free, however, when $X = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathrm{SO}_2(\mathbb{R})$. Let $A \in \mathrm{SO}_2(\mathbb{R})$. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since A has determinant 1, it's inverse is given by $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and since $A^{-1} = A^t$, this gives $a = d$ and $b = -c$. So $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$.

Suppose now that there exists a $v \in X$ such that $Av = v$. Then $(A - I)v = 0$. If $A - I$ is invertible, this implies that $v = 0 \notin X$. Hence $A - I$ is not invertible, that is $\det(A - I) = 0$. This gives $(a-1)^2 + b^2 = 0$, and hence $a^2 + b^2 - 2a + 1 = 0$. But $a^2 + b^2 = 1$, hence we get that $a = 1$, and $b = 0$, i.e. $A = I$. Hence the action is free.

Aufgabe 3. Sei G eine endliche Gruppe und sei $H \triangleleft G$ Normalteiler von G. Sei $\tau : G \times X \to X$ eine transitive Gruppenwirkung. Zeige, dass es zwischen je zwei Bahnen der Einschränkung von τ auf $H \times X$ eine Bijektion gibt.

Solution

Suppose $x, y \in X$. It will suffice to find a bijection between the orbits Hx and Hy. Since G acts transitively on X , there exists some $g \in G$ such that $gx = y$. Since H is a normal subgroup of G, $ghg^{-1} \in H$ for all $h \in H$, so we can define a map $f : Hx \to Hy$ by $f(hx) := ghg^{-1}y$ for all $h \in H$. We first show that this is well defined. Let $h,h' \in H$ such that $hx = h'x$. Then $ghg^{-1}y = ghx = gh'x = ghg^{-1}y$. Hence the map is well defined.

f is injective: If $h,h' \in H$ are such that $ghg^{-1}y = gh'g^{-1}y$, then $ghx = gh'x$, so multiplying on the right by g^{-1} gives $hx = h'x$. f is surjective: If $h \in H$ is such that $hy \in Hy$, then $g^{-1}hg \in H$, and $f(g^{-1}hg(x)) = hy$. Therefore f is a bijection.

Aufgabe 4. Sei G Gruppe und $H \triangleleft G$ abelsch. Zeige, durch $\tau(gH,h) := ghg^{-1}$ für $g \in G$ und $h \in H$ eine Abbildung $\tau : G/H \times H \to H$ definiert wird und dass diese Abbildung eine Wirkung der Gruppe G/H auf H ist.

Suche ein Beispiel für eine Untergruppe H, die nicht abelsch ist, so dass τ nicht wohldefiniert ist.

Solution

First we show that the map is well defined. For all $g \in G, h \in H$ we have $ghg^{-1} \in H$ as H is normal. Let $g,g' \in G$ such that $gH = g'H$, so $g = g'a$ for some $a \in H$. For all $h \in H$, we have

$$ghg^{-1} = g'ah(g'a)^{-1} = g'aha^{-1}g'^{-1},$$

but H is abelian, so $aha^{-1} = h$, hence

$$ghg^{-1} = g'h(g')^{-1},$$

and the map is well-defined.

Now we show that it is a group action. Firstly, we clearly have that $\tau(H,h) = h$ for all $h \in H$. Now let $g,g' \in G$. We have for all $h \in H$,

$$\tau(g'H,\tau(gH,h)) = \tau(g'H, ghg^{-1}) = g'(ghg^{-1})g'^{-1} = (g'g)h(g'g)^{-1} = \tau(g'gH,h),$$

hence τ is a group action.

Now D_3 is a subgroup in D_6 given by $\{e,r^2,r^4,s,sr^2,sr^4\}$, where

$$r := \begin{pmatrix} \cos(\frac{2\pi}{6}) & -\sin(\frac{2\pi}{6}) \\ \sin(\frac{2\pi}{6}) & \cos(\frac{2\pi}{6}) \end{pmatrix} \qquad \text{and} \qquad s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $D_3 \leqslant D_6$ has index 2, hence is normal, and that D_3 is nonabelian. Moreover, r and rs are distinct representatives of rD_3. However, $rr^2r^{-1} = r^2$ and $rsr^2(rs)^{-1} = r^4$, so that the action described above is not well defined.