BLOW-UP, CRITICAL EXPONENTS AND ASYMPTOTIC SPECTRA FOR NONLINEAR HYPERBOLIC EQUATIONS

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ABSTRACT. We prove nonexistence results for the Cauchy problem for the abstract hyperbolic equation in a Banach space X,

$$u_{tt} = f'(u), \ \ t > 0; \quad u(0) = u_0, \ \ u_t(0) = u_1,$$

where $f : X \to \mathbf{R}$ is a C^1-function. Several applications to the second and higher-order hyperbolic equations with local and nonlocal nonlinearities are presented. We also describe an approach to Kato's and John's critical exponents for the semilinear equations $u_t = \Delta u + b(x,t)|u|^p$, $p > 1$, which are responsible for phenomena of stability, unstability, blow-up and asymptotic behaviour.

We construct countable spectra of different asymptotic patterns of self-similar and non self-similar types for global and blow-up solutions for the autonomous equation $u_{tt} = \Delta u + |u|^{p-1}u$ in different parameter ranges.

1. Introduction

We consider some aspects of nonexistence, global existence and asymptotic behaviour of solutions of the Cauchy problem for the quasilinear second-order hyperbolic equations and equations of higher order. The list of equations to be studied includes the classical semilinear equation

$$u_{tt} = \Delta u + |u|^{p-1}u, \quad p > 1, \tag{1.1}$$

and the corresponding $2m$-th order one

$$u_{tt} = -(-\Delta)^m u + |u|^{p-1}u, \quad m \geq 1, \ p > 1, \tag{1.2}$$

as well as the quasilinear second-order equation with the p-Laplacian operator

$$u_{tt} = D(|Du|^{q-1}Du) + |u|^{p-1}u, \quad q > 0, \ \ p > 1, \tag{1.3}$$

and the higher-order one. We also consider higher-order Kirchhoff-type equations with a nonlocal nonlinearity

$$u_{tt} = -\left(\int |D^m u|^2\right)^q (-\Delta)^m u + |u|^{p-1}u, \quad m \geq 1, \ \ q > 0, \ p > 1. \tag{1.4}$$

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The paper consists of two parts, where Part I (Sections 2 - 4) is devoted to general global nonexistence (blow-up) results for the above equations and critical exponents, while in Part II (Sections 5 - 8) we study a detailed structure and present a classification of global and blow-up asymptotic patterns for the semilinear equation (1.1).

In Section 2 we prove a general abstract nonexistence theorem to be applied in Section 3 to several equations listed above. In Section 4 using an energy approach, we derive Kato's and John's critical exponents for the non-autonomous semilinear equation

$$u_{tt} = \Delta u + b(x,t)|u|^p, \quad x \in \mathbf{R}^3, \ t > 0. \tag{1.5}$$

In Sections 5 and 6 we consider globally decaying *very singular* self-similar solutions of the semilinear equation (1.1) of the form

$$u_*(x,t) = t^{-2/(p-1)}\theta(\eta), \quad \eta = x/t,$$

where the function θ solves a semilinear degenerate elliptic equation inside and outside the light cone $S_1(t) = \{|x| = t\}$. We prove existence of a countable spectrum of such self-similar solutions to equation (1.1) in $B_1 = \{|x| < t\}$ in the parameter range

$$1 < p \le p_1 = (N+3)/(N-1).$$

In the range

$$p_1 < p < p_S = (N+2)/(N-2),$$

where p_S is the critical Sobolev exponent for the elliptic operator $\Delta u + |u|^{p-1}u$, we prove existence of continuous weak self-similar solutions defined everywhere in $\mathbf{R}^N \times \mathbf{R}_+$. These solutions are invertible and the functions $u_*(x,-t)$ describe finite time blow-up as $t \to -0$ (the so-called nonlinear blow-up patterns).

In Section 7 we describe another countable spectrum of blow-up patterns for (1.1). These solutions are not self-similar and are constructed by matching involving structures on the stable subspace of the linearized rescaled self-similar operator in the *inner region* and self-similar solutions of a nonlinear second-order ordinary differential equation in the *intermediate region*. The third *outer region* then reveals a discrete spectrum of asymptotics of final-time profiles $u(x,-0)$ near singular points. In the final Section 8 we briefly discuss some common features of the general structures of singular blow-up spectra for nonlinear parabolic and hyperbolic equations.

PART I: GLOBAL NONEXISTENCE AND CRITICAL EXPONENTS

2. **Abstract global existence and nonexistence results**

The first general nonexistence result for abstract semilinear hyperbolic equations in a Hilbert space was presented in [27] on the basis of a concavity technique applied also for some types of quasilinear wave equations [29]. Several generalizations of such an approach are well-known, see a survey [28] and some more recent results in [38], [36], [37].

In this section we prove a general nonexistence result for a nonlinear hyperbolic equation of divergent structure. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Given a C^1-function $f : H \to \mathbf{R}$ with the Fréchet derivative $f' : \mathcal{D} \to H$, $\mathcal{D} \subset H$ being the domain of f', we consider the "hyperbolic" equation

$$u_{tt} = f'(u), \;\; t > 0, \tag{2.1}$$

with the initial data

$$u(0) = u_0 \in \mathcal{D}, \;\; u_t(0) = u_1 \in H. \tag{2.2}$$

An appropriate concept of the solution then follows from the necessary identities and inequalities we will use later on. These assumptions can be verified in particular examples to be considered in the next sections. We thus assume that the problem is solvable locally in time, and our main goal is to prove that under certain hypotheses, the solutions cannot be extended globally in time.

Consider the following functionals:

$$E(t) = 2^{-1}\|u_t(t)\|^2 - f(u(t)) \quad \text{and} \quad G(t) = \|u(t)\|^2. \tag{2.3}$$

From equation (2.1) we have that

$$E'(t) \equiv 0 \;\; \Longrightarrow \;\; 2^{-1}\|u_t\|^2 - f(u) \equiv E(0) = E_0. \tag{2.4}$$

On the other hand, there hold

$$G' = 2\langle u, u_t \rangle, \quad G'' = 2\|u_t\|^2 + 2\langle u, u_{tt}\rangle = 2\|u_t\|^2 + 2\langle u, f'(u)\rangle. \tag{2.5}$$

We assume that there exists a real constant $\lambda > 2$ such that

$$\langle u, f'(u)\rangle - \lambda f(u) \geq 0, \quad u \in \mathcal{D}, \tag{2.6}$$

and assume that the initial data satisfy

$$E_0 = E(0) \leq 0. \tag{2.7}$$

Then, using (2.4) and (2.6), one obtains

$$G'' \geq 2\|u_t\|^2 + 2\lambda(2^{-1}\|u_t\|^2 - E_0) = (2+\lambda)\|u_t\|^2 - 2\lambda E_0 \geq (2+\lambda)\|u_t\|^2, \;\; t > 0. \tag{2.8}$$

On the other hand, from the first identity (2.5) by the Cauchy-Bunyakovskii-Schwarz inequality we conclude that

$$(G')^2 \leq 4\|u\|^2\|u_t\|^2 = 4G\|u_t\|^2 \Longrightarrow \|u_t\|^2 \geq (G')^2/4G. \tag{2.9}$$

Then (2.8) implies the following ordinary differential inequality (an ODI in short) for the function $G(t)$:

$$G''(t) \geq (2+\lambda)(G')^2(t)/4G(t), \quad t > 0. \tag{2.10}$$

The existence/nonexistence conclusions are straightforward.

2.1. **Global existence: a priori uniform bound.** Assume now that

$$G'(0) = 2\langle u(0), u_t(0)\rangle \equiv 2\langle u_0, u_1\rangle < 0. \tag{2.11}$$

Integrating inequality (2.10) in the form $G''/(G')^2 \geq (2+\lambda)/4G(t)$, we obtain

$$(G'(t))^{-1} \leq (G'(0))^{-1} - [(2+\lambda)/4] \int_0^t d\tau/G(\tau) < 0, \tag{2.12}$$

so that $G'(t) \equiv 2\langle u(t), u_t(t)\rangle < 0$ for all $t > 0$. We thus arrive at the following *a priori* bound on the solution.

Lemma 2.1. *Assume that* (2.6)*,* (2.7) *and* (2.11) *hold. Then* $\|u(t)\|$ *is strictly decreasing and is uniformly bounded:* $\|u(t)\| < \|u(0)\|$ *for* $t > 0$.

Proof. Integrating (2.10) in the form $G''/G' \leq (2+\lambda)G'/4G$, we obtain $G'(t)/G'(0) \leq [G(t)/G(0)]^{(2+\lambda)/4}$. Integrating once more yields the following estimate from above:

$$\|u(t)\| \leq \|u_0\| \, [1 + (\lambda-2)|F_0|t/4]^{-2/(\lambda-2)}, \quad F_0 = 2\langle u_0, u_1\rangle/\|u_0\|^2 > 0. \tag{2.13}$$

□

2.2. **Global nonexistence.** If

$$G'(0) = 2\langle u_0, u_1\rangle > 0, \tag{2.14}$$

then (2.10) implies that $G'(t) > 0$ for all $t > 0$ and integrating it twice as above, we arrive at

$$G(t) \geq G(0)\, [1 - (\lambda-2)F_0 t/4]^{-4/(\lambda-2)}.$$

Under the above hypotheses on the local existence of the solutions, we obtain the following global nonexistence result.

Lemma 2.2. *Let* (2.6) *and* (2.7) *be satisfied and* (2.14) *hold. Then the solution blows up (in the sense that* $G(t) = \|u(t)\|^2$ *becomes unbounded) on the finite interval* $(0, T)$*, with* $T = 4/(\lambda-2)F_0$.

3. Applications

3.1. **Higher-order semilinear hyperbolic equation.** Consider in $\mathbf{R}_+^{N+1} = \mathbf{R}^N \times \mathbf{R}_+$ the following $2m$-th order hyperbolic equation:

$$u_{tt} = f'(u) \equiv -(-\Delta)^m u + |u|^{p-1}u, \quad m \geq 1, \; p > 1, \tag{3.1}$$

where

$$f(u) = -\frac{1}{2}\int |D^m u|^2 + \frac{1}{p+1}\int |u|^{p+1}, \quad \langle u, f'(u)\rangle = -\int |D^m u|^2 + \int |u|^{p+1}.$$

Then (2.6) takes the form

$$\langle u, f'(u)\rangle - \lambda f(u) = (\lambda/2 - 1)\int |D^m u|^2 + [1 - \lambda/(p+1)]\int |u|^{p+1}.$$

Setting $\lambda = p+1$, we deduce that

$$\langle u, f'(u)\rangle - (p+1)f(u) = [(p+1)/2 - 1]\int |D^m u|^2 \ge 0$$

and Lemma 2.2 establishes nonexistence (blow-up) for equation (3.1).

3.2. **Second-order quasilinear hyperbolic equation.** Consider now in $\mathbf{R}_+^{N+1}$ the quasilinear equation with the p-Laplacian operator

$$u_{tt} = D(|Du|^{q-1}Du) + |u|^{p-1}u, \quad q > 0,\ p > 1. \tag{3.2}$$

Then

$$f(u) = -\frac{1}{q+1}\int |Du|^{q+1} + \frac{1}{p+1}\int |u|^{p+1},\ \ \langle u, f'(u)\rangle = -\int |Du|^{q+1} + \int |u|^{p+1},$$

and (2.6) reads

$$\langle u, f'(u)\rangle - \lambda f(u) = [\lambda/(q+1) - 1]\int |Du|^{q+1} + [1 - \lambda/(p+1)]\int |u|^{p+1} \ge 0,$$

if $\lambda > 2$ satisfies the inequalities $\lambda \ge q+1,\ \ \lambda \le p+1$, which mean that Lemma 2.2 applies if $p \ge q$.

3.3. **Higher-order quasilinear equations.** The above two examples admit a natural generalization to the higher-order quasilinear hyperbolic equation

$$u_{tt} = -\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha(a_\alpha(x)|D^\alpha u|^{q-1}D^\alpha u) + \sum_{|\beta|\le k}(-1)^{|\beta|}D^\beta(b_\beta(x)|D^\beta u|^{p-1}D^\beta u), \tag{3.3}$$

where $p > 1$, $q > 0$ and $\alpha = (\alpha_1, ..., \alpha_N)$, $\beta = (\beta_1, ..., \beta_N)$ denote multiindices, $|\alpha| = \alpha_1 + ... + \alpha_N$, $|\beta| = \beta_1 + ... + \beta_N$. We have

$$f(u) = -\frac{1}{q+1}\int \sum_{|\alpha|\le m} a_\alpha(x)|D^\alpha u|^{q+1} + \frac{1}{p+1}\int \sum_{|\beta|\le k} b_\beta(x)|D^\beta u|^{p+1},$$

$$\langle u, f'(u)\rangle = -\int \sum_{|\alpha|\le m} a_\alpha(x)|D^\alpha u|^{q+1} + \int \sum_{|\beta|\le k} b_\beta(x)|D^\beta u|^{p+1}.$$

Consider the main hypothesis (2.6):

$$\langle u, f'(u)\rangle - \lambda f(u) = (\frac{\lambda}{q+1} - 1)\int \sum_{|\alpha|\le m} a_\alpha(x)|D^\alpha u|^{q+1}$$

$$+ (1 - \frac{\lambda}{p+1})\int \sum_{|\beta|\le k} b_\beta(x)|D^\beta u|^{p+1}. \tag{3.4}$$

Assume that both differential forms in (3.3) are nonnegatively defined:

$$\sum_{|\alpha|\le m} a_\alpha(x)|\xi_\alpha|^{q+1} \ge 0, \quad \sum_{|\beta|\le k} b_\beta(x)|\xi_\beta|^{p+1} \ge 0. \tag{3.5}$$

It then follows from (3.4) that the nonexistence Lemma 2.2 applies provided λ satisfies $\lambda \geq q+1$ and $\lambda \leq p+1$. This gives the same condition $p \geq q$. Note that the nonexistence result remains valid without any positivity-like hypothesis on the first operator on the right-hand side of (3.3) if we choose $\lambda = q+1 > 2$ provided that $p > q > 1$. If $p = q > 1$, then setting $\lambda = q+1$, we obtain nonexistence without any restriction on the both operators in (3.3).

3.4. **Kirchhoff equations with nonlocal nonlinearities.** Consider the second-order hyperbolic equation with nonlocal nonlinearity

$$u_{tt} = a\left(\int |Du|^2\right) \Delta u + h(u), \quad (x,t) \in \mathbf{R}_+^{N+1}, \tag{3.6}$$

with sufficiently smooth real valued functions a and h satisfying some necessary hypotheses. It is a generalized Kirchhoff equation. Here

$$f(u) = -\frac{1}{2} A\left(\int |Du|^2\right) + \int H(u),$$

where

$$A(s) = \int_0^s a(\tau)d\tau, \quad H(s) = \int_0^s h(\tau)d\tau.$$

In this case (2.6) takes the form

$$\langle u, f'(u)\rangle - \lambda f(u) = -a(\int |Du|^2) \int |Du|^2 + \int h(u)u + \frac{\lambda}{2} A(\int |Du|^2)v$$

$$-\lambda \int H(u) = \left[\frac{\lambda}{2} A(\int |Du|^2) - a(\int |Du|^2) \int |Du|^2\right] + \int (h(u)u - \lambda H(u)). \tag{3.7}$$

Therefore, if for some $\lambda > 2$ the right-hand side of (3.7) is nonnegative, the nonexistence theorem similar to Lemma 2.2 is valid. Such a result admits a natural extension to nonlocal higher-order hyperbolic equations.

3.5. **Higher-order Kirchhoff-type equations.** Consider, for instance, a higher-order Kirchhoff-type equation of the form

$$u_{tt} = -\left(\int |D^m u|^2\right)^q (-\Delta)^m u + |u|^{p-1}u, \quad m \geq 1, \ q > 0, \ p > 1, \tag{3.8}$$

where

$$a(s) = s^q, \ A(s) = (q+1)^{-1} s^{q+1} \ (s = \int |D^m u|^2); \quad H(u) = (p+1)^{-1}|u|^{p+1},$$

$$f(u) = -[2(q+1)]^{-1} \left(\int |D^m u|^2\right)^{q+1} + (p+1)^{-1} \int |u|^{p+1}.$$

The main inequality reads (cf. (3.7))

$$\langle u, f'(u)\rangle - \lambda f(u) = [(\lambda/2(q+1)) - 1](\int |D^m u|^2)^{q+1} + (1 - \lambda/(p+1)) \int |u|^{p+1} \geq 0.$$

Therefore, we impose the following assumptions:

$$\lambda > 2, \ \ \lambda \geq 2(q+1), \ \ \lambda \leq p+1. \tag{3.9}$$

Such a constant λ exists and the corresponding nonexistence result applies to equation (3.8) provided that $p \geq 2q+1$.

4. On an approach to critical exponents

We now study global nonexistence for the non-autonomous semilinear hyperbolic equation

$$u_{tt} = \Delta u + b(x,t)|u|^p \ \text{ in } \mathbf{R}^{N+1}_+; \quad u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x) \ \text{ in } \mathbf{R}^N, \tag{4.1}$$

with the exponent $p > 1$, where $b \in L^\infty_{\rm loc}(\mathbf{R}^{N+1}_+)$, $b \geq 0$, is a given function. We study the properties of solutions of (4.1) from the class $C^2_{\rm loc}(\mathbf{R}^{N+1}_+)$ with initial data $u_0 \in C^2_0(\mathbf{R}^N)$ and $u_1 \in C^1_0(\mathbf{R}^N)$. We impose the following condition:

$$\int_{\mathbf{R}^N} u_1(x)dx > 0. \tag{4.2}$$

We assume that initial data are compactly supported in the ball of radius R: $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subset \{|x| < R\}$. Let us introduce the functions

$$B(t) = \left(\int_{|x|\leq R+t} b^{1-p'} dx\right)^{p-1}, \quad B_1(t) = \left(\int_{t-R<|x|\leq R+t} b^{1-p'} dx\right)^{p-1}, \tag{4.3}$$

where $1/p+1/p' = 1$. Supposing that both functions are well-defined for $t > 0$, we assume that there exist exponents α, $\beta \in \mathbf{R}$ such that

$$\limsup_{t\to\infty} B(t)/t^\alpha < \infty, \quad \limsup_{t\to\infty} B_1(t)/t^\beta < \infty. \tag{4.4}$$

4.1. Nonexistence in the three dimensional case.

We first prove the following result.

Theorem 4.1. *Let $N = 3$ and*

$$\alpha < (p+2-\beta)(p-1)+2. \tag{4.5}$$

Then the maximal existence time T of the solution to (4.1) is finite.

Proof. It consists of three steps.

Step 1: an energy ordinary differential inequality. This step is true in $\mathbf{R}^N$. Consider the energy-like functional

$$E(t) = \int_{\mathbf{R}^N} u(x,t)dx.$$

Since by assumptions $\operatorname{supp} u(t) \subset \{|x| \leq R+t\}$, we have that

$$E(t) = \int_{|x|\leq R+t} u(x,t)dx$$

is a C^2-function on $(0,T)$. Integrating (4.1) over $\mathbf{R}^N$, we obtain

$$\ddot{E} = \int_{|x|\le R+t} b|u|^p dx. \tag{4.6}$$

By the Hölder inequality,

$$\left|\int_{|x|\le R+t} u(x,t)dx\right| \le \left|\int_{|x|\le R+t} b|u|^p dx\right|^{1/p} \left(\int_{|x|\le R+t} b^{1-p'} dx\right)^{1/p'}.$$

Therefore

$$\int_{|x|\le R+t} b|u|^p dx \ge B^{-1}(t) \left|\int_{|x|\le R+t} u(x,t)dx\right|^p = B^{-1}(t)E^p(t).$$

Finally, we arrive at the second-order ordinary differential inequality (ODI)

$$\ddot{E}(t) \ge B^{-1}(t)E^p(t), \quad t>0; \quad E'(0)>0. \tag{4.7}$$

In the case $b \equiv 1$ this inequality leads to Kato's critical exponent $p_K = (N+1)/(N-1)$ $(=2$ for $N=3)$ [22], so that any orbit $\{E(t)\}$ of (4.7) with $1<p<p_K$ blows up in finite time.

Step 2: a lower energy estimate. We need an extra estimate on $E(t)$ from below. We now consider the linear Cauchy problem with the same initial data $(N=3)$

$$v_{tt} = \Delta v \ \text{ in } \mathbf{R}_+^{N+1}, \quad v(0,x) = u_0(x), \ \ v_t(0,x) = u_1(x). \tag{4.8}$$

By the positivity property of the fundamental solution in dimension $N=3$,

$$F_3(x,t) = (2\pi)^{-1}\delta(t^2 - |x|^2) \ge 0 \quad (t>0),$$

where δ denotes Dirac's delta function, it follows that

$$u(x,t) \ge v(x,t) \ \text{ in } \mathbf{R}_+^4. \tag{4.9}$$

Hence

$$E(t) = \int_{|x|\le R+t} u(x,t)dx \ge \int_{|x|\le R+t} v(x,t)dx. \tag{4.10}$$

On the other hand, by the Huyghens Principle for $N=3$ we have that for $t>R$

$$\int_{|x|\le R+t} v(x,t)dx = \int_{t-R\le|x|\le R+t} v(x,t)dx. \tag{4.11}$$

It then follows from (4.9) and (4.10) that

$$\int_{t-R\le|x|\le R+t} v(x,t)dx \le \int_{t-R\le|x|\le R+t} u(x,t)dx \le \left(\int_{t-R\le|x|\le R+t} b|u|^p\right)^{1/p}$$
$$\times \left(\int_{t-R\le|x|\le R+t} b^{1-p'}\right)^{1/p'} \le \left(\int_{|x|\le R+t} b|u|^p\right)^{1/p} \left(\int_{t-R\le|x|\le R+t} b^{1-p'}\right)^{1/p'}.$$

This means that

$$\int_{|x|\le R+t} b|u|^p \ge \Big(\int_{t-R\le|x|\le R+t} v\Big)^p B_1^{-1}(t). \tag{4.12}$$

We need a lower energy estimate on v. Integrating (4.8) over $\mathbf{R}^3$, we deduce that the corresponding energy functional

$$E_0(t)=\int_{\mathbf{R}^N} v(x,t)dx=\int_{|x|\le t+R} v\equiv \int_{t-R\le|x|\le t+R} v, \;\; t>R,$$

satisfies

$$\ddot E_0=0,\;\; t>0;\;\; E_0(0)=\int_{\mathbf{R}^3} u_0\equiv a_0,\;\; E_0'(0)=\int_{\mathbf{R}^3} u_1\equiv a_1,$$

and hence $E_0(t)=a_0+a_1t$. We have $a_1>0$ by (4.2), and (4.12) takes the form

$$\int_{|x|\le R+t} b(x,t)|u|^p \ge (a_0+a_1t)^p B_1^{-1}(t). \tag{4.13}$$

Then (4.6) implies that $\ddot E(t)\ge (a_0+a_1t)^pB_1^{-1}(t)$, and integrating yields the lower estimate

$$E(t)\ge a_0+a_1t+H_0(t), \quad H_0(t)=\int_0^t (t-\tau)(a_0+a_1\tau)^pB_1^{-1}(\tau)d\tau. \tag{4.14}$$

Step 3: analysis of the ODI. Finally, for large times, we arrive at the ODI for the energy orbits $\{E(t),\, t>0\}$ with an extra lower bound (a constraint)

$$\ddot E\ge B^{-1}(t)E^p,\;\; E(t)\ge H_0(t),\quad t\gg R \quad (a_1>0). \tag{4.15}$$

Under hypotheses (4.4) we derive the system ($c>0$ is a constant)

$$\ddot E(t)\ge c\,t^{-\alpha}E^p(t),\;\; E(t)\ge c\,t^{-\beta+p+2},\;\; t\gg 1. \tag{4.16}$$

Integrating the ODI, one can see that under condition (4.5) the energy functional $E(t)$ blows up in finite time. See Lemma 4 in [45]. Observe also that the corresponding ODE $\ddot E(t)=c\,t^{-\alpha}E^p(t)$ is the classical Emden-Fowler equation. By a standard transformation it reduces to a first-order ODE. Its phase-plane indicates the above blow-up condition. This completes the proof of the theorem.

□

4.2. **Examples for N = 3.** *Time-dependent operators.* Consider equation (4.1), where

$$b(x,t)=c\,t^l,\;\; t\gg 1,\quad l>0,\;\; c>0.$$

Then function (4.3) satisfies

$$B(t)=\Big(\int_{|x|\le R+t}(ct^l)^{1-p'}\Big)^{p-1}=c_1(R+t)^{3(p-1)}t^{-l},$$

so that $\alpha=3(p-1)-l$. Similarly, $B_1(t)$ given by (4.4) is such that

$$B_1(t)\le c_2(R+t)^{2(p-1)-l} \implies \beta=2(p-1)-l.$$

The inequality (4.5) on the exponents reads $3(p-1)-l<(p+2+l-2p+2)(p-1)+2$, which is equivalent to the quadratic one $p^2-(2+l)p-1<0$. This gives the following blow-up interval (the subcritical range):

$$1<p<p_c=1+l/2+[(1+l/2)^2+1]^{1/2}. \tag{4.17}$$

In the autonomous case $l=0$ and $p_c=1+\sqrt{2}$ gives John's critical exponent [21].

Spatially non-autonomous operators. Consider the hyperbolic equation (4.1) with $b(x,t)$ depending on the spatial variable:

$$u_{tt}=\Delta u+c|x|^k|u|^p, \;\; k<N(p-1)=3(p-1) \;\; (c>0). \tag{4.18}$$

Then

$$B(t)=c_1\left(\int_{|x|\le R+t}|x|^{k(1-p')}\right)^{p-1}=c_2(R+t)^\alpha, \;\; \alpha=3(p-1)-k,$$

$$B_1(t)=c_3\left(\int_{t-R\le|x|\le R+t}|x|^{k(1-p')}\right)^{p-1}$$

$$=c_3\left(\int\frac{|x|^{N-1}}{|x|^{k(p'-1)}}\right)^{p-1}=c_4[(t+R)^{N-k(p'-1)}-(t-R)^{N-k(p'-1)}]^{p-1}\le c_5(R+t)^\beta,$$

where $\beta=(N-1)(p-1)-k=2(p-1)-k$. From (4.5), $3(p-1)-k<(p+2+k-2p+2)(p-1)+2$, which is equivalent to the following quadratic inequality on the exponent p: $p^2-(2+k)p-1<0$, we derive the subcritical blow-up range

$$1<p<p_c=1+k/2+[(1+k/2)^2+1]^{1/2} \quad (k<3(p-1)). \tag{4.19}$$

If $k=0$, this coincides with John's exponent $p_c=1+\sqrt{2}$ [21].

4.3. **Critical blow-up exponents for $\mathbf{N>3}$.** We consider the Cauchy problem (4.1) in dimensions $N>3$. We will use some estimates from [45]. We set

$$m=(N-5)/2 \text{ if } N \text{ is odd}; \quad m=(N-4)/2 \text{ if } N \text{ is even},$$

and define for $t>R$

$$F(t)=\int_{t-R}^t(t-s)^m\int_{\mathbf{R}^N}u(x,s)\,dxds \Longrightarrow \ddot{F}=\int_{t-R}^t(t-s)^m\int_{\mathbf{R}^N}u_{tt}\,dxds. \tag{4.20}$$

Then equation (4.1) implies that

$$\ddot{F}(t)=\int_{t-R}^t(t-s)^m\int_{\mathbf{R}^N}b(x,s)|u(x,s)|^p\,dxds. \tag{4.21}$$

Since $\operatorname{supp} u(t) \subset S(t) \equiv \{|x| < t + R\}$, by the Hölder inequality we obtain

$$
\begin{aligned}
|F(t)| &= \left| \int_{t-R}^{t} \int_{S(t)} (t-s)^m u(x,s)\, dxds \right| \le \left| \int_{t-R}^{t} \int_{S(t)} (t-s)^m b|u|^p\, dxds \right|^{1/p} \\
&\times \left| \int_{t-R}^{t} \int_{S(t)} (t-s)^m b^{1-p'}\, dxds \right|^{1/p'} \le \left| \int_{t-R}^{t} \int_{S(t)} (t-s)^m b|u|^p dxds \right|^{1/p} \\
&\times R^{m/p'} \left(\int_{t-R}^{R} \int_{S(t)} b^{1-p'}\, dxds \right)^{1/p'} . \qquad (4.22)
\end{aligned}
$$

Denoting

$$
B(t) = R^{m(p-1)} \left(\int_{t-R}^{t} \int_{S(t)} b^{1-p'}(x,s) dxds \right)^{p-1}, \qquad (4.23)
$$

we have

$$
G(t) \equiv \int_{t-R}^{t} \int_{S(t)} (t-s)^m b(x,s) |u|^p dxds \ge B^{-1}(t) F^p(t),
$$

and (4.21) gives the ODI

$$
\ddot{F}(t) \ge B^{-1}(t) F^p(t), \quad t > R. \qquad (4.24)
$$

In order to derive a lower estimate on $G(t)$ we consider the linear problem (4.8). By Lemma 5 in [45] we have (cf. (4.9))

$$
\int_0^t (t-s)^m u(x,s) ds \ge \int_0^t (t-s)^m v(x,s) ds, \qquad (4.25)
$$

and hence

$$
\int_0^t (t-s)^m \int_{|x|>t} u(x,s) dxds \ge \int_0^t (t-s)^m \int_{|x|>t} v(x,s) dxds.
$$

Since supports of both solutions $u(\cdot,t)$ and $v(\cdot,t)$ are contained in the ball $S(t)$, we conclude that for $t > R$

$$
\int_0^t u\, ds = \int_{t-R}^t u\, ds, \quad \int_0^t v\, ds = \int_{t-R}^t v\, ds.
$$

Then from the last inequality

$$
\int_{t-R}^t (t-s)^m \int_{|x|>t} u(x,s) dxds \ge \int_{t-R}^t (t-s)^m \int_{|x|>t} v(x,s) dxds \equiv H(t). \qquad (4.26)
$$

We now apply Lemma 6 in [45].

Lemma 4.1. *Assume that*

$$
\int_{\mathbf{R}^N} |x|^{\eta-1} u_0(x) dx > 0, \quad \int_{\mathbf{R}^N} |x|^{\eta-1} u_1(x) dx > 0, \qquad (4.27)
$$

where $\eta = 0$ *if* N *is odd and* $\eta = 1/2$ *if* N *is even. Then there exists a constant* $C > 0$ *such that for* $t \gg 1$

$$H(t) \geq C(R+t)^{(N-1)/2}. \tag{4.28}$$

In view of (4.22) and (4.25), by the Hölder inequality we derive that

$$C(R+t)^{(N-1)/2} \leq H(t) \leq \int_{t-R}^{t} (t-s)^m \int_{|x|>t} u(x,s)dxds$$

$$= \int_{t-R}^{t} (t-s)^m \int_{t<|x|<t+R} u(x,s)dxds \leq B_1^{1/p}(t) \left(\int_{t-R}^{t} (t-s)^m \int_{S(t)} u(x,s)dxds \right)^{1/p},$$

$$B_1(t) = R^{m(p-1)} \left(\int_{t-R}^{t} \int_{t<|x|<t+R} b^{1-p'}(x,s)dxds \right)^{p-1}. \tag{4.29}$$

Hence,

$$G(t) \geq C^p B_1^{-1}(t)(R+t)^{(N-1)p/2}. \tag{4.30}$$

It then follows from (4.21) that $\ddot{F}(t) \geq C^p B_1^{-1}(t)(R+t)^{(N-1)p/2}$. Integrating this inequality, we arrive at the lower bound

$$F(t) \geq A + Bt + C^p \int_0^t (t-\tau)(R+\tau)^{(N-1)p/2} B_1^{-1}(\tau)d\tau. \tag{4.31}$$

We now assume that there exist exponents α, β such that (4.4) hold. Then (4.24) and (4.31) form the system

$$\ddot{F}(t) \geq C_1 t^{-\alpha} F^p(t), \quad F(t) \geq C_2 t^{-\beta+2+(N-1)p/2}, \quad t \gg 1. \tag{4.32}$$

We suppose also that

$$-\beta + 2 + (N-1)p/2 > 1. \tag{4.33}$$

Then the linear term $A + Bt$ in (4.31) is negligible for large times. Using Lemma 4 in [45] on the global nonexistence for ODIs like (4.32), we arrive at the following result.

Theorem 4.2. *Let for* $N > 3$ *initial data satisfy* (4.27). *Assume* (4.33) *and*

$$\alpha < [-\beta + 2 + (N-1)p/2](p-1) + 2 \tag{4.34}$$

hold. Then the maximal existence time T *of the solutions to* (4.1) *is finite.*

4.4. **Examples for** $\mathbf{N > 3}$. *Time-dependent operators.* Consider

$$u_{tt} = \Delta u + ct^l |u|^p, \quad l \leq 1/(p'-1).$$

Then for $t > R$

$$B(t) \leq C_R \left(\int_{t-R}^{t} \int_{S(t)} s^{-l(p'-1)} dxds \right)^{p-1} = C_R (|R+t|^N s^{1-l(p'-1)} \Big|_{t-R}^{t})^{p-1}$$

$$\leq C_R (|R+t|^N t^{-l(p'-1)})^{p-1} \leq C_R t^{\alpha}, \quad \alpha = N(p-1) - l,$$

where C_R denotes different positive constants depending on R. Estimating $B_1(t)$, we get

$$B_1(t) \le C_R \left(\int_{t-R}^{t} \int_{t<|x|<t+R} s^{-l(p'-1)} dx ds \right)^{p-1} \le C_R t^{\beta},$$

where $\beta = (N-1)(p-1) - l$. Then (4.33) and (4.34) yield

$$(N-1)(p-1) - l < 1 + (N-1)p/2,$$
$$N(p-1) - l < [(N-1)p/2 + 2 - (N-1)(p-1) + l](p-1) + 2,$$

i.e., $p < 2\,(N+l)/(N-1)$ and

$$(N-1)p^2 - (N+1+2l)p - 2 < 0. \tag{4.35}$$

Since the positive root of this quadratic trinomial satisfies the first bound, we arrive at a single condition (4.35) of blow-up in the Cauchy problem. For the autonomous equation with $l = 0$ we obtain the same critical exponent as in [45].

Spatially non-autonomous operators. Consider

$$u_{tt} = \Delta u + c|x|^k |u|^p, \;\; k < N(p-1) = N/(p'-1),$$

where for $t > R$

$$B(t) \le C_R \left(\int_{t-R}^{t} \int_{S(t)} |x|^{-k(p'-1)} dx ds \right)^{p-1} \le C_R t^{\alpha}, \;\; \alpha = N(p-1) - k,$$

$$B_1(t) \le C_R \left(\int_{t-R}^{t} \int_{t<|x|<t+R} |x|^{-k(p'-1)} dx ds \right)^{p-1} \le C_R t^{\beta}, \;\; \beta = (N-1)(p-1) - k.$$

From (4.33), (4.34) we obtain the conditions

$$(N-1)(p-1) - k < (N-1)p/2 + 1,$$
$$N(p-1) - k < [(N-1)p/2 + 2 - (N-1)(p-1) + k]\,(p-1) + 2.$$

If $1 < p < 2\,(N+k)/(N-1)$, $k < N(p-1)$ and $(N-1)p^2 - (N+1+2k)p - 2 < 0$, then the solution of the Cauchy problem blows up in finite time.

One can see that the above results apply to hyperbolic inequalities $u_{tt} \ge \Delta u + b(x,t)|u|^p$.

PART II: ASYMPTOTIC BEHAVIOUR AND SPECTRA

5. On spectra of asymptotic patterns

We now begin to study the asymptotic behaviour of global and blow-up solutions of the autonomous semilinear equation

$$u_{tt} = \Delta u + f(u) \quad \text{in} \;\; \mathbf{R}_+^{N+1}, \tag{5.1}$$

where $f(u)$ is a homogeneous nonlinearity of order $p > 1$:

$$\text{either} \;\; f(u) = |u|^p \;\; \text{or} \;\; f(u) = |u|^{p-1}u, \;\; p > 1.$$

Particular similarity solutions often represent different types of asymptotic patterns as $t \to \infty$. It is well-known in the parabolic asymptotic theory (see references to Chapt. 2 and

4 in [44]), the construction of similarity solutions generated by *nonlinear eigenfunctions* (solutions of nonlinear elliptic equations) is a necessary step for general understanding of asymptotic properties of the nonlinear evolution problem under consideration. It is important that nonexistence or finiteness of the nonlinear spectrum of such similarity patterns usually means that the rest of the patterns are not self-similar and should be constructed by a matching with a spectrum of linearized patterns. Such a transition between nonlinear and linear spectra of patterns is well understood in the blow-up analysis for quasilinear parabolic equations, see [13], [7] and references therein.

In Sections 6 and 6 we will present a construction of different finite and countable spectra of similarity patterns as well as of a countable spectrum of linearized blow-up patterns in Section 7. There are many important results on the asymptotic behaviour of global and blow-up solutions of such semilinear wave equations, see e.g. references presented below and in the book [1]. In view of the finite propagation along straight characteristics, such asymptotic hyperbolic problems are more definite and exhibit less sensitivity than similar parabolic ones for semilinear heat equations

$$u_t = \Delta u + f(u). \tag{5.2}$$

In particular, equation (5.1) is invariant under the reflection $t \mapsto -t$, so that the asymptotic similarity patterns as $t \to \infty$ are somehow essentially the same as those which blow-up in finite time, $t \to -0$. Obviously, this is not the case for the parabolic equations (5.2), where the stable patterns as $t \to \infty$ and $t \to -0$ are entirely different. Moreover, for the hyperbolic equations in the normal form, the problems on formation of singularities and singular blow-up surfaces can be covered by classical approaches based on the Cauchy-Kovalevskaya Theorem, see a general singularity classification in [25] or on p. 6 in [1]. Obviously, these ideas are not applicable to the parabolic equations (5.2), not normal in the time variable.

Nevertheless, regardless their good properties and extra advantages of the analysis, a detailed description of the spectra of the asymptotic patterns for the semilinear hyperbolic equations is still not available. Viewing (5.1) as a standard nonlinear evolution equation admitting a symmetry group of scalings, we present an approach to constructing of its asymptotic patterns, based on the same general principles as for the evolution parabolic PDEs like (5.2).

On the other hand, the similarity analysis is related to the question on the stability of the trivial stationary solution $u \equiv 0$ which is *unstable* in the range

$$1 < p \le p_c = [N + 1 + (N^2 + 10N + 7)^{1/2}]/2(N-1).$$

Moreover, solutions with arbitrarily small initial data blow-up in finite time and is *stable* in the supercritical range $p > p_c$. It is known that for equation (5.1) with $f = |u|^p$ in the stability supercritical range $p > p_c$, there exists another critical exponent $p_1 = (N+3)/(N-1) > p_c$, and there hold:

(i) For $p \ge p_1$ the asymptotic behaviour of small global solutions as $t \to \infty$ is described by the linear equation

$$u_{tt} = \Delta u,$$

with certain initial data, so that the nonlinear term $|u|^p$ is negligible for large times. See [31] where a complete list of related references on the preceeding results and a historical survey are presented.

(ii) If $p_c < p < p_1$, then the global solvability for small initial data takes place in the weighted Strichartz functional classes [17].

In what follows, we show that in different parameters ranges the global self-similar structures play an important role for classes of both global (this section) and blow-up (see the next section) solutions of semilinear hyperbolic equations.

5.1. **Self-similar structures.** Let us fix the subscritical Sobolev range, $1 < p < p_S$. As a first step in the asymptotic analysis, we consider the following global self-similar solutions of equation (5.1):

$$u_*(x,t) = t^{-\alpha}\theta(\eta), \ \ \eta = x/t, \quad \alpha = 2/(p-1) > 0, \tag{5.3}$$

where the function θ solves the following elliptic (stationary) equation:

$$\mathbf{A}\theta \equiv \Delta\theta - \nabla(\nabla\theta \cdot \eta)\cdot\eta - (1+2\alpha)\nabla\theta\cdot\eta - \alpha(1+\alpha)\theta = -f(\theta), \ \ \eta \in \mathbf{R}^N. \tag{5.4}$$

Using the identity

$$\nabla(\nabla\theta\cdot\eta)\cdot\eta = \sum_{(i,j)} \theta_{\eta_i\eta_j}\eta_i\eta_j + \nabla\theta\cdot\eta,$$

the linear operator is written down in the form

$$\mathbf{A}\theta = \Delta\theta - \sum_{(i,j)} \theta_{\eta_i\eta_j}\eta_i\eta_j - 2(1+\alpha)\nabla\theta\cdot\eta - \alpha(1+\alpha)\theta. \tag{5.5}$$

We begin with the first properties of $\mathbf{A}$ and θ.

Proposition 5.1. *The quadratic form of the linear operator* (5.5) *satisfies*

$$B_{\mathbf{A}}(z,z) \equiv \sum_{(i,j)} (\delta_{ij} - \eta_i\eta_j) z_i z_j \ge |z|^2(1-|\eta|^2), \tag{5.6}$$

so that $\mathbf{A}$ *is uniformly elliptic on any compact subset from the unit ball* B_1.

Proof. By the Cauchy-Bunyakovskii-Schwarz inequality we have $B_{\mathbf{A}}(z,z) = |z|^2 - (z\cdot\eta)^2 \ge |z|^2(1-|\eta|^2)$. □

Proposition 5.2. *Let there exist a sufficiently regular, compactly supported solution* θ *of equation* (5.8). *Then*

$$\operatorname{supp}\theta = \overline{B}_1. \tag{5.7}$$

Proof. Let $\operatorname{supp}\theta \subseteq \{|\eta| \le c\}$. Then $\operatorname{supp} u_*(x,t) \subseteq \{|x| \le ct\}$, and passing to the limit $t \to +0$ in (5.3), we see that $\operatorname{supp} u_*(x,+0) = \{0\}$, i.e., the initial data for $u_*(x,t)$ are concentrated at the origin, whence the result by the characteristic propagation. On the other hand, the equality in (5.7) follows by the strong Maximum Principle applied to equation (5.4), which is uniformly elliptic in any domain bounded away from the unit sphere $S_1 = \{|\eta| = 1\}$. □

Let us look for a solution θ of the elliptic equation (5.8) with the radially symmetric support B_1. This suggests to consider this problem in the radial setting, where θ depends on the single radial variable $\xi = |\eta|$. Then (5.4) reduces to the ODE

$$\mathbf{A_r}\theta \equiv (1-\xi^2)\theta'' + \frac{N-1}{\xi}\theta' - 2(1+\alpha)\theta'\xi - \alpha(1+\alpha)\theta = -f(\theta), \quad \xi \neq 1. \tag{5.8}$$

We impose a standard symmetry condition at the origin $\theta'(0)=0$ (a natural functional restriction for the radial setting of the Laplacian operator). The linear operator $\mathbf{A}$ degenerates on the sphere $\xi = |\eta| = 1$ which is the *light cone* $\{|x|=t\}$ for the linear hyperbolic equation. In the global sense, we are looking for a weak solution $\theta(\xi)$ of (5.8) in $\mathbf{R}_+$, which, in particular, is continuous at the degeneracy point $\xi = 1$.

It is already known that in the *critical* and *supecritical* Sobolev ranges, $p \geq p_S$, nontrivial self-similar solutions with *finite-energy* ($f(\theta) = |\theta|^{p-1}\theta$)

$$E(\theta) = \frac{1}{2}\|2\theta/(p-1) + \xi\theta'\|_2^2 + \frac{1}{2}\|\theta'\|_2^2 - \frac{1}{p+1}\|\theta\|_{p+1}^{p+1}$$

do not exist. See the results in the paper [23], where a detailed asymptotic analysis of the ODE (5.8) at the critical points as well as other local and global properties of the solutions are available. We will use some of the results from [23] in our construction of nontrivial solutions in the subcritical Sobolev range.

5.2. **The first spatially flat similarity pattern.** Let $f(\theta) = |\theta|^{p-1}\theta$. One can see that equation (5.4) or (5.8) admits a constant solution

$$\theta_1(\eta) \equiv c_* = [\alpha(1+\alpha)]^{1/(p-1)} \equiv [2(p+1)/(p-1)^2]^{1/(p-1)}. \tag{5.9}$$

This profile gives the first self-similar pattern

$$u_1(x,t) = t^{-\alpha}c_*. \tag{5.10}$$

Obviously, the piece-wise pattern

$$\tilde{\theta}_1(\eta) = \{c_*, \ |\eta| \leq 1; \quad 0, \ |\eta| > 1\},$$

has a strong discontinuity on S_1 and $\tilde{u}_1(x,t) = t^{-\alpha}\tilde{\theta}_1(\eta)$ is not a proper weak solution of the semilinear hyperbolic equation in the sense that it does not satisfy the corresponding integral equation given by the D'Alambert-Poisson-Kirchhoff formulae. In order to choose a proper solution of hyperbolic equations with *a priori* known singularity propagation, the concept of the wave front set in the microlocal analysis applies, see Chapt. 8 in [20]. On the other hand, such a discontinuous solution $\tilde{u}_1$ cannot be obtained as a limit of a sequence of global regular solutions $\{u_\varepsilon\}$ obtained by a suitable truncation of the equation, $u = \lim_{\varepsilon\to 0} u_\varepsilon$, with continuous (truncated) initial data $u_{0\varepsilon}, u_{1\varepsilon} \to u_0, u_1$ as $\varepsilon \to 0$ in a certain metric (say, in L^1). Such an extended semigroup theory is quite natural in singular (blow-up) parabolic problems, see [16], Sect. 3 and references therein.

5.3. **Very singular self-similar solution (VSS).** Assuming that a compactly supported continuous profile θ exists and integrating (5.3) over $\mathbf{R}^N$ (or B_1), we obtain that

$$E_*(t) = \int_{\mathbf{R}^N} u_*(x,t)dx = t^\gamma c_*, \quad c_* = \int_{\mathbf{R}^N} \theta(\eta)d\eta,$$

where $\gamma = -2/(p-1)+N$. Therefore, if $\gamma < 0$, i.e., $1 < p < p_F = 1 + 2/N$, and $c_* \ne 0$, then u_* is a *very singular solution* in the sense that it satisfies non-integrable initial data: as $t \to +0$, $u_*(x,t) \to u_{*0}(x) \notin L^1(\mathbf{R}^N)$. On the other hand, in the critical case $p = p_F$ we have $u_{*0} = c_*\delta(x)$. In a similar way, one can treat the type of singularity of the derivative:

$$u_{*t}(x,t) = -t^{-\alpha-1}(\alpha\theta + \nabla\theta\cdot\eta),$$

so that $u_{*t}(x,t) \to u_{*1}(x) \in L^1_{\rm loc}(\mathbf{R}^N)$ as $t \to +0$ provided that $-\alpha - 1 + N \ge 0$, i.e., $p \ge p_K = (N+1)/(N-1) > p_F$, where p_K is Kato's critical exponent [22]. If $p = p_K$, then $u_{*1} = c_1\delta(x)$, $c_1 \ne 0$. In the second supercritical range, $p > p_K$, self-similar solutions $u_* \not\equiv 0$ satisfy the trivial initial conditions $u_{*0} = 0 = u_{*1}$ in $\mathcal{D}'(\mathbf{R}^N)$ or $L^1(\mathbf{R}^N)$, a nonuniqueness of the solution in a natural sense. It is curious that the exponent p_F coincides with the critical Fujita exponent [11] for the semilinear parabolic equation

$$u_t = \Delta u + u^p, \quad p > 1, \tag{5.11}$$

where p_F determines the subcritical range $p < p_F$ and the supercritical one $p > p_F$ of unstability and stability of the trivial stationary solution $u \equiv 0$, see Chapt. 4 in [44] and an extended list of references therein.

5.4. **Properties of the linear operator in the radial setting.** In order to construct asymptotic patterns for the semilinear wave equation, we need some analysis of the linear and linearized degenerate elliptic operators. We begin with the properties of the radial linear operator (5.8) in the unit ball $B_1 = \{|\eta| < 1\}$ which is identified with the interval $\xi \in B_1 = [0,1)$ for sufficiently smooth even functions. Its Sturm-Liouville form

$$\mathbf{A}_\mathbf{r} = \frac{1}{\rho}[(d/d\xi)(a\, d/d\xi) - \alpha(1+\alpha)\rho I], \tag{5.12}$$

with the positive coefficients on $(0,1)$,

$$\rho(\xi) = \xi^{N-1}(1-\xi^2)^\nu > 0, \quad a(\xi) = \xi^{N-1}(1-\xi^2)^{1+\nu} > 0, \tag{5.13}$$

$$\nu = \alpha - (N-1)/2 = 2/(p-1) - (N-1)/2, \tag{5.14}$$

where $\nu > 0$ for $1 < p < p_1$ and $\nu = 0$ if $p = p_1$. Operator $\mathbf{A}_\mathbf{r}$ is symmetric in the weighted class $C^\infty_{0\rho}(B_1)$ of symmetric functions with compact support in B_1, $-\mathbf{A}_\mathbf{r} > 0$ is semibounded below and admits a unique Friedrichs extension [4], which is a self-adjoint operator in the weighted Hilbert space $L^2_\rho(B_1)$ of even functions.

Let us study the spectrum of the linear operator. As usual for the radial setting of the Laplacian, the left-hand singular end point $\xi = 0$ is in the limit-point case for $N \ge 4$. Indeed, since $\rho \sim a \sim \xi^{N-1}$ as $\xi \to 0$, we have two possible expansions of the solutions near the singularity: $\theta_1 \sim 1$, $\theta_2 \sim \xi^{2-N}$, and obviously $\theta_1 \in L^2_\rho$ and $\theta_2 \notin L^2_\rho$ provided that $N \ge 4$. For $N = 2$ and $N = 3$ we need to impose a natural condition of boundednes at the origin or, which is equivalent, we assume that $\xi^{N-2}\theta(\xi) \to 0$ as $\xi \to 0$. In the one-dimensional case $N = 1$ the origin is not a singular point, where we put the symmetry condition $\theta'(0) = 0$.

We next put $\xi = \sin\zeta : I = [0, \pi/2) \to B_1$. Then

$$\mathbf{A_r} = \frac{1}{\bar\rho}[(d/d\xi)(\bar\rho\, d/d\xi) - \alpha(1+\alpha)\bar\rho I], \tag{5.15}$$

where

$$\bar\rho(\zeta) = \sin^{N-1}\zeta \cos^{\mu}\zeta, \quad \mu = 1 + 2\nu = 2\alpha - (N-2) \tag{5.16}$$

($\mu > 0$ if $p < p_S$). $\mathbf{A_r}$ is symmetric and semibounded in $C^\infty_{0\bar\rho}(I)$ and its self-adjoint extension in $L^2_{\bar\rho}(I)$ has the domain $H^2_{\bar\rho}(I)$

Consider the right-hand singular end point $\xi = 1$, i.e., $\zeta = \pi/2$. We use the representation (5.15). Similar to the analysis at $\xi = 0$, using a standard procedure from the theory of ODEs, see the books [3] or [30], we have to study the asymptotics of two linearly independent solutions near this singular point. Setting for convenience $s = \pi/2 - \zeta$ and using the expansion of the coefficients $\bar\rho \sim s^{\mu}$, we have that in the first approximation

$$\mathbf{A_r} Y \sim \frac{1}{s^{\mu}}[(s^{\mu}Y')' + ...].$$

Therefore, any solution of the eigenvalue problem

$$\mathbf{A_r} Y = -\lambda Y \tag{5.17}$$

can exhibit the following two linearly independent expansions as $s \to 0$: $Y_1 \sim 1$ and $Y_2 \sim s^{1-\mu}$. We have that $Y_1 \in L^2_\rho$ if $p < \bar p = (N+1)/(N-3) > p_S$, and $Y_2 \notin L^2_\rho$ provided that $\mu \geq 3$, i.e., $p \leq p_2 = (N+5)/(N+1) < p_1$. In this case the end point $\xi = 1$ is in the limit-point case and we arrive at the following result, cf. [30].

Proposition 5.3. *(i) For $p \in (1, p_2]$ the self-adjoint operator $\mathbf{A_r}$ has a purely discrete countable spectrum of real simple eigenvalues*

$$\sigma(\mathbf{A_r}) = \{\lambda_k = 2k(2k+1+2\alpha) + \alpha(1+\alpha),\ k = 0, 1, 2, ...\}, \tag{5.18}$$

the corresponding eigenfunctions ψ_k are $2k$-th order polynomials, and $\{\psi_k\}_{k\geq 0}$ form an orthonormal basis in the weighted space $L^2_\rho(B_1)$ of symmetric functions.

(ii) If $p \in (p_2, p_1]$, then $\xi = 1$ is in the limit-circle case, and the properties in (i) remain valid in the domain $\mathcal{D}(\mathbf{A_r})$ of symmetric functions satisfying the extra boundary condition as $\xi \to 1$

$$(1-\xi)^{\nu}\theta(\xi) \to 0 \ \ (\nu > 0), \quad |\ln(1-\xi)|^{-1}\theta(\xi) \to 0 \ \ (\nu = 0). \tag{5.19}$$

(iii) If $p > p_1$, then the discrete spectrum (5.18) *exists under the condition*

$$\psi,\ \psi' \ \text{are bounded as}\ \xi \to 1. \tag{5.20}$$

Proof. For any $\lambda \in \mathbf{R}$, the symmetric solutions of equation (5.17) with analytic coefficients in B_1 are given by Kummer's series [3]

$$\psi(\xi) = \sum_{k=0}^{\infty} C_{2k}\xi^{2k}.$$

Substituting into the equation, we obtain a typical recursion on the expansion coefficients

$$C_{2k+2} = A_k C_{2k}, \quad k = 1, 2, \dots, \tag{5.21}$$

where

$$A_k = \frac{2k(2k+1+2\alpha) + \alpha(1+\alpha) - \lambda}{2(k+1)(2k+N)} = 1 + \frac{a_1}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty,$$

and $a_1 = (2\alpha - N - 1)/2$. It then follows that this series converges uniformly on any interval $[0, 1-\varepsilon]$ bounded away from 1 and is not uniformly converging on $[0,1]$. Moreover, we have that at the end point $\xi = 1$, for $k \gg 1$, there holds $C_{2k} = O(k^{-1})$ with no sign changes if $p \neq p_2$ and $C_{2k} = 1 + O(k^{-2})$ if $p = p_2$. Hence, the series diverges at $\xi = 1$ unless $\lambda = \lambda_k$ so that $A_j = 0$ for all $j \geq k$. The corresponding eigenfunctions ψ_k are $2k$-th order polynomials

$$\psi_k(\xi) = c_k(1 + a_1\xi^2 + \dots + a_k\xi^{2k}).$$

These eigenfunctions are defined inside the rescaled light cone $\{\xi < 1\}$ but also outside, for $\xi > 1$, so that $\{\psi_k\}$ are polynomial eigenfunctions of the eigenvalue problem posed in the complement $\{\xi > 1\}$ and therefore these are global eigenfunctions in $\mathbf{R}_+$. In both cases (ii) and (iii) extra boundary conditions at the degeneracy point $\xi = 1$ follow from the above asymptotic analysis. Similar to the classical Hermite polynomials, the higher order term $\psi_k(\xi) \sim \xi^{2k}$ gives the spectrum (5.18). The rest of the analysis is standard for self-adjoint second-order ordinary differential operators [30]. □

The first two $\{\lambda_k, \psi_k\}$ pairs are:

$$\lambda_0 = \alpha(1+\alpha), \quad \psi_0(\xi) = c_0 > 0,$$

$$\lambda_1 = 2(3 + 2\alpha) + \alpha(1+\alpha), \quad \psi_1(\xi) = c_1[1 - (2\alpha+3)\xi^2/N].$$

Here $c_0, c_1, \dots, c_k, \dots$ denote normalization constants.

5.5. **Internal variational problem for $f = |u|^p$: the first pattern.** Let $f(\theta) = |\theta|^p$. Consider the following functional:

$$F(v) = -\frac{1}{2}\int a(\xi)(v')^2 - \frac{1}{2}\alpha(1+\alpha)\int v^2 + \frac{1}{p+1}\int \rho|v|^p v. \tag{5.22}$$

We will study its conditional critical points on the unit sphere in the Hilbert space of real-valued functions

$$H_a^1((0,1)) = \left\{ v : \ \|v\|_a^2 = \int a(\xi)(v')^2 + \int v^2 < \infty \right\},$$

naturally endowed with the corresponding inner product. This functional changes sign, is not uniform and the corresponding operator of the Euler equation $\mathbf{A_r} = F'$ is non-coercive. In order to apply the fibering method [41], [42], we need some embedding estimates.

Proposition 5.4. *The embedding $H^1_a \subset L^{p+1}_\rho$ is compact if and only if*

$$1 < p < p_S = (N+2)/(N-2), \quad \textit{and} \tag{5.23}$$

$$1 < p < 1 + 2/\nu, \ \nu > 0 \quad (p \in (1, \infty) \ \textit{if} \ \nu = 0). \tag{5.24}$$

Proof. The first condition (5.23) is indeed coming from the standard analysis of the operator in a small neighbourhood of the origin $\xi = 0$ and it is the same as for the semilinear elliptic operator $\Delta\theta + |\theta|^p$, see [39], so that p_S is the Sobolev critical exponent. The second one (5.24) is derived in a similar way by using the expansions of the coefficients a and ρ for $\xi \approx 1$. ☐

Substituting $\nu \geq 0$ given in (5.14) into (5.24) one can see that it is always true provided $\nu \geq 0$, i.e., $p \leq p_1 < p_S$. Thus, $1 < p \leq p_1$ is necessary and sufficient condition for compact embedding of the functional spaces. We then apply the fibering method proposed in [40], see also a detailed description in [42], and prove the following result.

Theorem 5.1. *Let $p \in (1, p_1]$. Then equation* (5.23) *admits a weak solution $\theta_1 \in H^1_a((0,1))$, which is a bounded continuous function on* $[0,1]$*, positive on* $[0,1)$.

As usual, the positivity property follows from the variational statement of the elliptic (ODE) problem. One can expect that Theorem 5.1 establishes existence of the constant weak solution θ_1 given in (5.9).

5.6. **On a countable spectrum of internal self-similar patterns.** It is essential that Theorem 5.1 establishes existence of a single nonnegative self-similar profile (θ_1) for the hyperbolic equation (5.1) and, in general, the corresponding elliptic ODE (5.8) with non-monotone nonlinearity $|\theta|^p$ admits no other nontrivial solutions, see examples in [42]. Consider the hyperbolic equation with the monotone nonlinear term

$$u_{tt} = \Delta u + |u|^{p-1}u \quad \text{in } \mathbf{R}^{N+1}_+. \tag{5.25}$$

Looking for the same self-similar solutions (5.3), we arrive at the ODE

$$\mathbf{A_r}\theta \equiv (1-\xi^2)\theta'' + \frac{N-1}{\xi}\,\theta' - 2(1+\alpha)\theta'\xi - \alpha(1+\alpha)\theta = -|\theta|^{p-1}\theta, \quad \xi \in (0,1), \tag{5.26}$$

with quite similar functional setting. Several results remain valid for such a new equation. Nevertheless, there exists an important difference. Namely, in the corresponding functional

$$F(v) = -\frac{1}{2}\int a(\xi)(v')^2 - \frac{1}{2}\alpha(1+\alpha)\int v^2 + \frac{1}{p+1}\int \rho|v|^{p+1}, \quad v \in H^1_a, \tag{5.27}$$

the last potential is homogeneous and nonnegative and both linear and nonlinear operators are monotone. Using the fibering method, we reduce the problem to the study of conditional critical points of a homogeneous nonnegative functional on the unit sphere and the Lyusternik-Shnirel'man theory applies, see [41]. We then arrive at the following result.

Theorem 5.2. *For $p \in (1, p_1]$, (5.26) admits a countable spectrum of weak self-similar solutions $\{\theta_k(\xi),\ k = 1, 2, ...\}$, where $\theta = \theta_k \in H^1_a((0,1))$ satisfy*

$$\|\theta_k\|_a \to \infty, \quad k \to \infty.$$

As usual for such ODEs, the first solution $\theta_1 \equiv c_*$ is positive in the non-degeneracy domain $(0,1)$ unlike the other profiles $\theta_k(\xi)$ which change sign exactly k times in $(0,1)$. We do not know if the parameter range $p \in (1, p_1]$ the above internal patterns can be extended beyond the cone B_1 in a proper weak sense. In fact, due to the singularity behaviour of the solutions near S_1 to be described below, we suspect that in general such an extension is not possible.

5.7. **Example: nonexistence of a VSS.** The similarity VSS were first constructed for the semilinear parabolic equation with absorption

$$u_t = \Delta u - |u|^{p-1}u, \quad 1 < p < 1 + 2/N,$$

see [14], [6] and related references in Chapt. 2 in [44]. For a similar hyperbolic equation

$$u_{tt} = \Delta u - |u|^{p-1}u, \tag{5.28}$$

such a construction of the symmetric VSS (5.3) leads to the stationary equation

$$\mathbf{A}_r\theta - |\theta|^{p-1}\theta \equiv \frac{1}{\rho}\,[(a\theta')' - \alpha(1+\alpha)\rho\theta] - |\theta|^{p-1}\theta = 0, \tag{5.29}$$

with a monotone operator. Therefore, it admits a unique solution $\theta \equiv 0$ in H^1_a, and we arrive at the nonexistence result.

Proposition 5.5. *The semilinear hyperbolic equation (5.28) with $p \in (1, p_1]$ does not admit a nontrivial weak VSS.*

6. Nonlinear self-similar patterns for $p_1 < p < p_S$

In the parameter range

$$p_1 = (N+3)/(N-1) < p < p_S = (N+2)/(N-2) \tag{6.1}$$

the embedding is not compact and the variational methods do not apply. In order to construct global weak continuous solutions $\theta(\xi)$ we will use a shooting procedure.

6.1. **Construction inside the light cone.** We need some local properties of the solutions of the ODE (5.8) near the singularity point $\xi = 1$. The asymptotic expansion shows how dramatically the singularity formation changes when p passes through the critical exponent p_1. Such local solvability results can be proved by Bahach's Contraction Principle applied to the equivalent integral equation, see also [23] where the proofs of a variety of asymptotic and global results are presented.

Proposition 6.1. *Let $f(\theta) = |\theta|^{p-1}\theta$ and denote $G(\theta) = f(\theta) - \alpha(1+\alpha)\theta$. (i) If $1 < p < p_1$ then* (5.26) *admits the singular behaviour*

$$\theta(\xi) = (C + o(1))(1-\xi)^{\omega}, \quad \xi \to 1-0, \tag{6.2}$$

where, for $1 < p < p_3 = (N+1)/(N-1)$

$$\omega = -1/(p-1), \quad C^{p-1} = [2/(p-1) - N + 1]/(p-1) > 0,$$

and, for $p_3 < p < p_1$, $C \in \mathbf{R}$ is arbitrary and $\omega = -\nu = (N-1)/2 - \alpha < 0$. For any $\theta_0 \in \mathbf{R}$ there exists a unique solution with a regular (analytic) expansion as $\xi \to 1-0$ (corresponding to $C = 0$ in (6.2)*)*

$$\theta(\xi) = \theta_R(\xi) \equiv \theta_0 + \gamma_0(1-\xi) + \delta_0(1-\xi)^2 + ..., \tag{6.3}$$

$$\gamma_0 = G(\theta_0)/2(1+\nu), \quad \delta_0 = \nu^{-1}[\nu + N - G'(\theta_0)G(\theta_0)/4(1+\nu)].$$

(ii) If $p_1 < p < p_4 = (N+1)/(N-3) > p_S$, then all the orbits are bounded near $\xi = 1$, and for any fixed $\theta_0 = \theta(1) \in \mathbf{R}$, as $\xi \to 1-0$,

$$\theta(\xi) = \theta_R(\xi) + [C(1-\xi)^{-\nu} + ...], \tag{6.4}$$

where the regular expansion θ_R is as given in (6.3) *and the term in square brackets describes a rational bundle with the exponent $0 < -\nu < 1$ and arbitrary $C \in \mathbf{R}$.*

It follows from (ii) that in this parameter range, the point $\xi = 1$ is a removable singularity and the orbits pass through it staying uniformly bounded and continuous. The result in (i) shows that it is not the case for $p < p_1$. We now prove the main result of this section.

Theorem 6.1. *Let $p_1 < p < p_S$. Then there exists a smooth solution $\theta^*(\xi) > 0$ of the ODE* (5.8) *on $(0,1)$ satisfying $(\theta^*)'(0) = 0$ and $\theta^*(1) = 0$. The extension $\theta^*(\xi) \equiv 0$ for $\xi > 1$ determines a continuous weak solution of* (5.8) *in $\mathbf{R}_+$.*

Proof. Given arbitrary value $\mu \in \mathbf{R}$, denote by $\theta_\mu(\xi)$ the unique local in $\xi > 0$ solution of (5.8) satisfying $\theta_\mu(0) = \mu$. Since the behaviour of θ_μ for $\mu \gg 1$ near the origin is governed by the elliptic operator $\Delta\theta + \theta^p$ in the subcritical range $p < p_S$, by a standard scaling argument (see e.g. p. 188 in [44]) we conclude that

$$\theta_\mu(\xi) \text{ vanishes on } (0,1) \text{ if } \mu \gg 1. \tag{6.5}$$

The function $\theta_\mu(\xi)$ depends continuously on μ on any interval $[0, 1-\varepsilon]$, $\varepsilon > 0$, where equation (5.8) is not degenerate. On the other hand, since $\xi = 1$ is a removable singularity and all the orbits stay continuous at the singularity, by a standard argument we conclude that such a continuous dependence is available on the closed interval $[0,1]$.

Consider now θ_μ for $\mu \approx c_* + 0$ where such a behaviour is described by the linearized equation, $\theta_\mu(\xi) = c_* + (\mu - c_*)Y(\xi)$. To the first approximation, Y solves the linear equation

$$(\mathbf{A_r} + p\alpha(1+\alpha)I)Y = 0, \ \xi > 0; \quad Y(0) = 1, \ Y'(0) = 0,$$

cf. [44], p. 191. By Proposition 5.3 we have that the first two eigenvalues of the self-adjoint operator $\mathbf{A_r} + p\alpha(1+\alpha)I$ satisfy $\tilde{\lambda}_0 = -(p-1)\alpha(1+\alpha) < 0$ and $\tilde{\lambda}_1 = 4p/(p-1) > 0$

(see the next section where we study this spectrum in detail). Therefore Y has a unique zero on $(0,1)$. By the continuous dependence, this means that

$$\theta_\mu(1) \to c_* \pm 0 \quad \text{as} \quad \mu \to c_* \mp 0. \tag{6.6}$$

It follows from (6.5) and (6.6) that there exists a $\mu^* > c_*$ such that $\theta_{\mu^*}(1) = 0$. Then $\theta^* \equiv \theta_{\mu^*}$ has the behaviour (6.4), $\theta_0 = 0$, near the singularity and the trivial extension for $\xi > 1$ gives a weak solution of (5.8). □

In order to complete the analysis of the orbits in B_1, we note that if $\mu \to 0$ then the linearized operator is $\mathbf{A_r}$ with the positive spectrum, see Proposition 5.3. Therefore, for any sufficiently small $|\mu| \neq 0$ the function $\theta_\mu(\xi)$ does not change sign in $(0,1)$. Combining the above asymptotic properties and using the continuous dependence, we prove the following result which will be used later on for a suitable nontrivial extension of the solutions outside the light cone $\{\xi > 1\}$.

Proposition 6.2. *Let* $f(\theta) = |\theta|^{p-1}\theta$ *and* $p_1 < p < p_S$. *Given arbitrary* $\theta_0 \in \mathbf{R}$ *there exists a solution* $\theta(\xi)$ *of* (5.8) *such that* $\theta(1) = \theta_0$.

Proof. We use a shooting argument with respect to the parameter C in (6.4). Due to well-known singularity properties of the radial elliptic operator $\Delta\theta + |\theta|^{p-1}\theta$ as $\xi \to 0$, we have that $\theta_C(\xi) \to +\infty$ as $\xi \to 0$ for $C \gg 1$ and $\theta_C(\xi) \to -\infty$ as $\xi \to 0$ for $C \ll -1$. By the continuous dependence on C, for some C^*, $\theta_{C^*}(\xi)$ is a bounded solution, hence satisfying the symmetry condition at the origin. □

In particular, there exists a positive solutions $\tilde{\theta}$ such that $\tilde{\theta}(1) = c_*$ and $\tilde{\theta}(0) \in (0, c_*)$. Extending it by c_* in the supplement $\xi > 1$, we obtain a nontrivial (not entirely flat) weak similarity solution $\tilde{u}(x,t) = t^{-\alpha}\tilde{\theta}(x/t)$ describing a weak singularity propagation on the light cone $\{|x| = t\}$.

6.2. **Construction outside the light cone.** We now briefly discuss possible types of nontrivial extension of the solutions θ outside the light cone $\{\xi > 1\}$. We consider the ODE (5.8) in $(1,\infty)$ in the parameter range $p_1 < p < p_S$. The singularity end point $\xi = 1$ is removable as in Proposition 6.1, (ii), and we have to study the end point $\xi = \infty$. The linearized operator is $\mathbf{A_r}$. One can see that the homogeneous linearized equation $\mathbf{A_r}Y = 0$ admits two different types of solutions (see also [23])

$$Y_+(\xi) = \xi^{-\alpha} + ..., \quad Y_-(\xi) = \xi^{-(p+1)/(p-1)} + ...$$

It is important that the initial traces of both self-similar solutions $u_\pm(x,t) = t^{-\alpha}Y(\xi) = t^{-\alpha}Y_\pm(\xi) + ...$ as $t \to +0$ are quite different:

$$u_+(x,+0) = |x|^{-2/(p-1)}, \quad u_-(x,+0) = 0, \ \ x > 0.$$

Similar to Proposition 5.3 one can obtain the discrete point spectrum of $\mathbf{A_r}$ written down in the symmetric form

$$\mathbf{A_r} = \frac{1}{\rho}[(d/d\xi)(a\, d/d\xi) + \alpha(1+\alpha)\rho I], \tag{6.7}$$

with the positive coefficients on $(1, \infty)$, $\rho(\xi) = \xi^{N-1}(\xi^2 - 1)^\nu$, $a(\xi) = \xi^{N-1}(\xi^2 - 1)^{1+\nu}$, and the same exponent $\nu = \alpha - (N-1)/2 > 0$. One can see that $Y_- \in L^2_\rho$ and $Y_+ \notin L^2_\rho$ so that the singular end point $\xi = \infty$ is in the limit-point case for $\mathbf{A_r}$. Therefore, its Friedrich extension is a self-adjoint operator with a discrete spectrum

$$\lambda_k = 2k(2k + 1 + 2\alpha) - \alpha(1 + \alpha), \quad k = 0, 1, 2, ..., \tag{6.8}$$

so that $\lambda_0 = -\alpha(1+\alpha) < 0$. Using a similar shooting technique and considering solutions $\theta_C(\xi)$ of (5.8) with the asymptotic behaviour corresponding to $Y_-(\xi) \in L^2_\rho$:

$$\theta_C(\xi) = CY_-(\xi) + o(Y_-(\xi)), \quad \xi \to \infty; \quad C \in \mathbf{R},$$

we conclude that for all sufficiently small $|C| \neq 0$ the function $\theta_C(\xi)$ is not monotone and has at least one zero for $\xi > 1$. On the other hand, if $C \gg 1$, by a suitable scaling, we obtain that solution θ_C is governed by the ODE operator $-\xi^2\theta'' - 2(1 + \alpha)\theta'\xi + |\theta|^{p-1}\theta$ so that θ_C blows up at a finite $\xi \to \xi_C + 0$, where $\xi_C \to \infty$ as $C \to \infty$ with the singular behaviour given by the monotone operator $-\xi_C^2\theta'' + |\theta|^{p-1}\theta$. Therefore, by a standard continuity argument, there exists a C_* (say, positive) such that the solution $\theta^* \equiv \theta_{C_*}$ is monotone and does not blow-up. Hence, it is a bounded continuous solution defined for any $\xi \geq 1$. Such a construction is quite similar to that for quasilinear heat equations, see p. 193 in [44] and necessary references to Chapt. 4. Fixing $\theta_0 = \theta^*(1)$, by Proposition 6.2 we can extend it inside the light cone B_1 and arrive at a global weak solution θ^* in $\mathbf{R}_+$. Recall that the corresponding self-similar solution $u_*(x, t)$ satisfies $u_*(x, +0) = 0$, $x \neq 0$.

We note that, due to the properties of the spectrum (6.8), for p sufficiently close to $1+0$ (i.e., $\alpha \gg 1$), the linearized solution $Y(\xi)$ can exhibit more than one zero and in fact an arbitrarily large number of zeros. By a similar continuity argument, we conclude that outside the light cone there exists a *finite spectrum* of nonmonotone self-similar patterns. Unfortunately, a weak continuation of such solutions inside the light cone is not always possible. Anyway, the continuation result in Proposition 6.2 is valid for $p \in (p_1, p_S)$ and both critical exponent $p_1, p_S \to 1 + 0$ as $N \to \infty$. Therefore, we can guarantee existence of an arbitrarily large finite number $M = M(N)$ of such non-monotone bounded patterns in $\mathbf{R}^{N+1}_+$ provided that $N \gg 1$.

More self-similar profiles in $\mathbf{R}_+$ can be constructed if we use the shooting argument for the two-parametric family of solutions of (5.8) with the asymptotic expansion

$$\theta_{C_\pm}(\xi) = C_-Y_-(\xi) + C_+Y_+(\xi) + ..., \quad \xi \to \infty; \quad C_\pm \in \mathbf{R}.$$

We do not study such (highly unstable) self-similar patterns and will concentrate on the analysis of another countable spectrum of blow-up patterns.

7. Linearized countable spectra of blow-up patterns

We now consider blow-up solutions of the semilinear hyperbolic equation (5.1). In the hyperbolic equations any blow-up singularity propagates with finite speed and forms a smooth blow-up surface in $\mathbf{R}^{N+1}$. Therefore, blow-up solutions admit a natural extension beyond blow-up singularity, and blow-up is always *incomplete*, unlike the semilinear and quasilinear parabolic equations, where the problem of continuation beyond blow-up is mathematically consistent. See [16] and references therein.

7.1. **Self-similar occurrence of blow-up.** The phenomena of the *propagation of blow-up singularities* and properties of blow-up surfaces for equations like (5.1) are well-known, see [8], [9], [25] and Chapt. 3 in [1]. Locally, the blow-up propagation is self-similar, see a comment below. As we have pointed out, the phenomenon of the *occurrence of blow-up singularities* from uniformly bounded solutions for semilinear wave equations is also an important question of the general theory of nonlinear evolutions equations and a detailed description remained an open question.

Let us impose a necessary restriction on the geometry of the occurrence of blow-up. We always assume that the solution blows up first time at $t = -0$ at the origin $x = 0$, so that it is uniformly bounded on any subset $\mathbf{R}^N \times [-1, -\varepsilon]$, $\varepsilon > 0$. Our main goal is to describe possible spectra of blow-up patterns near the point $x = 0$, $t = -0$. For simplicity, we assume that the blow-up behaviour is radially symmetric, a typical blow-up assumption which makes it possible to study the blow-up singularity by using eigenfunctions and matching procedures based on an ODE analysis instead of the more general and technically more involved elliptic one for non-symmetric blow-up. Nevertheless, we expect that, similar to the parabolic problems, the blow-up behaviour can be considered as a symmetric one in the first approximation.

Using the invariant reflection transformation $t \mapsto -t$, we have that the blow-up self-similar solutions

$$u_*(x,t) = (-t)^{-\alpha}\theta(\eta), \quad \eta = x/(-t), \tag{7.1}$$

are governed by the same continuous profiles θ satisfying the elliptic equation (5.4) or its radial restriction (5.8). Therefore, the existence results inside the (*artificial*) light cone $\{|\eta| < 1\}$ on the global self-similar solutions in Theorems 5.1, 5.2 and in Section 6 coincide with those for the blow-up solutions.

We will establish that the first blow-up pattern

$$u_1(x,t) = (-t)^{-2/(p-1)} c_* \tag{7.2}$$

inside $\xi \le 1$ and outside, $\xi > 1$, of the light cone, to the *first approximation* describes a generic occurrence of blow-up singularity from bounded symmetric initial data. Below, we show how to derive a *refined* countable spectrum of different positive patterns for this semilinear wave equation by using a linearization and a nonlinear matching procedure.

7.2. **A comment on self-similar blow-up propagation.** One can see that the symmetric ODE (5.8) admits an exact singular solution in the open supplement $\{|\eta| > 1\}$ of the unit ball:

$$\theta_S(\eta) = C_S(|\eta|^2 - 1)_+^{-1/(p-1)} > 0, \quad C_S^{p-1} = 2[N - 1 - 2/(p-1)]/(p-1), \tag{7.3}$$

where $\theta_S \to \infty$ as $|\eta| \to +1$. It follows that such a singular solution exists in the supercritical range $p > p_1 = (N+3)/(N-1)$. The corresponding self-similar solution

$$u_S(x,t) = t^{-2/(p-1)} C_S \left[(|x|/t)^2 - 1\right]_+^{-1/(p-1)}$$

is known to describe a generic propagation of blow-up singularities on the moving sphere $\{|x| = t\}$, see the general results in [8] and [9].

7.3. **Countable spectrum of linearized patterns in the inner region.** We recall that a countable subset of similarity blow-up patterns (7.1), constructed in Theorem 5.2 for the monotone nonlinear term, is composed from the profiles $\theta_k(\xi)$ changing sign for any $k \geq 2$, so that these solutions tend to $\pm\infty$ as $t \to -0$. Such single point non-uniform blow-up is expected to be highly unstable. In order to construct a spectrum of new positive patterns, we first consider the linearized problem describing possible types of approaching as $t \to -0$ of the evolution orbit $u(x,t)$ to the first pattern (7.2). We then introduce the corresponding rescaled variables by setting

$$u(x,t) = (-t)^{-\alpha} g(\xi, \tau), \quad \tau = -\ln(-t) \to \infty \text{ as } t \to -0, \tag{7.4}$$

where the rescaled solution g solves the semilinear hyperbolic equation

$$\mathbf{P}g \equiv g_{\tau\tau} + (1+2\alpha) g_\tau + 2 g_{\tau\xi}\xi = \mathbf{A_r} g + f(g), \tag{7.5}$$

where, as usual, $f(g) = |g|^p$ or $f(g) = |g|^{p-1}g$. We assume that $g(\cdot,\tau) \to \theta_1(\cdot) = c_*$ as $\tau \to \infty$ uniformly on compact subsets. We next perform a standard linearization in equation (7.5) by setting

$$g(\xi,\tau) = c_* + Y(\xi,\tau), \tag{7.6}$$

where Y solves the perturbed equation

$$\mathbf{P}Y = \mathbf{C_r} Y + \mathbf{D}(Y), \quad \mathbf{C_r} = \mathbf{A_r} + p\alpha(1+\alpha) I, \tag{7.7}$$

where the perturbation term $\mathbf{D}$ is quadratic: $\mathbf{D}(Y) = O(Y^2)$ as $Y \to 0$. Let us remind that by Proposition 5.3, the linear operator on the right-hand side of (7.7) admits a self-similar Friedrichs extension with the discrete spectrum

$$\sigma(\mathbf{C_r}) = \{\tilde{\lambda}_k = \lambda_k - p\alpha(1+\alpha), \ k = 0,1,2,...\}. \tag{7.8}$$

In the domain in H^2_ρ operator $\mathbf{C_r}$ admits a one-dimensional unstable subspace $E^u(0)$, no centre subspace and a stable one $E^s(0) = (E^u(0))^\perp$ of codimension 1. Indeed,

$$\tilde{\lambda}_0 = -(p-1)\alpha(1+\alpha) = -2(p+1)/(p-1) < 0,$$

$$\tilde{\lambda}_1 = 2(3+2\alpha) - (p-1)\alpha(1+\alpha) = 4p/(p-1) > 0.$$

$\mathbf{C_r}$ is a self-adjoint (sectorial) operator and $\mathbf{C_r}^{-1}$ is a compact integral operator in L^2_ρ. We apply a formal stable manifold approach which is well established for a wide class of abstract nonlinear evolution equations of parabolic type, see [32] and references therein. By the above asymptotic analysis in $\{\xi < 1\}$ and in $\{\xi > 1\}$, the set of eigenfunctions of $\mathbf{C_r}$ and of similar operators is complete in L^2_ρ, taking into account that the nonlinear perturbation $\mathbf{D}$ is actually quadratic in L^2_ρ on the exponential orbits, we may conclude that all possible bounded orbits behave in accordance with the structure of the stable manifold tangent to E^s, so that in the inner region there exists a countable subset of essentially different patterns with the behaviour as $\tau \to \infty$ driven by the eigenfunctions:

$$Y_k(\xi, \tau) = Ce^{-\mu_k \tau} \phi_k(\xi) + o(e^{-\mu_k \tau}) \quad (C = \text{const} \neq 0), \tag{7.9}$$

uniformly on any compact subset in ξ. The completeness of the spectrum of such asymptotic expansion assumes that solutions cannot decay to zero faster than exponentially. We note that such a problem was considered in [18] in connection with Morse-Smale systems generated by parabolic equations. It is important that the approach in [18], Sect. 3, contains the operator and semigroup statements applied to more general equations and uses general Agmon's estimates. Though (7.6) is a second-order evolution equations, we expect that such an approach can be applied to the present equation which is a quadratic perturbation of a smooth flow associated with an (analytic) semigroup generated by the self-adjoint operator $\mathbf{C_r}$.

Substituting (7.9) into equation (7.7) and keeping the main exponential terms, we arrive at the following eigenvalue problem:

$$\mathbf{C_r} \phi_k = \mu_k^2 \phi_k - \mu_k (1 + 2\alpha) \phi_k - 2\mu_k \phi_k' \xi, \tag{7.10}$$

in the space of bounded sufficiently smooth symmetric functions. Here we see both linear, μ_k and quadratic, μ_k^2, dependence on the eigenvalue μ_k, which influences the functional setting including boundary conditions (the weight function in the symmetric representation depends on μ_k as well). In general, this leads to a quadratic operator pencil, see [33]. For the ordinary differential operators such eigenvalue problem are well-known, see e.g. Sect. 2 in [35]. Let us state the result.

Proposition 7.1. *For $p > 1$, the eigenvalue problem (7.10) in $\mathbf{R}_+$ admits two spectra $\{\phi_k^\pm\}_{k \geq 0}$ of analytic eigenfunctions ($2k$-th order polynomials) corresponding to the eigenvalues*

$$\mu_k^- = 2k - 1, \quad \mu_k^+ = 2k + 2(p+1)/(p-1), \quad k = 0, 1, 2, \ldots . \tag{7.11}$$

Proof. As in the proof of Proposition 5.3, using Kummer's series in the construction of analytic solutions, we arrive at the recursion (5.21) with the coefficient

$$A_k = \frac{2k[2k + 1 + 2\alpha - 2\mu] + (p-1)\alpha(1+\alpha) - \mu^2 + \mu(1 + 2\alpha)}{2(k+1)(2k+N)}. \tag{7.12}$$

The truncation condition of the series implies the following quadratic equation on the eigenvalues $\mu = \mu_k$:

$$\mu^2 - \mu(1 + 2\alpha + 4k) - (p-1)\alpha(1+\alpha) + 2k(2k + 1 + 2\alpha) = 0, \tag{7.13}$$

whence the result. The $2k$-th order polynomials $\phi_k^\pm(\xi)$ are well-defined for all $\xi \geq 0$. $\square$

The positive eigenvalues

$$\mu_k^+ > 0, \;\; k = 0, 1, 2, \ldots; \quad \mu_k^- > 0, \;\; k = 1, 2, \ldots, \tag{7.14}$$

correspond to the *stable manifold* of the origin for the nonlinear operator on the right-hand side of (7.7). There exists a single negative eigenvalue $\mu_0^- = -1$ with $\phi_0^- = \text{const}$

describing a certain linear instability. Obviously, this one-dimensional *unstable manifold* is due to the time translational invariance of the semilinear equation (corresponds to shifting of the blow-up time). Thus, the first blow-up pattern $\theta_1 = c_*$ is *stable* in the linear approximation.

7.4. **Unstability of the uniform global pattern.** It follows from the above linearized analysis that the uniform profile $\theta_1 \equiv c_*$ studied in the previous section is entirely *unstable* in the class of global solutions defined for all $t > 0$. Indeed, introducing the rescaled variables (cf. (7.4))

$$u(x,t) = t^{-\alpha} g(\xi, \tau), \quad \xi = x/t, \quad \tau = \ln t, \tag{7.15}$$

we arrive at the equation (cf. (7.5))

$$\mathbf{P}g \equiv g_{\tau\tau} - (1+2\alpha) g_\tau - 2 g_{\tau\xi}\xi = \mathbf{A}_\tau g + f(g), \tag{7.16}$$

where linear operators containing the first-order derivative $d/d\tau$ change sign. Via linearization (7.6) this leads to the eigenvalue problem (cf. (7.10))

$$\mathbf{C_r}\phi_k = \mu_k^2 \phi_k + \mu_k(1+2\alpha)\phi_k + 2\mu_k \phi_k' \xi, \tag{7.17}$$

and to the following quadratic equation for the eigenvalues (cf. (7.13)):

$$\mu^2 + \mu(1+2\alpha+4k) - (p-1)\alpha(1+\alpha) + 2k(2k+1+2\alpha) = 0, \tag{7.18}$$

with the opposite sign in the second term. This implies that all $\mu_k^\pm < 0$ (unstability) except the single one $\mu_0^- > 0$ describing the trivial stability of the self-similar behaviour as $t \to \infty$ under the time-translation $t \mapsto t + T$.

7.5. **Transition to the intermediate ODE region: a perturbed dynamical system.** In order to extend expansion (7.9), which is C^∞-smooth and analytic at the light cone $\{\xi = 1\}$, into the *nonlinear intermediate region* to be specified later on, we use the fact that each eigenfunction $\phi_k(\xi)$ is a polynomial of the order $2k$ so that for $\xi \gg 1$, the first term in (7.9) written in the form

(7.19)
$$Y_k(\xi,\tau) = C e^{-\mu_k \tau} \phi_k(\xi) + ... = C c_k e^{-\mu_k \tau} \xi^{2k} + ... = C c_k (e^{-\beta_k \tau}\xi)^{2k} + ... \equiv C c_k \zeta^{2k} + ...,$$

where $\beta_k = \mu_k/2k$ and c_k denote the normalization constants, suggests rescaling $\zeta = e^{-\beta_k \tau}\xi$. Then equation (7.5) for the rescaled function $h(\zeta,\tau) = g(\zeta e^{\beta_k \tau}, \tau)$ reduces to the following perturbed one:

$$\mathbf{P}h - 2\beta_k h_{\tau\zeta}\zeta = \mathbf{D_k} h + f(h) + e^{-2\beta_k \tau}\Delta_\zeta h, \tag{7.20}$$

with the linear operator

$$\mathbf{D_k} h = -(\beta_k - 1)^2 h_{\zeta\zeta}\zeta^2 + (\beta_k - 1)[2(1+\alpha) - \beta_k] h_\zeta \zeta - \alpha(1+\alpha) h. \tag{7.21}$$

It is important that in these rescaled variables, on sufficiently smooth evolution orbits, the Laplace operator Δu in the hyperbolic equation reduces to an exponentially small perturbation that is negligible as $\tau \to \infty$ ($t \to -0$). In fact, in the first approximation, in the intermediate region the smooth global flow is governed by the second-order ODE

$$u_{tt} = |u|^{p-1}u. \tag{7.22}$$

As above, we first study the corresponding elliptic (stationary) equation obtained by the passage to the limit $\tau \to \infty$ in (7.20) under the natural regularity assumptions on smooth compact orbits:

$$\mathbf{D_k} H + f(H) = 0. \tag{7.23}$$

The function H is the rescaled self-similar solution of the ODE (7.22). In the radial case, it is Euler's equation with a nonlinear term. Setting $y = \ln \zeta$, we arrive at the autonomous second-order ODE

$$(\beta_k - 1)^2 H'' - (\beta_k - 1)(2\alpha + 1)H' + \alpha(1 + \alpha)H - f(H) = 0, \quad y \in \mathbf{R}. \tag{7.24}$$

It reduces to a first-order ODE and a standard phase-plane analysis applies. It admits a one-parameter family of profiles

$$H_D(\zeta) = c_*(1 + D\zeta^{\nu_k})^{-2/(p-1)}, \quad \nu_k = 1/(1 - \beta_k), \tag{7.25}$$

where $D \in \mathbf{R}_+$ is a parameter. We now choose the appropriate pairs $\{\mu_k^-, \psi_k^-\}$ corresponding to the occurrence of blow-up from bounded classical solutions. Then $\beta_k = \mu_k^-/2k = 1 - 1/2k < 1$, and (7.25) gives a monotone decreasing function $H_D(\zeta) \to 0$ as $\zeta \to \infty$. This makes it possible to match such a localized blow-up behaviour with smooth bounded flow beyond the blow-up set. Substituting β_k into (7.25), we get the profiles

$$H_D(\zeta) = c_*(1 + D\zeta^{2k})^{-2/(p-1)}. \tag{7.26}$$

The matching condition of (7.26) as $\zeta \to 0$ and (7.19) takes the form $D = -\frac{p-1}{2}\frac{c_k}{c_*}C$. This uniquely chosen profile $H_D(\zeta)$ gives the desired behaviour of the solution in the intermediate region.

7.6. **On equivalent parabolic intermediate flow.** The explicit self-similar solution

$$u(x,t) = (-t)^{-\alpha} H_D(\zeta), \tag{7.27}$$

of the ODE (7.22) also satisfies the first-order ODE (obtained multiplying by u_t and integrating)

$$u_t = a_* u^{(p+1)/2}, \quad a_* = \sqrt{2/(p+1)}. \tag{7.28}$$

Integrating again gives the solution

$$u(x,t) = (-t)^{-\alpha} c_*[1 + D(x)/(-t)]^{-2/(p-1)},$$

and choosing the constant of integration $D(x) = Dx^{\nu_k}$ gives the similarity profile (7.25) and solution (7.27). The first-order ODE (7.28) implies that the leading structure of the intermediate ODE region for the semilinear hyperbolic equation under consideration is essentially the same as for the parabolic heat equation (cf. (5.11) with the exponent p replaced by $(p+1)/2$)

$$u_t = \Delta u + a_* \, u^{(p+1)/2}, \tag{7.29}$$

where the Laplacian Δu is known to be negligible near the blow-up time.

7.7. **Transition to the outer region, a spectrum of final-time profiles.** We now extend the behaviour in the intermediate ODE region into the outer region where $0 < |x| \ll 1$. We use the asymptotic behaviour of $H_D(\zeta)$ as $\zeta \to \infty$. Such type of extension is well established for a wide class of quasilinear second-order heat equations, see Sect. 8 in [15] and references therein. It follows from (7.26) that

$$H_D(\zeta) = c_* D^{-2/(p-1)} \zeta^{-4k/(p-1)} + \dots, \quad \zeta \to \infty, \quad k = 1, 2, \dots .$$

Since $\zeta = (-t)^{\beta_k} \xi = (-t)^{\beta_k - 1}|x|$ we then obtain, passing to the limit $t \to -0$, that the asymptotic behaviour in the outer region inherits (by extension) the behaviour in the neighbour intermediate one:

$$u(x,t) = (-t)^{-\alpha} H_D(\zeta)(1 + o(1)) = (-t)^{-\alpha} c_* D^{-2/(p-1)}((-t)^{\beta_k - 1}|x|)^{-4k/(p-1)} + \dots ,$$

where the exponent of $(-t)$ vanishes, $-\alpha - 4k(\beta_k - 1)/(p-1) = 0$, so that $u(x, -0) = c_* D^{-2/(p-1)}|x|^{-4k/(p-1)} + \dots$. The countable set

$$\{|x|^{-4k/(p-1)}, \ k = 1, 2, \dots\} \tag{7.30}$$

determines possible types of asymptotics of final-time profiles near a single singular point. Than $k = 1$ corresponds to the stable (generic) blow-up pattern, while for $k = 2, 3, \dots$ such final-time profiles correspond to more flat unstable blow-up, when the orbit blowing-up at $x = 0$, $t = -0$ originates an analytic blow-up surface on the $\{x, t\}$-plane with the local behaviour of the type $t = \psi_k(x) = a_0 + b_0 x^{2k} + \dots$, $a_0, b_0 > 0$, with $x = 0$ being a local minimum. Such a discretization is not a straightforward consequence of the general description of the singularity formation available in [25], p. 1891 or in [1], p. 6.

8. Comparison of spectra of singular blow-up patterns for parabolic and hyperbolic equations

The results in Section 7 show that the main concepts of the *three-region matching* construction (with a specific asymptotic solution structure in each of them) of the countable spectra of blow-up patterns remain similar for semilinear evolution equations of different types including:

(i) the second-order reaction-diffusion equation (5.11), see [2], [5], [10], [19], [34], [46], [47], and references in [44], Chapt. 4,

(ii) the higher-order semilinear parabolic equation $u_t = -(-\Delta)^m u + |u|^p$, $m > 1$ [12],

(iii) the semilinear hyperbolic equation (1.1).

On the other hand, for the essentially quasilinear equations like $u_t = \nabla \cdot (|\nabla u|^{\sigma+1} \nabla u) + u^p$ (or $+e^u$), $\sigma > 0$, or with the porous medium diffusion operator $\Delta u^{\sigma+1}$ replacing the p-Laplacian, the first finite or infinite number (depending on $\sigma > 0$) of patterns can be entirely self-similar and then a *two-region matching* applies. If a finite number of such essentially nonlinear similarity patterns exist, the rest of patterns is constructed by linearization and step by step extension via a similar three-region matching. In this case we observe a transitional behaviour between nonlinear and linear spectra when of the exponent $\sigma > 0$ passes through the critical values $\{\sigma_k,\ k = 1, 2, ...\}$; see [13], [7] and Chapt. 4 in [44].

The above results reveal some common features of the microstructural discretization of singular asymptotics of different infinite-dimensional dynamical systems, a phenomenon which play a fundamental role in the theory of self-organization of dissipative and other structures, see e.g. [26] and references therein. Actually, such countable spectra are one of the most important property of the nonlinear evolution equations, a property occurring near singularity for rather arbitrary initial data. These essentially exhaust internal self-organizing microstructural ("turbulent") properties of nonlinear evolution systems under consideration.

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