Local Solutions to Quasilinear Weakly Hyperbolic Differential Equations

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Introduction

Let us consider the differential operator of order m

$$P(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j,$$

where we adopted the usual notation $D_t = -i\partial_t$, $D_x = -i\nabla_x$. This operator P is called *hyperbolic in the direction* t if the roots $\tau_j = \tau_j(x,t,\xi)$ of the equation

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\xi^\alpha \tau^j = 0$$

are real for all x, t, ξ. The operator P is called *strictly hyperbolic in the direction* t, if the roots τ_j are real and *distinct* for $\xi \in \mathbb{R}^n \setminus \{0\}$. If P is hyperbolic, but not (necessarily) strictly hyperbolic, it is called *weakly hyperbolic*.

The hyperbolicity is a necessary condition for C^∞ well–posedness of the Cauchy problem (see [Lax57], [Miz61]). Well–posedness (with respect to chosen topological spaces for the data, right–hand side and the solution) of a Cauchy problem means, as usual, the existence, uniqueness and continuous dependence (in the topologies of the given spaces) of the solution. However, the hyperbolicity does not guarantee the well–posedness in, e.g., C^∞ or Sobolev spaces. A sufficient condition for the well–posedness in C^∞ and in Sobolev spaces is the strict hyperbolicity, see [Pet38], [Ler54] and [Gar57].

Therefore it is a natural goal to find classes of weakly hyperbolic Cauchy problems which are C^∞ well–posed.

In the weakly hyperbolic case, new phenomena occur which may prevent the well–posedness. These phenomena are the following:

Oscillations in the coefficients with respect to time

- Colombini and Spagnolo [CS82] constructed a function $a(t) \ge 0$ from C^∞, smooth data $u_0(x)$, $u_1(x)$ and a number $T > 0$ with the property that the solution $u = u(x,t)$ of

  $$u_{tt} - a(t)u_{xx} = 0, \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

belongs to $C^\infty([0,T), C^\infty(\mathbb{R}))$, but not to $C([0,T], \mathcal{D}'(\mathbb{R}))$. This coefficient $a(t)$ is positive for $t < T$, oscillating for $t \to T-0$ and identical to zero for $t > T$.

- Let $b(t)$ be a positive, 1-periodic, smooth and non–constant function. In [Tar95] it was proved that the Cauchy problem

$$u_{tt} - \exp(-2t^{-\alpha}) b(t^{-1})^2 u_{xx} = 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

is C^∞ well–posed if and only if $\alpha \geq 1/2$.

The influence of lower order terms

- Gevrey [Gev13] proved that the Cauchy problem for the equation

$$u_{tt} - u_x = 0$$

is well–posed in Gevrey spaces if and only if the Gevrey exponent does not exceed 2. Especially, this Cauchy problem is not well–posed in C^∞.

- In [IP74] it was shown that necessary conditions for the C^∞ well–posedness of

$$v_{tt} - t^{2l} v_{xx} + t^k v_x = 0, \quad l, k \in \mathbb{N}_0, \tag{0.0.1}$$
$$u_{tt} - x^{2n} u_{xx} + x^m u_x = 0, \quad n, m \in \mathbb{N}_0, \tag{0.0.2}$$

are $k \geq l-1$ and $m \geq n$. The sufficiency of these conditions was proved in [Ole70].

- If one wants to study well–posedness in Sobolev spaces, one has to spend attention to another phenomenon, which occurs in the border case $k = l-1$ of the C^∞ well–posedness: *the loss of Sobolev regularity*. Qi Min-You [Qi58] showed by an explicit representation of the solution to the Cauchy problem

$$u_{tt} - t^2 u_{xx} = a u_x, \tag{0.0.3}$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = 0, \quad a = 4n+1, \quad n \in \mathbb{N},$$

that $u(.,t) \in H^{s-n}$ if $\varphi \in H^s$. Let us describe this phenomenon in another way. Let us be given a datum function $\varphi(x)$ with high Sobolev smoothness $s \gg 1$. Then a number a exists with the property that there is no classical solution of (0.0.3). The solution only exists in spaces of distribution.

There are different ways to exclude the phenomenon of oscillations and to restrict the influence of the lower order terms. We take the equation

$$u_{tt} - a(x,t)u_{xx} + b(x,t)u_x + d(x,t)u_t + e(x,t)u = f(x,t)$$

as a model problem. Let us first consider the oscillations.

- One may assume the nonlocal condition

  $$\int_0^T \frac{|a'(\tau)|}{a(\tau)+\varepsilon} d\tau \le C|\ln\varepsilon|, \quad 0 < \varepsilon \le \varepsilon_0$$

  for the coefficient $a = a(t)$, see [CJS83], [DR98a].

- It may be presumed that $a = a(t)$ with a analytic, see [Spa88], [CJS83], [D'A95].

- If the degeneracy occurs for $t = 0$ only, we may assume ([Ole70])

  $$0 \le C(a(x,t) + \partial_t a(x,t)), \quad t \ge 0.$$

- We can suppose that the Levi conditions of C^∞ type $a(x,t) = a_0(x,t)\sigma(x)^2\lambda(t)^2$ hold with smooth $a_0(x,t) \ge \alpha > 0$, $\lambda(0) = 0$, $\lambda'(t) > 0$ $(t > 0)$. The degeneration appears at the points (x_0, t_0) with $\sigma(x_0)\lambda(t_0) = 0$. Assumptions of this type were made, e.g., in [Ner66], [Yag78], [RY93], [Yag96], [Yag97b], [Yag97a] and [DR97], [DR98b]. *We will take this idea and generalise it to quasilinear higher order equations in higher dimensions.*

Let us consider the lower order terms. Conditions which restrict the influence of these terms are called *Levi conditions*. The following Levi conditions have been used widely in the past:

- If the degeneracy occurs for $t = 0$ only, then we may take the condition

  $$Btb(x,t)^2 \le Aa(x,t) + \partial_t a(x,t), \quad t \ge 0 \tag{0.0.4}$$

  from [Ole70]; A and B are some positive constants. This Levi condition is sharp in the case of finite degeneracy: if one takes $a(x,t) = x^{2n}t^{2l}$ and $b(x,t) = x^m t^k$, this condition implies $m \ge n$, $k \ge l-1$. These are exactly the necessary and sufficient conditions from Ivrii, Petkov and Oleinik. However, this condition is not sharp in the case of infinite time degeneracy. It exists an explicit representation of the solutions to

  $$u_{tt} - e^{-\frac{2}{t}}\frac{1}{t^4}u_{xx} + ke^{-\frac{1}{t}}\frac{1}{t^4}u_x = 0, \quad t \ge 0, \quad k = \text{const}, \tag{0.0.5}$$

see [Ale84], which implies that the Cauchy problem for this equation is C^∞ well–posed. Yet, the coefficients from this equation do not satisfy (0.0.4).

- If one wants to include more general degenerations, one may assume the rather general conditions

$$
\begin{aligned}
&b(x,t)^2 \leq Ca(x,t),\\
&a_t(x,t) \leq Ca(x,t) \text{ or } a_t(x,t) \geq -Ca(x,t),
\end{aligned}
$$

or, similarly,

$$Bb(x,t)^2 \leq Aa(x,t) + a_t(x,t), \quad A, B > 0,$$

compare [Man96], [MT96], [D'A94b]. However, this condition is not sharp.

- It can be presumed $a(x,t) = a_0(x,t)\sigma(x)^2\lambda(t)^2$ with $a_0(x,t) \geq \alpha > 0$ and $|b(x,t)| \leq C|\sigma(x)|\lambda'(t)$. This Levi condition is sharp for finitely and infinitely degenerated λ, cf. (0.0.1) and (0.0.5). *We will follow this way and generalise these conditions to the higher order case.*

Let us have a look at the main results of this Ph.D. thesis. We consider the Cauchy problem

$$
\begin{aligned}
&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k u\})\lambda(t)^{|\alpha|} D_x^\alpha D_t^j(\sigma(x)^{|\alpha|}u)\\
&\quad = f(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\}), \quad k+|\beta| \leq m-1, \quad m \geq 2,\\
&u(x,0) = \varphi_0(x), \ldots, D_t^{m-1}u(x,0) = \varphi_{m-1}(x)
\end{aligned}
$$

for $(x,t) \in M \times [0,T]$; M is an n-dimensional torus. The functions $c_{k,\beta} = c_{k,\beta}(x,t)$ are weight functions which describe the Levi conditions.
Let the data belong to Sobolev spaces. We will find a local solution which suffers from a loss of Sobolev regularity as motivated by the Qi Min-You Example.
The methods for proving this result are a unification of ideas taken from [Tay91] who studied quasilinear strictly hyperbolic equations and [KY98] who studied quasilinear weakly hyperbolic equations with time degeneracy.
We will also give a criterion for the possible blow–up. Let $a_{j,\alpha}$ and f be defined on some set $M \times [0,T] \times K$; $K \subset \mathbb{R}^{n_0}$ is bounded. Then a blow–up cannot happen as long as the Zygmund norm $\|.\|_{C_*^1}$ of certain $(m-1)$th order weighted derivatives of u is bounded and the arguments of $a_{j,\alpha}$ and f do not intersect

the set $M \times [0,T] \times \partial K$. *We point out that we take the same comparison space (C^1_*) as in the strictly hyperbolic theory*, see [Tay91].
In order to show the local existence of solutions to quasilinear equations with spatial and time degeneracy, we will linearise the equation and apply a fixed point argument. In this step we need the so–called *strictly hyperbolic type property* for the linearised equations, (see [RY99]). Let L be a weakly hyperbolic linear operator of order 2 (for simplicity) with time degeneracy. We consider

$$Lu = f(x,t), \quad u(x,0) = D_t u(x,0) = 0$$

and say that the *strictly hyperbolic type property* is fulfilled if for some suitably chosen Banach space B the relation $f \in B$ implies

$$\begin{aligned} &\frac{\lambda'}{\lambda}u, \frac{\lambda'}{\lambda}u_t, \sigma\lambda' u_x \in B, \\ &\left\| \frac{\lambda'}{\lambda}u \right\|_B + \left\| \frac{\lambda'}{\lambda}u_t \right\|_B + \|\sigma\lambda' u_x\|_B \leq C \|f\|_B . \end{aligned}$$

This property holds in the strictly hyperbolic theory with $B = C([0,T], H^s)$, $\sigma(x) = 1$ and $\lambda(t) = 1 + t$. If such a strictly hyperbolic type property holds for the linear problem, one can mostly construct a sequence of approximate solutions of the quasilinear problem in the usual way. This sequence will converge to a solution for small times.
In Chapter 2 we present and prove some properties of pseudodifferential operators with limited smoothness. Results about smoothing operators, commutator estimates, adjoint operators and compositions of operators are given. These results base on [Tay91] and serve as a tool for the investigations in the following two chapters.
Quasilinear equations with pure spatial degeneracy on a torus are studied in Chapter 3. We prove energy estimates, an existence result and a blow–up criterion. Additionally, the stability of global solutions and the life–span are studied. A more detailed introduction into these results can be found at the beginning of that chapter.
Chapter 4 is devoted to equations with spatial *and* time degeneracy. We approximate such equations by equations with pure spatial degeneracy, apply the results of the previous chapter and prove the local existence of a solution. For a comprehensive presentation the reader is referred to the beginning of Chapter 4.
In Chapter 5 we describe domains of dependence for quasilinear weakly hyperbolic equations with spatial degeneracy. Geometrically spoken, these domains can be described by the condition that the principal part of the operator be hyperbolic at each point of the boundary of the domain in the normal direction of the boundary. Since the coefficients of the principal part depend on

the solution, the domain of dependence for the solution will be dependent on the solution itself. The properties of existence, uniqueness and C^∞ regularity of solutions are studied. A more comprehensive presentation of these results can be found at the beginning of Chapter 5.

Finally, in Chapter 6 the propagation of singularities for semilinear weakly hyperbolic equations is studied. We take a semilinear equation whose solution suffers from the loss of regularity, cf. the Example of Qi Min–You. We construct function spaces for the data, the right–hand side and the solution in such a way that the *strictly hyperbolic type property* holds and that we regain the known loss of regularity in the cases in which we know an explicit representation of the solution to the linear problem. In other words, this general description of the behaviour of the solutions is sharp in these special cases. Using these newly defined spaces we are able to show that the solution to the semilinear problem and the solution of a suitably linearised problem belong to the same space and that their difference belongs to a space of higher regularity. This result can be interpreted in the way that the singular supports of these two solutions coincide, modulo certain weaker singularities.

A general theory for weakly hyperbolic differential equations of higher order and in higher dimensions is presented in the case that sharp Levi conditions are satisfied. The equations to be studied also cover the case of *strict* hyperbolicity. For these equations the following problems are studied:

Local existence in Sobolev spaces The Cauchy problems to quasilinear weakly hyperbolic operators with spatial and time degeneracy are proved to have a local solution in Sobolev spaces. Due to the sharp Levi conditions, the solution suffers from a loss of Sobolev regularity. The results of this Ph.D. thesis coincide with the known results for the strictly hyperbolic case if the operator is strictly hyperbolic, see e.g. [Dio62], [Tay91].

Blow–up criterion We will prove that a blow–up of the solution in high order Sobolev spaces is only possible if the C^1_* Zygmund norm of certain weighted derivatives (up to the order $m-1$) of the solution blows up. This is exactly the same criterion as in the strictly hyperbolic case, see [Tay91].

Local existence in C^∞ That blow–up criterion leads to the local existence of solutions in C^∞ immediately.

Stability of solutions and life–span of solutions Let us be given a solution of a quasilinear weakly hyperbolic Cauchy problem which persists up to some time $T > 0$. We consider an additional Cauchy problem which has perturbed data, right–hand side and coefficients. Then the solution of this perturbed problem persists up to T and differs from the first solution by an arbitrary small value (in certain norms), if the perturbation is sufficiently small (in certain norms). This stability of solutions leads to an estimate of the life–span immediately. The continuous dependence of the solution from the data and right–hand side also means, that the quasilinear weakly hyperbolic Cauchy problem is well–posed in Sobolev classes. For the strictly hyperbolic case, see e.g. [Pet38], [Ler54], [Dio62] and [Gar57].

Domains of dependence In this Ph.D. thesis domains of dependence for quasilinear weakly hyperbolic operators will be defined and studied. These domains will be used to prove the C^∞ regularity. Our concept of domains of dependence extends the concept of [AM84] from the strictly hyperbolic to the weakly hyperbolic case.

Propagation of singularities The singular support of the solution to a semilinear wave equation and the singular support of the solution to a suitably linearised wave equation coincide, modulo weaker singularities; see, e.g. [Rau79]. A similar result will be proved in the weakly hyperbolic case. However, since the Levi conditions are sharp, the solutions of both Cauchy problems suffer from a loss of Sobolev regularity in comparison with the initial data. This is a severe difficulty because the loss of regularity makes it impossible to prove even the existence of a solution to the semilinear problem by the standard iteration approach. As far as it is known, there are no results in the literature stating that the solution to the semilinear and the linearised problem at least belong to the same space. In this Ph.D. thesis a concept will be introduced which seems to be quite new. Generalising ideas of [RY99], function spaces (adjusted to the hyperbolic operator) will be defined and studied. These spaces generalise the classes of Sobolev spaces. We will show that both the solutions belong to the same space and that the difference of the solutions has a higher regularity. In other words, the singular supports of the solutions coincide, modulo weaker singularities. These spaces turn out to exactly describe the loss of regularity in all examples of operators for which we know an explicit representation of the solution.

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Chapter 1

Notations

Let M be a closed smooth compact n-dimensional manifold or $M = \mathbb{R}^n$.
By $C^k(M) = C_b^k(M)$, $k \in \mathbb{N}_0$, we denote the set of bounded and continuous functions whose derivatives up to the order k are bounded and continuous. The norm of this space is given by

$$\|u\|_{C_b^k(M)} := \sup_{x \in M} \sum_{|\alpha| \le k} |D_x^\alpha u(x)|.$$

Here the standard multi index notation is used:

$$x = (x_1, \ldots, x_n) \in M, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,$$
$$D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}, \quad D_{x_j} = -i \frac{\partial}{\partial x_j}, \quad i^2 = -1.$$

The Hölder spaces $C_b^s(M)$, $s \in \mathbb{R}^+$, are defined in a similar way.
We write $Lip^1(M)$ for the space of Lipschitz continuous functions defined in M.
The symbols $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}$ and $\mathcal{S}'$ denote the usual topological spaces of distributions, distributions with compact support, Schwartz functions and temperate distributions.
By $\langle . \rangle$ we denote a function from $C^\infty(\mathbb{R}^n)$ with $\langle \xi \rangle \ge C_1 > 0$ for all $\xi \in \mathbb{R}^n$ and $\langle \xi \rangle = |\xi|$ for $|\xi| \ge C_2 > 0$. Then positive constants c, C exist with

$$c\langle \xi \rangle \le (1 + |\xi|^2)^{1/2} \le C\langle \xi \rangle \quad \forall \xi \in \mathbb{R}^n.$$

The pseudodifferential operator $\langle D \rangle$ with symbol $\langle \xi \rangle$ can be used to define the Sobolev spaces $H^{s,p}(M)$ by

$$H^{s,p}(M) := \langle D \rangle^{-s} L^p(M).$$

A thorough representation of the theory of pseudodifferential operators can be found in [Hör85]. We write $H^{s,2}(M) =: H^s(M)$ in the case of $p = 2$. If $s = 0$, then we get the usual L^2 space with scalar product $(.,.)$:

$$H^0(M) = L^2(M), \quad \|u\|_{L^2} = \sqrt{(u,u)}.$$

The function $\langle . \rangle$ and its constants C_1, C_2, C, c can be chosen in such a way that

$$\begin{aligned} &\left\| D_x^\beta u \right\|_{L^2} \leq \left\| \langle D \rangle^{|\beta|} u \right\|_{L^2} \quad \forall |\beta| \geq 0, \quad \forall u \in C_0^\infty(M), \\ &\|u\|_{H^s} \leq \|u\|_{H^{s+1}} \quad \forall s \geq 0, \quad \forall u \in H^{s+1}(M). \end{aligned}$$

Let $S_{1,0}^m$ be the set of symbols $p(x,\xi) \in C^\infty(M \times \mathbb{R}^n)$ with

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|} \quad \forall (x,\xi) \in M \times \mathbb{R}^n$$

and let $OPS_{1,0}^m$ denote the set of operators with symbols in $S_{1,0}^m$.
Very often we will employ the so–called Hadamard formula:

$$\begin{aligned} f(v(x)) - f(w(x)) &= \int_0^1 \partial_s f(w(x) + s(v(x) - w(x)))\, ds \\ &= (v(x) - w(x)) \int_0^1 f'(w(x) + s(v(x) - w(x)))\, ds \end{aligned}$$

for all functions $f \in C^1$. There is a canonical generalisation to the case of vector valued functions $v(x)$, $w(x)$.

Chapter 2

Pseudodifferential Operators with Finite Smoothness

2.1 Introduction

In the Chapters 3 and 4 quasilinear hyperbolic equations and their solutions in Sobolev spaces will be studied. Natural tools for the investigation of such equations are energy estimates in Sobolev spaces. These energy estimates will be proved using the theory of pseudodifferential operators. However, since the coefficients of the principal part of the hyperbolic operator depend on the solution and its derivatives itself and because the solution will be from some Sobolev space, the coefficients of this hyperbolic operator will not have C^∞ smoothness. Thus, the theory of pseudodifferential operators with symbols of *infinite* smoothness seems not to be applicable. For this reason we present a theory of pseudodifferential operators with symbols of *finite* smoothness in this chapter. This theory is taken from [Tay91] and includes results about mapping properties, commutator estimates, adjoints and compositions. The proofs in [Tay91] used model operators of special structure. We will give the proofs for general operators especially taking care of the remainders.

2.2 Definition and Mapping Properties

Definition 2.2.1 (Microlocalisable scale). *The set of Banach spaces* $\{X^s : s \in \sum\}$ *is called a* microlocalisable scale *if the following hold:*

- $\sum = [\sigma_0, \infty)$ *or* $\sum = (\sigma_0, \infty)$, $\quad \sigma_0 \in \mathbb{R}$,
- $C_0^\infty(M) \subset X^s \subset C_b^0(M) \quad \forall s \in \sum$,

- $X^s \subset X^t \quad (t < s)$,
- $f \in C^\infty(\mathbb{R}), u \in X^s \Longrightarrow f(u) \in X^s$, *$f$ maps bounded sets into bounded sets,*
- *If* $m \in \mathbb{R}, s, s+m \in \sum$ *and* $P \in OPS^m_{1,0}$, *then* P *maps* X^{s+m} *into* X^s *continuously.*

Example 2.2.2. $X^s = C^s_*(\mathbb{R}^n), \quad \sum = (0,\infty)$. *Here* C^s_* *are the Hölder spaces for* $s \notin \mathbb{N}$ *and coincide with the* Zygmund spaces *for* $s \in \mathbb{N}$. *The Zygmund spaces* C^s_*, $s \in \mathbb{N}^+$, *consist of all functions* u *with the property that*

$$u \in C^{s-1}_b(\mathbb{R}^n), \ \sup_{x \neq y} \sum_{|\alpha|=s-1} \frac{\left|D^\alpha u(x) - 2(D^\alpha u)\left(\frac{x+y}{2}\right) + D^\alpha u(y)\right|}{|x-y|} < \infty.$$

The continuous embedding $C^k_b \subset C^k_*$, $k \in \mathbb{N}^+$, *holds. More properties of Hölder spaces and Zygmund spaces can be found in [Tri78].*

Example 2.2.3. $X^s = H^{s,p}(M), \quad \sum = (n/p, \infty), \quad 1 < p < \infty$. *The Sobolev Embedding Theorem implies* $X^s \subset C^0_b(M)$ *for* $s \in \Sigma$.

Definition 2.2.4 (Space of symbols of finite smoothness). *The space* $X^s S^m_{1,0}$ *consists of all symbols* $p(x,\xi)$ *with*

$$\left\| D^\alpha_\xi p(.,\xi) \right\|_{X^s} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|} \quad \forall \alpha \geq 0.$$

In other words, for all $N \in \mathbb{N}_0$ *it holds*

$$\pi^m_{N,X^s}(p) := \sup\left\{ \left\| D^\alpha_\xi p(.,\xi) \right\|_{X^s} \langle \xi \rangle^{-m+|\alpha|} : \xi \in \mathbb{R}^n, |\alpha| \leq N \right\} < \infty.$$

Definition 2.2.5 (Classical symbols of finite smoothness). *We say that* $p(x,\xi) \in X^s S^m_{cl}$, *if* $p(x,\xi) \in X^s S^m_{1,0}$ *and*

$$p(x,\xi) \sim \sum_{j \geq 0} \chi(\xi) p_j(x,\xi)$$

where $p_j(x,\xi)$ *are positive homogeneous of degree* $m-j$ *in* ξ *and* $p - \sum_{j=0}^{N-1} \chi p_j \in X^s S^{m-N}_{1,0}$. *The function* $\chi \in C^\infty(\mathbb{R}^n_\xi)$ *vanishes in a neighbourhood of 0 and equals 1 for* $|\xi| \geq C > 0$.

Definition 2.2.6 (Operators of finite smoothness). *Let* $M = \mathbb{R}^n$. *The spaces* $OPX^s S^m_{1,0}$, $OPX^s S^m_{cl}$*,respectively, consist of all operators* $p(x,D)$ *whose symbols* $p(x,\xi)$ *belong to* $X^s S^m_{1,0}$, $X^s S^m_{cl}$, *respectively, and satisfy*

$$(p(x,D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi) \hat{u}(\xi) d\xi \quad \forall u \in C^\infty_0(\mathbb{R}^n).$$

If M is a smooth closed compact manifold, then the operator $p(x,D)$ is defined in the following way:
A continuous linear operator $P : C_0^\infty(M) \to \mathcal{D}'(M)$ is called a pseudodifferential operator of the class $OPX^sS^m_{1,0}$ if its Schwartz Kernel is locally X^s–smooth off the diagonal in $M \times M$, and if there is a locally finite open cover of M with sets Ω_j, a subordinate partition of unity φ_j and diffeomorphisms $F_j : \Omega_j \to V_j \subset \mathbb{R}^n$ that transform the operators $\varphi_k P \varphi_j : C^\infty(\Omega_j) \to \mathcal{E}'(\Omega_k)$ into pseudodifferential operators in $OPX^sS^m_{1,0}$, see [Tay91], Section 0.12.

The following mapping property is cited from [Tay91] (Prop. 2.1.D.).

Proposition 2.2.7. *Let $p(x,D) \in OPC^s_bS^m_{1,0}$. Then $p(x,D)$ maps C^{r+m}_*, $H^{r+m,p}$ continuously into C^r_*, $H^{r,p}$, respectively, if $1 < p < \infty$ and*

$$-s < r < s.$$

We point out that this proposition cannot be applied in the cases $r = -s$ or $r = s$. The following proposition partially closes this gap. The idea behind this proposition is to impose stronger conditions on the order of the operator. This allows to reach the highest smoothness (of the image space) that is gainable keeping in mind the finite smoothness of the symbol. We will use the proposition to estimate the remainders of the expansions of classical pseudodifferential operators with symbols of finite smoothness.

Proposition 2.2.8. *Let $p(x,D) \in OPC^s_bS^{-N}_{1,0}$, $s \in \mathbb{N}_0$, $N > s_0 \geq s$. Then $p(x,D)$ maps continuously*

$$L^2 \to H^s, \quad H^{-s_0} \to L^2.$$

Proof. The second statement is contained in Proposition 2.2.7, if $s > 0$. But sometimes we will need it in the case $s = 0$. The Proposition 2.2.7 gives no information in this case.
We choose a partition of unity $\{\psi_j(\xi)\}_{j=0}^\infty$ in the frequency space with the following properties:

$$\operatorname{supp}\psi_0 \subset \{|\xi| < 2\}, \quad \operatorname{supp}\psi_j \subset \{2^{j-1} < |\xi| < 2^{j+1}\}, \quad j > 0,$$
$$\psi_j(\xi) = \psi_1(2^{1-j}\xi), \quad \sum_{j=0}^\infty \psi_j(\xi) = 1,$$

see [Bon81]. Then we define symbols $p_j(x,\xi)$ by the formula $p_j(x,\xi) = p(x,\xi)\psi_j(\xi)\langle\xi\rangle^N$ and see that

$$p(x,\xi) = \sum_{j=0}^\infty p(x,\xi)\psi_j(\xi) = \sum_{j=0}^\infty p_j(x,\xi)\langle\xi\rangle^{-N}.$$

The functions $p_j(x,\xi)$ have compact support with respect to ξ, which is located near $|\xi| = 2^j$ for $j > 0$. If $\xi \in \{2^{j-1} < |\xi| < 2^{j+1}\}$, then $\eta := 2^{-j}\xi \in \{1/2 < |\eta| < 2\}$. Let $\{\beta_l(\eta)\}_{l=0}^{\infty}$ be a basis of $L^2(\{1/2 < |\eta| < 2\})$, see Proposition B.0.1. We can write

$$\begin{aligned} p_j(x,\xi) &= p_j(x,2^j\eta) = \tilde{p}_j(x,\eta) = \sum_{l=0}^{\infty} \tilde{p}_{jl}(x)\beta_l(\eta) = \sum_{l=0}^{\infty} \tilde{p}_{jl}(x)\beta_l(2^{-j}\xi), \\ \tilde{p}_{jl}(x) &= \int_{1/2<|\eta|<2} \tilde{p}_j(x,\eta)\beta_l(\eta)\,d\eta = \int_{1/2<|\eta|<2} p_j(x,2^j\eta)\beta_l(\eta)\,d\eta. \end{aligned}$$

Due to Proposition B.0.1 we have

$$\begin{aligned} &\|\tilde{p}_{jl}\|_{C_b^s} \le C_k \langle l\rangle^{1-\frac{2}{n}k} \cdot 2^{2jk} \\ &\quad \times \sup\{|D_\xi^\alpha D_x^\beta p_j(x,\xi)| : x \in M, |\alpha| = 2k, |\beta| \le s, \xi \in \operatorname{supp}\psi_j\}. \end{aligned}$$

We estimate the supremum:

$$\begin{aligned} |D_\xi^\alpha D_x^\beta p_j(x,\xi)| &\le C \sum_{\gamma\le\alpha} |D_\xi^\gamma D_x^\beta (p(x,\xi)\langle\xi\rangle^N)||D_\xi^{\alpha-\gamma}\psi_j(\xi)| \\ &\le C \sum_{\gamma\le\alpha} \pi^0_{|\gamma|,C^{|\beta|}}(p(x,\xi)\langle\xi\rangle^N)\langle\xi\rangle^{-\gamma}|D_\xi^{\alpha-\gamma}\psi_1(2^{1-j}\xi)| \\ &\le C\pi^0_{2k,C_b^s}(p(x,\xi)\langle\xi\rangle^N) \sum_{\gamma\le\alpha} 2^{-j|\gamma|}2^{(1-j)(|\alpha|-|\gamma|)} \\ &\le C\pi^{-N}_{2k,C_b^s}(p)2^{-2jk}. \end{aligned}$$

This gives

$$\|\tilde{p}_{jl}\|_{C_b^s} \le C_k\langle l\rangle^{1-\frac{2}{n}k}\pi^{-N}_{2k,C_b^s}(p) \quad \forall k \ge 0.$$

Now we are in a position to estimate appropriate norms of each summand of

$$p(x,D)u = \sum_{j,l} \tilde{p}_{jl}(x)\beta_l(2^{-j}D)\langle D\rangle^{-N}u.$$

For $M = \mathbb{R}^n$ and $j > 0$ it holds

$$\begin{aligned} &\left\|\tilde{p}_{jl}(x)\beta_l(2^{-j}D)\langle D\rangle^{-N}u\right\|^2_{H^s} \le C\,\|\tilde{p}_{jl}\|^2_{C_b^s}\left\|\beta_l(2^{-j}D)\langle D\rangle^{-N}u\right\|^2_{H^s} \\ &\quad \le C\,\|\tilde{p}_{jl}\|^2_{C_b^s}\int_{2^{j-1}<|\xi|<2^{j+1}} \langle\xi\rangle^{2(s-N)}|\beta_l(2^{-j}\xi)|^2|\hat{u}(\xi)|^2\,d\xi \\ &\quad \le C_k\langle l\rangle^{4-\frac{4}{n}k}\left(\pi^{-N}_{2k,C_b^s}(p)\right)^2 2^{2j(s-N)}\int_{2^{j-1}<|\xi|<2^{j+1}} |\hat{u}(\xi)|^2\,d\xi. \end{aligned}$$

A similar estimate is true in the case of $j = 0$. We conclude that

$$\begin{aligned}
\|p(x,D)u\|_{H^s} &\le \sum_{j,l} \left\|\tilde{p}_{jl}(x)\beta_l(2^{-j}D)\langle D\rangle^{-N}u\right\|_{H^s} \\
&\le C_k \sum_{j,l} \langle l\rangle^{2-\frac{2}{n}k} \pi^{-N}_{2k,C^s_b}(p) 2^{j(s-N)} \left(\int_{2^{j-1}<|\xi|<2^{j+1}} |\hat{u}(\xi)|^2\, d\xi\right)^{1/2} \\
&\le C_k \pi^{-N}_{2k,C^s_b}(p) \sum_{j,l} \langle l\rangle^{2-\frac{2}{n}k} 2^{j(s-N)} \|u\|_{L^2} \\
&\le C_k \pi^{-N}_{2k,C^s_b}(p) \|u\|_{L^2},
\end{aligned}$$

if $k > \frac{3n}{2}$ and $s < N$. On the other hand,

$$\begin{aligned}
\left\|\tilde{p}_{jl}(x)\beta_l(2^{-j}D)\langle D\rangle^{-N}u\right\|^2_{L^2} &\le C \|\tilde{p}_{jl}\|^2_{C^0_b} \left\|\beta_l(2^{-j}D)\langle D\rangle^{-N}u\right\|^2_{L^2} \\
&\le C \|\tilde{p}_{jl}\|^2_{C^s_b} \int_{2^{j-1}<|\xi|<2^{j+1}} \langle\xi\rangle^{-2N} |\beta_l(2^{-j}\xi)|^2 |\hat{u}(\xi)|^2\, d\xi \\
&\le C_k \langle l\rangle^{4-\frac{4}{n}k} \left(\pi^{-N}_{2k,C^s_b}(p)\right)^2 2^{2j(s_0-N)} \int_{2^{j-1}<|\xi|<2^{j+1}} \langle\xi\rangle^{-2s_0} |\hat{u}(\xi)|^2\, d\xi \\
&\le C_k \langle l\rangle^{4-\frac{4}{n}k} \left(\pi^{-N}_{2k,C^s_b}(p)\right)^2 2^{2j(s_0-N)} \|u\|^2_{H^{-s_0}}.
\end{aligned}$$

The same argument as above gives $\|p(x,D)u\|_{L^2} \le C\pi^{-N}_{2k,C^s_b}(p) \|u\|_{H^{-s_0}}$. The proof for the case $M \ne \mathbb{R}^n$ runs similarly. □

Remark 2.2.9. *If the Littlewood–Paley–characterisation of the $H^{s,p}$ norm is applied, then the result of this proposition can be extended to $L^p \to H^{s,p}$, $H^{-s_0,p} \to L^p$ for $1 < p < \infty$. We will not follow this line, since our aims are energy estimates. The natural spaces for such estimates are the spaces L^2 and $H^{s,2} = H^s$.*

Remark 2.2.10. *One can also characterise the Hölder spaces and Zygmund spaces by the aid of the operators $\psi_k(D)$ introduced in the proof of the previous proposition (cf. [Tay91], Appendix A): These spaces consist of all functions u with*

$$\sup_k 2^{ks} \|\psi_k(D)u\|_{L^\infty} < \infty.$$

In the case of operators with coefficients from Sobolev spaces we have no problems with the borderline case $r = s$, cf. Proposition 2.2.7. For the following proposition see e.g. [Tay91].

Proposition 2.2.11. *If $p(x,\xi) \in H^s S^m_{1,0}$, then $p(x,D)$ maps H^{r+m} continuously into H^r for*

$$-s < r \le s.$$

2.3 Special Smoothing Operators

Definition 2.3.1 (Smoothing operator). *Let $\Phi \in C_0^\infty(\mathbb{R}^n)$ be a real-valued function with $\Phi(\xi) = 1$ for $|\xi| \le 1$. The operator J_ε $(0 < \varepsilon \le 1)$ is defined by*

$$J_\varepsilon f(x) = \Phi(\varepsilon D) f(x).$$

Lemma 2.3.2. *The smoothing operator J_ε commutes with $\langle D\rangle^s$, $s \in \mathbb{R}$,*

$$[J_\varepsilon, \langle D\rangle^s] = 0.$$

Proof. The symbols do not depend on x. □

Lemma 2.3.3. *The smoothing operators J_ε are self-adjoint.*

Lemma 2.3.4. *If $\{X^s : s \in \sum\}$ is a microlocalisable scale, then*

$$\begin{aligned}
&\left\|D_x^\beta J_\varepsilon f\right\|_{X^s} \le C_\beta \varepsilon^{-|\beta|} \left\|f\right\|_{X^s},\\
&\left\|f - J_\varepsilon f\right\|_{X^{s-t}} \le C\varepsilon^t \left\|f\right\|_{X^s}, \quad s, s-t \in \sum, \quad t \ge 0.
\end{aligned}$$

Proof. See [Tay91], Lemma 1.3 A. □

Corollary 2.3.5. *Let $X^s = H^{s,p}(M)$. Then the assertions of the previous lemma hold for all $H^{s,p}$, $H^{s-t,p}$, $s \in \mathbb{R}$, $t \ge 0$.*

Proof. For all $s \in \mathbb{R}$ we have

$$\begin{aligned}
\left\|D_x^\beta J_\varepsilon f\right\|_{H^{s,p}} &= \left\|\langle D\rangle^s D_x^\beta J_\varepsilon f\right\|_{L^p} = \left\|\langle D\rangle^{\frac{n}{p}+1} \langle D\rangle^{s-\frac{n}{p}-1} D_x^\beta J_\varepsilon f\right\|_{L^p}\\
&= \left\|D_x^\beta J_\varepsilon \langle D\rangle^{s-\frac{n}{p}-1} f\right\|_{H^{\frac{n}{p}+1,p}} \le C_\beta \varepsilon^{-|\beta|} \left\|\langle D\rangle^{s-\frac{n}{p}-1} f\right\|_{H^{\frac{n}{p}+1,p}}\\
&= C_\beta \varepsilon^{-|\beta|} \left\|\langle D\rangle^{\frac{n}{p}+1} \langle D\rangle^{s-\frac{n}{p}-1} f\right\|_{L^p} = C_\beta \varepsilon^{-|\beta|} \left\|f\right\|_{H^{s,p}}.
\end{aligned}$$

If $s \in \mathbb{R}$ and $t \ge 0$, then

$$\begin{aligned}
\left\|f - J_\varepsilon f\right\|_{H^{s-t,p}} &= \left\|\langle D\rangle^{s-t}(f - J_\varepsilon f)\right\|_{L^p}\\
&= \left\|\langle D\rangle^{\frac{n}{p}+1} \langle D\rangle^{s-t-\frac{n}{p}-1}(f - J_\varepsilon f)\right\|_{L^p}\\
&= \left\|(I - J_\varepsilon)\langle D\rangle^{s-t-\frac{n}{p}-1} f\right\|_{H^{\frac{n}{p}+1,p}} \le C\varepsilon^t \left\|\langle D\rangle^{s-t-\frac{n}{p}-1} f\right\|_{H^{\frac{n}{p}+1+t,p}}\\
&= C\varepsilon^t \left\|\langle D\rangle^{\frac{n}{p}+1+t} \langle D\rangle^{s-t-\frac{n}{p}-1} f\right\|_{L^p} = C\varepsilon^t \left\|f\right\|_{H^{s,p}}. \quad \square
\end{aligned}$$

Lemma 2.3.6. *For all $0 < \varepsilon \le 1$ it holds*

$$\left\|J_\varepsilon\right\|_{C_b^0 \to C_b^0} \le C.$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$. Since $\hat{u} \in \mathcal{S}$, we can write

$$(J_\varepsilon(D)u)(x) = (2\pi)^{-n} \int e^{ix\xi} \Phi_\varepsilon(\xi)\hat{u}(\xi)d\xi = \int \check{\Phi}_\varepsilon(x-y)u(y)dy$$
$$= \varepsilon^{-n} \int \check{\Phi}\left(\frac{x-y}{\varepsilon}\right) u(y)dy. \tag{2.3.1}$$

Here we introduced the notation $\Phi_\varepsilon(\xi) := \Phi(\varepsilon\xi)$. Now we assume $u \in C_b^0(\mathbb{R}^n)$. Then $u \in \mathcal{S}'$, since $u(\varphi) := \int u(x)\varphi(x)dx$ defines a linear continuous functional on $\mathcal{S}$. We choose a cut–off function

$$\chi \in C_0^\infty(\mathbb{R}^n), \quad 0 \le \chi(x) \le 1, \quad \chi(x) := \begin{cases} 1 & : |x| \le 1, \\ 0 & : |x| \ge 2 \end{cases}$$

and Friedrich's mollifiers $h_\varepsilon(x)$,

$$h_\varepsilon(x) = \varepsilon^{-n} h\left(\frac{x}{\varepsilon}\right), \quad h \in C_0^\infty(\mathbb{R}^n), \quad 0 \le h \le 1, \quad \int_{\mathbb{R}^n} h(x)\,dx = 1.$$

They enable us to define functions $u_k(x) := h_{1/k}(x) * (u(x)\chi(x/k))$. The Convergence Theorem of Lebesgue gives

$$u_k(\varphi) \to u(\varphi) \quad (k \to \infty) \quad \forall \varphi \in \mathcal{S},$$

hence $u_k \to u$ in the topology of $\mathcal{S}'$. Since $u_k \in C_0^\infty$, (2.3.1) implies

$$a_k(x) := (J_\varepsilon(D)u_k)(x) = \varepsilon^{-n} \int \check{\Phi}\left(\frac{x-y}{\varepsilon}\right) u_k(y)dy.$$

Due to the Convergence Theorem of Lebesgue, the right–hand side converges (pointwise) for $k \to \infty$ to

$$a(x) := \varepsilon^{-n} \int \check{\Phi}\left(\frac{x-y}{\varepsilon}\right) u(y)dy.$$

From $\|u_k\|_{L^\infty} \le \|u\|_{L^\infty}$ we deduce that $\|a_k\|_{L^\infty} \le C$. The functions $a_k(x)$ define distributions of $\mathcal{S}'$ in a natural way: $a_k(\varphi) = \int a_k(x)\varphi(x)dx$. Then the Convergence Theorem of Lebesgue gives

$$a_k(\varphi) \to a(\varphi) \quad (k \to \infty) \quad \forall \varphi \in \mathcal{S}, \quad a(\varphi) := \int a(x)\varphi(x)\,dx.$$

This proves that $(J_\varepsilon(D)u_k)(x)$ converges to $a(x)$ in the topology of $\mathcal{S}'$.
Since pseudodifferential operators are continuous mappings from $\mathcal{S}'$ into $\mathcal{S}'$, we conclude that $(J_\varepsilon(D)u_k(x))$ converges to $(J_\varepsilon(D)u(x))$ in the topology of $\mathcal{S}'$. Then it follows that the formula (2.3.1) holds for all functions u from C_b^0. The function $\check{\Phi}$ belongs to L^1. This results in the estimate

$$|(J_\varepsilon(D)u)(x)| \le \left\|\check{\Phi}\right\|_{L^1} \|u\|_{L^\infty}. \quad \square$$

2.4 Commutator Estimates

Let us start with the well–known commutator estimates of Coifman–Meyer and Kato–Ponce,

$$\|P(fu) - fPu\|_{L^p} \leq C\,\|f\|_{Lip^1}\,\|u\|_{L^p}\,, \tag{2.4.1}$$
$$P \in OPS^1_{1,0}, \quad 1 < p < \infty, \quad u \in L^p, \quad f \in Lip^1,$$

$$\|P(fu) - fPu\|_{L^p} \leq C\,\|f\|_{Lip^1}\,\|u\|_{H^{s-1,p}} + C\,\|f\|_{H^{s,p}}\,\|u\|_{L^\infty}\,, \tag{2.4.2}$$
$$P \in OPS^s_{1,0}, \quad s > 0, \quad 1 < p < \infty,$$
$$u \in L^\infty \cap H^{s-1,p}, \quad f \in Lip^1 \cap H^{s,p},$$

see [CM78], [KP88] and [Tay91], Subsection 3.6. In [Tay91] a useful generalisation of (2.4.2) was proved, see Proposition 3.6.A:

$$\|P(fu) - fPu\|_{H^{\sigma,p}}$$
$$\leq C\,\|f\|_{Lip^1}\,\|u\|_{H^{s-1+\sigma,p}} + C\,\|f\|_{H^{s+\sigma,p}}\,\|u\|_{L^\infty}\,, \tag{2.4.3}$$
$$P \in OPS^s_{1,0}, \quad \sigma \geq 0, \quad s > 0, \quad 1 < p < \infty,$$
$$u \in L^\infty \cap H^{s-1+\sigma,p}, \quad f \in Lip^1 \cap H^{s+\sigma,p}.$$

We generalise these results to operators with finite smoothness and restrict ourselves to the case $p = 2$. By small modifications of the proofs the case $p \neq 2$ can be handled. We are not interested in this field since our goals are energy estimates.

This section is organised as follows. The central results are Proposition 2.4.4 and Proposition 2.4.5, which will be used extensively in the Sections 3.2 and 3.3. In Proposition 2.4.4 classical operators with non–smooth symbols of order 0 or 1 are studied. In Proposition 2.4.5 arbitrary orders are allowed, but one operator must have a smooth symbol and the estimate of the commutator is more complicated. A suitable function space $C^\alpha_{\sharp,K_0}$ has to be defined. In the Sections 3.2 and 3.3 we will pay much attention to the exponents of the C^1_b norms of certain functions. These exponents should not exceed 1. On the other hand, there are no restrictions on the exponents of C^0_b norms. Therefore it is convenient to have a result about commutators of an operator with C^0_b smooth symbol and a function. Such a result is provided by Proposition 2.4.3. Later commutators of the form $[J_\varepsilon, p(x,D)]$ must be estimated and the estimates should be independent of ε. Therefore we provide Lemma 2.4.2. Finally, the Proposition 2.4.1 is a tool to prove the Propositions 2.4.4 and 2.4.5. Later it will be used only in Section 2.6. At last, we give a lemma about special properties of the newly defined spaces $C^\alpha_{\sharp,K_0}$.

At first we want to make (2.4.1) more precise in the case of $p = 2$ and operators of order 0.

Proposition 2.4.1. *If $P \in OPS^0_{1,0}$, then a constant C exists with*

$$\|[P,f]\|_{L^2\to H^1} \le C\,\|f\|_{Lip^1}, \quad \|[P,f]\|_{H^{-1}\to L^2} \le C\,\|f\|_{Lip^1} \quad \forall f \in Lip^1.$$

Proof. We consider the auxiliary operators

$$\begin{aligned} Q_1 &:= \langle D\rangle\,[P,f] = [\langle D\rangle P, f] - [\langle D\rangle, f]\,P,\\ Q_2 &:= [P,f]\,\langle D\rangle = [P\langle D\rangle, f] - P\,[\langle D\rangle, f]\,. \end{aligned}$$

From (2.4.1) we conclude that

$$\|Q_1\|_{L^2\to L^2} \le C\,\|f\|_{Lip^1}, \quad \|Q_2\|_{L^2\to L^2} \le C\,\|f\|_{Lip^1}\,.$$

This gives

$$\left\|\langle D\rangle^{-1}Q_1\right\|_{L^2\to H^1} \le C\,\|f\|_{Lip^1}, \quad \left\|Q_2\langle D\rangle^{-1}\right\|_{H^{-1}\to L^2} \le C\,\|f\|_{Lip^1}\,. \quad \square$$

Lemma 2.4.2. *Let J_ε be the smoothing operator from Definition 2.3.1. Then the assertions of the previous proposition hold for $P = J_\varepsilon$ with constants C independent of ε, $0 < \varepsilon \le 1$.*

Proof. The operator J_ε has the symbol $\Phi(\varepsilon\xi)$ with $\Phi \in C_0^\infty(\mathbb{R}^n)$ and $\Phi(\xi) = 1$ for $|\xi| \le 1$. The uniform estimate

$$|D_\xi^\alpha \Phi(\varepsilon\xi)| \le C_\alpha \langle\xi\rangle^{-|\alpha|} \quad \forall\varepsilon$$

reveals that the operators $J_\varepsilon(D)$ belong to $OPS^0_{1,0}$ with uniformly bounded norms $\|.\|_{H^t\to H^t}$ for any $t \in \mathbb{R}$, see e.g. [Hör85], vol. 3, Chapter XVIII. $\square$

Proposition 2.4.3. *If $P \in OPC_b^0 S_{cl}^0$, then*

$$\|[P,f]\|_{H^{-1}\to L^2} \le C\,\|f\|_{Lip^1} \quad \forall f \in Lip^1.$$

Proof. The condition $P \in OPC_b^0 S_{cl}^0$ means

$$\begin{aligned} &p(x,\xi) - \chi(\xi)\sum_{j=0}^{N-1} p_j(x,\xi) \in C_b^0 S_{1,0}^{-N},\\ &\chi \in C^\infty(\mathbb{R}^n), \quad \chi(\xi) = \begin{cases} 0 & : |\xi| \le C_1,\\ 1 & : |\xi| \ge C_2 > C_1, \end{cases}\\ &p_j(x,\lambda\xi) = \lambda^{-j} p_j(x,\xi) \quad \forall \xi \ne 0, \quad \forall\lambda > 0,\\ &\left\|D_\xi^\alpha p_j(.,\xi)\right\|_{C_b^0} \le C_{j,\alpha}\langle\xi\rangle^{-j-|\alpha|} \quad \forall|\alpha| \ge 0, \quad \forall|\xi| \ge C_1. \end{aligned}$$

We can write

$$p_j(x,\xi) = |\xi|^{-j} p_j\left(x, \frac{\xi}{|\xi|}\right) = |\xi|^{-j} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} p_{jlm}(x) Y_{lm}(\xi),$$

where Y_{lm} are the spherical harmonics, see the Appendix A. A constant C_2 exists with

$$\chi(\xi) p_j(x,\xi) = \langle\xi\rangle^{-j} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} p_{jlm}(x) Y_{lm}(\xi)$$

for $|\xi| \geq C_2$. This shows, by Theorem A.0.3, that

$$\chi(\xi) p_j(x,\xi) - \langle\xi\rangle^{-j} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} p_{jlm}(x) Y_{lm}(\xi) \in C_b^0 S_{1,0}^{-\infty}, \quad p_{jlm} \in C_b^0.$$

Hence we can write

$$p(x,\xi) = \sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} p_{jlm}(x) Y_{lm}(\xi) \langle\xi\rangle^{-j} + r_N(x,\xi)$$

with $r_N(x,\xi) \in C_b^0 S_{1,0}^{-N}$. The beginning of Appendix A and Theorem A.0.3 give $h(l,n-2) = O(\langle l\rangle^{n-2})$ and the estimate

$$\|p_{jlm}(x)\|_{C_b^0} \leq C(n,k) \langle l\rangle^{-2k} \sup\left\{ \left\| D_\xi^\beta p_j(.,\xi) \right\|_{C_b^0} : |\beta| \leq 2k, |\xi| = 1 \right\}$$

for all $k \geq 0$. This implies that

$$\left(\sum_{m=1}^{h(l,n-2)} \|p_{jlm}(x)\|_{C_b^0} \right)_l$$

is a rapidly decreasing sequence in l. The commutator $[P,f]$ can be written in the form

$$[P,f] = \sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} \left[p_{jlm}(x) Y_{lm}(D) \langle D\rangle^{-j}, f \right] + \left[r_N(x,D), f \right].$$

Proposition 2.4.1 leads to

$$\begin{aligned}
\left\| \left[p_{jlm}(x) Y_{lm}(D) \langle D\rangle^{-j}, f \right] u \right\|_{L^2} &= \left\| p_{jlm}(x) \left[Y_{lm}(D) \langle D\rangle^{-j}, f \right] u \right\|_{L^2} \\
&\leq C \left\| p_{jlm} \right\|_{C_b^0} \left\| \left[Y_{lm}(D) \langle D\rangle^{-j}, f \right] u \right\|_{L^2} \\
&\leq C_M \langle l\rangle^{-M} \left\| u \right\|_{H^{-1}} \quad \forall M > 0.
\end{aligned}$$

From Proposition 2.2.8 with $s = 0$, $s_0 = 1$ and $N > 1$ we conclude that

$$\begin{aligned}\|[r_N(x,D), f]\,u\|_{L^2} &\leq \|r_N(x,D)(fu)\|_{L^2} + \|f r_N(x,D)u\|_{L^2}\\ &\leq C\,\|fu\|_{H^{-1}} + C\,\|f\|_{C_b^0}\,\|r_N(x,D)u\|_{L^2}\\ &\leq C\,\|f\|_{Lip^1}\,\|u\|_{H^{-1}}.\end{aligned}$$

The proposition is proved. □

The following proposition is one of the central results of this section.

Proposition 2.4.4. *Let $a(x,D) \in OPC_b^1 S_{cl}^{\alpha}$, $b(x,D) \in OPC_b^1 S_{cl}^{\beta}$ with $\alpha, \beta \in \{0,1\}$. Then it holds (with some N)*

$$\begin{aligned}&\|[a(x,D), b(x,D)]\|_{H^{\alpha+\beta-1}\to L^2}\\ &\leq C\left(\sum_{j=0}^{N-1} \pi_{N,C_b^1}^{\alpha-j}(a_j) + \pi_{N,C_b^1}^{\alpha-N}(r_{a,N})\right)\left(\sum_{j=0}^{N-1} \pi_{N,C_b^1}^{\beta-j}(b_j) + \pi_{N,C_b^1}^{\beta-N}(r_{b,N})\right),\end{aligned}$$

where a_j, b_j are the homogeneous components of the expansions of a, b with remainders $r_{a,N}$, $r_{b,N}$, respectively. If $\alpha = \beta = 0$, then we additionally have

$$\begin{aligned}&\|[a(x,D), b(x,D)]\|_{L^2\to H^1}\\ &\leq C\left(\sum_{j=0}^{N-1} \pi_{N,C_b^1}^{\alpha-j}(a_j) + \pi_{N,C_b^1}^{\alpha-N}(r_{a,N})\right)\left(\sum_{j=0}^{N-1} \pi_{N,C_b^1}^{\beta-j}(b_j) + \pi_{N,C_b^1}^{\beta-N}(r_{b,N})\right).\end{aligned}$$

Proof. Similar to the proof of Proposition 2.4.3 we can write

$$\begin{aligned}a(x,\xi) &= \sum_{j=0}^{N-1}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,n-2)} a_{jlm}(x) Y_{lm}(\xi)\langle\xi\rangle^{\alpha-j} + r_{a,N}(x,\xi),\\ b(x,\xi) &= \sum_{j=0}^{N-1}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,n-2)} b_{jlm}(x) Y_{lm}(\xi)\langle\xi\rangle^{\beta-j} + r_{b,N}(x,\xi),\\ r_{a,N}(x,\xi) &\in C_b^1 S_{1,0}^{\alpha-N}, \quad r_{b,N}(x,\xi) \in C_b^1 S_{1,0}^{\beta-N}.\end{aligned}$$

Consequently, the commutator $[a(x,D), b(x,D)]$ can be written as

$$\begin{aligned}&[a(x,D), b(x,D)]\\ &= \sum_{j,l,m}\sum_{i,p,q}\left[a_{jlm}(x)Y_{lm}(D)\langle D\rangle^{\alpha-j}, b_{ipq}(x)Y_{pq}(D)\langle D\rangle^{\beta-i}\right]\\ &\quad + \sum_{j,l,m}\left[a_{jlm}(x)Y_{lm}(D)\langle D\rangle^{\alpha-j}, r_{b,N}(x,D)\right]\\ &\quad + \sum_{i,p,q}\left[r_{a,N}(x,D), b_{ipq}(x)Y_{pq}(D)\langle D\rangle^{\beta-i}\right] + r_{a,N}(x,D)r_{b,N}(x,D).\end{aligned}$$

We denote the four commutators by I_1, I_2, I_3, I_4, neglecting the dependence on j, l, m, i, p, q for a moment. The commutator I_1 satisfies

$$\begin{aligned} I_1 &= a_{jlm}(x)Y_{lm}(D)\langle D\rangle^{-j}\left[\langle D\rangle^{\alpha}, b_{ipq}(x)\right]Y_{pq}(D)\langle D\rangle^{\beta-i} \\ &+ a_{jlm}(x)\left[Y_{lm}(D)\langle D\rangle^{-j}, b_{ipq}(x)\right]Y_{pq}(D)\langle D\rangle^{\alpha+\beta-i} \\ &+ b_{ipq}(x)\left[a_{jlm}(x), Y_{pq}(D)\langle D\rangle^{\beta-i}\right]Y_{lm}(D)\langle D\rangle^{\alpha-j}. \end{aligned}$$

From (2.4.1) and Proposition 2.4.1 it can be deduced that

$$\begin{aligned} \|I_1\|_{H^{\alpha+\beta-1}\to L^2} &\le C_{l,p}\|a_{jlm}\|_{Lip^1}\|b_{ipq}\|_{Lip^1} \\ &\le C_M\langle l\rangle^{-M}\langle i\rangle^{-M}\pi^{\alpha-j}_{N(M),C^1_b}(a_j)\pi^{\beta-i}_{N(M),C^1_b}(b_i) \quad \forall M>0. \end{aligned}$$

If $\alpha=\beta=0$, then we additionally have

$$\begin{aligned} \|I_1\|_{L^2\to H^1} &\le C_{l,p}\|a_{jlm}\|_{Lip^1}\|b_{ipq}\|_{Lip^1} \\ &\le C_M\langle l\rangle^{-M}\langle i\rangle^{-M}\pi^{\alpha-j}_{N(M),C^1_b}(a_j)\pi^{\beta-i}_{N(M),C^1_b}(b_i), \end{aligned}$$

see Proposition 2.4.1. The commutator I_2 allows the representation

$$\begin{aligned} I_2 =& a_{jlm}(x)Y_{lm}(D)\langle D\rangle^{\alpha-j}r_{b,N}(x,D) \\ &- r_{b,N}(x,D)a_{jlm}(x)Y_{lm}(D)\langle D\rangle^{\alpha-j}. \end{aligned}$$

Proposition 2.2.8 gives the continuity of

$$r_{b,N}(x,D): L^2\to H^1, \quad r_{b,N}(x,D): H^{-1}\to L^2.$$

Hence we conclude that

$$\begin{aligned} &\|I_2\|_{H^{\alpha+\beta-1}\to L^2} \le C_M\langle l\rangle^{-M}\pi^{\alpha-j}_{N(M),C^1_b}(a_j)\pi^{\beta-N}_{N(M),C^1_b}(r_{b,N}), \\ &\|I_2\|_{L^2\to H^1} \le C_M\langle l\rangle^{-M}\pi^{\alpha-j}_{N(M),C^1_b}(a_j)\pi^{\beta-N}_{N(M),C^1_b}(r_{b,N}) \quad (\alpha=\beta=0). \end{aligned}$$

The commutator I_3 can be estimated in the same way and the estimate of I_4 is trivial. Summing up and choosing M sufficiently large we complete the proof. □

Now we want to show a generalisation of (2.4.2), replacing $f\in Lip^1\cap H^s$ by $A(x,D)\in OPC^1_bS^{\alpha}_{cl}\cap OPH^{s_0}S^{\alpha}_{cl}$ with $s_0>n/2$, $\alpha\in\mathbb{N}_0$. Here we run into a problem, since the operator P from $OPS^s_{1,0}$ does not map $C^s_b(M)$ into $C^0_b(M)$. For this reason we introduce the space $C^{\alpha}_{\sharp,K_0}$ of all functions u satisfying $\langle D\rangle^{\alpha}Y_{lm}(D)u\in C^0_b$ for all l,m and with the property that

$$\sup_{l,m}\langle l\rangle^{-K_0}\|\langle D\rangle^{\alpha}Y_{lm}(D)u\|_{C^0_b}<\infty.$$

The constant K_0 is fixed in such a manner that

$$\|Y_{lm}(D)u\|_{C_b^0} \leq C\langle l\rangle^{K_0} \|u\|_{C_b^0} \quad \forall u \in \mathcal{S}.$$

The use of this definition is the property that

$$B \in OPS_{cl}^{\alpha} \Longrightarrow B : C_{\sharp,K_0}^{\alpha} \to C_b^0 \subset L^{\infty}.$$

The embedding

$$C_b^{\alpha+\delta} \subset C_{\sharp,K_0}^{\alpha} \tag{2.4.4}$$

is continuous for any positive δ, see [Tay91], p.126. We have the (set–theoretical) inclusions

$$C_{\sharp,K_0}^{\alpha} \subset C_b^{\alpha} \subset C_*^{\alpha}.$$

Proposition 2.4.5. *Let $P \in OPS_{1,0}^s$, $A(x,D) \in OPC_b^1 S_{cl}^{\alpha} \cap OPH^{s_0} S_{cl}^{\alpha}$ with $s_0 > n/2$, $0 < s \leq s_0$, $\alpha \in \mathbb{N}_0$ and $K \geq K_0$. Then it holds*

$$\begin{aligned}\|[P, A(x,D)]\, u\|_{L^2} \leq C &\left(\sum_{j=0}^{N-1} \pi_{N,C_b^1}^{\alpha-j}(a_j) + \pi_{N,C_b^1}^{\alpha-N}(r_N)\right) \|u\|_{H^{s+\alpha-1}} \\ &+ C_K \left(\sum_{j=0}^{N-1} \pi_{N,H^s}^{-j}(a_j) + \pi_{N,H^s}^{\alpha-N}(r_N)\right) \|u\|_{C_{\sharp,K}^{\alpha}}\end{aligned}$$

with some constant N and the terms a_j, r_N from the asymptotic expansion of the classical operator A.

Proof. The assumptions imply that

$$A(x,\xi) = \sum_{j=0}^{N-1} a_j(x,\xi) + r_N(x,\xi), \quad r_N(x,\xi) \in C_b^1 S_{cl}^{\alpha-N} \cap H^{s_0} S_{cl}^{\alpha-N},$$

$$a_j(x,\xi) = \sum_{l,m} a_{jlm}(x) Y_{lm}(\xi)\langle\xi\rangle^{\alpha-j},$$

$$\left(\sum_{m=1}^{h(l,n-2)} \|a_{jlm}\|_{C_b^1} + \|a_{jlm}\|_{H^s}\right)_l \text{ is a rapidly decreasing sequence in } l.$$

Then the commutator $[P, A]$ can be split into parts:

$$\begin{aligned}[P, a_j(x,D)]\, u = &\sum_{l,m} [P, a_{jlm}(x)]\, Y_{lm}(D)\langle D\rangle^{\alpha-j} u \\ &+ \sum_{l,m} a_{jlm}(x) \left[P, Y_{lm}(D)\langle D\rangle^{\alpha-j}\right] u.\end{aligned}$$

From (2.4.2) follows that

$$\left\| [P, a_{jlm}(x)]\, Y_{lm}(D)\langle D\rangle^{\alpha-j} u \right\|_{L^2} \leq C \left\| a_{jlm} \right\|_{C_b^1} \left\| Y_{lm}(D)\langle D\rangle^{\alpha-j} u \right\|_{H^{s-1}} + C \left\| a_{jlm} \right\|_{H^s} \left\| Y_{lm}(D)\langle D\rangle^{\alpha-j} u \right\|_{L^\infty}.$$

Theorem A.0.3 shows that, for some K_1 and every positive M

$$\left\| a_{jlm} \right\|_{C_b^1} \left\| Y_{lm}(D)\langle D\rangle^{\alpha-j} u \right\|_{H^{s-1}} \leq C \langle l\rangle^{K_1 - M} \pi^{\alpha-j}_{M', C_b^1}(a_j) \left\| u \right\|_{H^{s+\alpha-1}}$$

holds with some constant $M' = M'(M)$. From $Y_{lm}(D)\langle D\rangle^{\alpha-j} \in OPS^{\alpha}_{cl}$, the definition of $\|.\|_{C^{\alpha}_{\sharp,K}}$ and from Theorem A.0.3 we conclude that

$$\left\| a_{jlm} \right\|_{H^s} \left\| Y_{lm}(D)\langle D\rangle^{\alpha-j} u \right\|_{L^\infty} \leq C \langle l\rangle^{K - M} \pi^{\alpha-j}_{M', H^s}(a_j) \left\| u \right\|_{C^{\alpha}_{\sharp,K}}.$$

It is standard to show

$$\left\| \left[P, Y_{lm}(D)\langle D\rangle^{\alpha-j} \right] u \right\|_{L^2} \leq C \langle l\rangle^{K_2} \left\| u \right\|_{H^{s+\alpha-1}}.$$

Summing up and choosing M large we get

$$\left\| \left[P, \sum_{j=0}^{N-1} a_j(x,D) \right] u \right\|_{L^2} \leq C \sum_{j=0}^{N-1} \pi^{\alpha-j}_{N, C_b^1}(a_j) \left\| u \right\|_{H^{s+\alpha-1}} + C \sum_{j=0}^{N-1} \pi^{\alpha-j}_{N, H^s}(a_j) \left\| u \right\|_{C^{\alpha}_{\sharp,K}}$$

for large N. It remains to estimate $\left\| [P, r_N(x,D)]\, u \right\|_{L^2}$. If $\alpha - N \leq -1$, then $\left\| P r_N u \right\|_{L^2} \leq C \left\| r_N u \right\|_{H^s} \leq C \left\| u \right\|_{H^{s-1}}$ because of Proposition 2.2.11. Finally, we have $\left\| r_N P u \right\|_{L^2} \leq C \left\| P u \right\|_{H^{-1}} \leq C \left\| u \right\|_{H^{s-1}}$, which completes the proof. □

Now we list some properties of the spaces $C^{\alpha}_{\sharp,K}$.

Lemma 2.4.6. *For every $\alpha \in \mathbb{N}_0$ a positive constant C exists with the property that*

$$\left\| \langle D\rangle^{\alpha} u \right\|_{C_b^0} \leq C \left\| u \right\|_{C^{\alpha}_{\sharp,K_0}} \quad \forall u \in C^{\alpha}_{\sharp,K_0}, \tag{2.4.5}$$

$$\left\| u \right\|_{C_b^0} \leq C \left\| u \right\|_{C^{\alpha}_{\sharp,K_0}} \quad \forall u \in C^{\alpha}_{\sharp,K_0}, \tag{2.4.6}$$

$$\left\| u \right\|_{C_b^\alpha} \leq C \left\| u \right\|_{C^{\alpha}_{\sharp,K_0}} \quad \forall u \in C^{\alpha}_{\sharp,K_0}. \tag{2.4.7}$$

Let $\sigma \in C_b^\infty$, $u \in C^{\alpha}_{\sharp,K_0}$ and $K_1 > K_0$ be sufficiently large. Then $\sigma u \in C^{\alpha}_{\sharp,K_1}$ and a constant $C = C(\sigma,\alpha)$ (independent of u) exists with

$$\left\| \sigma u \right\|_{C^{\alpha}_{\sharp,K_1}} \leq C \left\| u \right\|_{C^{\alpha}_{\sharp,K_0}} \tag{2.4.8}$$

Proof. The assertions (2.4.5) and (2.4.6) follow from $\langle D\rangle^\alpha \in OPS^\alpha_{cl}$, $Id \in OPS^\alpha_{cl}$; and (2.4.7) is a consequence of $D^\beta_x \in OPS^\alpha_{cl}$, (2.4.6) and

$$\|u\|_{C^\alpha_b} \le C\Bigg(\sum_{|\beta|\le\alpha} \left\|D^\beta_x u\right\|_{C^0_b} + \|u\|_{C^0_b}\Bigg).$$

In order to show (2.4.8), we note that

$$\begin{aligned}&\|\langle D\rangle^\alpha Y_{lm}(D)(\sigma u)\|_{C^0_b}\\ &\quad\le \|\sigma\langle D\rangle^\alpha Y_{lm}(D)u\|_{C^0_b} + \|[\langle D\rangle^\alpha Y_{lm}(D),\sigma]\,u\|_{C^0_b}\\ &\quad\le \|\sigma\|_{C^0_b}\,\|\langle D\rangle^\alpha Y_{lm}(D)u\|_{C^0_b} + \|[\langle D\rangle^\alpha Y_{lm}(D),\sigma]\,u\|_{C^0_b}\,.\end{aligned}$$

The commutator $[\langle D\rangle^\alpha Y_{lm}(D),\sigma]$ belongs to $OPS^{\alpha-1}_{cl} \subset OPS^\alpha_{cl}$ and fulfils

$$\|[\langle D\rangle^\alpha Y_{lm}(D),\sigma]\,v\|_{C^0_b} \le C\langle l\rangle^M\,\|v\|_{C^\alpha_{\sharp,K_0}}$$

for all $v \in \mathcal{S}$. Then we obtain

$$\|[\langle D\rangle^\alpha Y_{lm}(D),\sigma]\,u\|_{C^0_b} \le C\langle l\rangle^M\,\|u\|_{C^\alpha_{\sharp,K_0}}\,,$$

which results in

$$\begin{aligned}\|\sigma u\|_{C^\alpha_{\sharp,K_1}} &\le \sup_{l,m}\langle l\rangle^{-K_1}\Big(\|\sigma\|_{C^0_b}\,\|\langle D\rangle^\alpha Y_{lm}(D)u\|_{C^0_b} + C\langle l\rangle^M\,\|u\|_{C^\alpha_{\sharp,K_0}}\Big)\\ &\le C\,\|u\|_{C^\alpha_{\sharp,K_0}}\,,\quad K_1 := K_0 + M.\end{aligned}$$

□

2.5 Adjoint Operators

Proposition 2.5.1. *Let $k(x,D) \in OPC^1_bS^1_{cl}$ be an operator whose symbol is positive homogeneous of order 1 for $|\xi| \ge C$. Then the adjoint operator $k^*(x,D)$ satisfies, for some N,*

$$(k^*u,v) = \big(\overline{k}u,v\big) + (Ru,v)\,,\quad \|Ru\|_{L^2} \le C\pi^1_{N,C^1_b}(k)\,\|u\|_{L^2}\,.$$

Proof. We have $k(x,\lambda\xi) = \lambda k(x,\xi)$ for $|\xi| > C$, $\lambda > 1$. Hence we can write

$$k(x,\xi) = k_0(x,\xi) + k_1(x,\xi),$$

where k_0 is positive homogeneous for all $\xi \in \mathbb{R}^n$ and all $\lambda > 0$ and k_1 has bounded support with respect to ξ. That is to say,

$$\begin{aligned}k_0(x,\xi) &= k_0\left(x,\frac{\xi}{|\xi|}\right)|\xi| = \sum_{l,m} k_{0,lm}(x)Y_{lm}\left(\frac{\xi}{|\xi|}\right)|\xi|\\ &=: \sum_{l,m} k_{0,lm}(x)Z_{lm}(\xi),\end{aligned}$$

$$\operatorname{supp} k_1(x,\xi) \subset M \times \{|\xi| \le C\},\quad k_1(x,\xi) \in C^1_bS^{-\infty}_{1,0}.$$

For the "main part" k_0 we compute the adjoint k_0^*:

$$\begin{aligned}
(u, k_0 v) &= \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} u(x) e^{-ix\xi} \overline{k_0(x,\xi)} e^{iy\xi} \overline{v(y)}\, dy\, d\xi\, dx \\
&= \sum_{l,m} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_x} \overline{v(y)} e^{iy\xi} \overline{Z_{lm}(\xi) k_{0,lm}(x)} u(x) e^{-ix\xi}\, dx\, d\xi\, dy \\
&= \sum_{l,m} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} \overline{v(y)} e^{iy\xi} \overline{Z_{lm}(\xi)} \left(\overline{k_{0,lm}} u\right)^\wedge(\xi)\, d\xi\, dy \\
&= \sum_{l,m} \left(\overline{Z_{lm}} (\overline{k_{0,lm}} u), v\right) \\
&= \left(\overline{k_0} u, v\right) + \sum_{l,m} \left(\left[\overline{Z_{lm}}, \overline{k_{0,lm}}\right] u, v\right).
\end{aligned}$$

By (2.4.1) and Theorem A.0.3, each term of the last sum satisfies

$$\begin{aligned}
\left\| \left[\overline{Z_{lm}}, \overline{k_{0,lm}}\right] u\right\|_{L^2} &\leq C \left\| k_{0,lm} \right\|_{C_b^1} \left\| u \right\|_{L^2} \\
&\leq C_R \pi^1_{N,C_b^1}(k_0) \langle l \rangle^{-R} \left\| u \right\|_{L^2} \quad \forall R > 0, \quad N = N(R).
\end{aligned}$$

Summing up we get

$$\left\| \sum_{l,m} \left[\overline{Z_{lm}}, \overline{k_{0,lm}}\right] u \right\|_{L^2} \leq C \pi^1_{N,C_b^1}(k_0) \left\| u \right\|_{L^2}.$$

Finally, from Proposition 2.2.8 ($s_0 = s = 0$) it can be concluded that

$$\left| \left(\overline{k_1} u, v\right) \right| + \left| (u, k_1 v) \right| \leq C \pi^0_{N,C_b^1}(k_1) \left\| u \right\|_{L^2} \left\| v \right\|_{L^2}.$$

This proves the assertion. □

From this proposition we easily derive the following corollary:

Corollary 2.5.2. *Let $K(x, D) \in OPC_b^1 S^1_{cl}$ be a matrix pseudodifferential operator whose symbol is positive homogeneous of order 1 for $|\xi| \geq C$. Then the adjoint operator $K^*(x, D)$ satisfies*

$$sym(K^*(x, D) - R) = \overline{K(x,\xi)^T}, \quad \left\| RU \right\|_{L^2} \leq C \pi^1_{N,C_b^1}(K) \left\| U \right\|_{L^2}.$$

with some operator R and some $N > 0$.

2.6 Compositions

We provide an estimate which is useful for handling the product of an hyperbolic differential matrix operator and its symmetrizer, see Corollary 2.6.2. However, at first let us consider the scalar case.

Proposition 2.6.1. *Let $a(x,D) \in OPC_b^0 S_{cl}^j$, $b(x,D) \in OPC_b^1 S_{cl}^{1-j}$ ($j=0$ or $j=1$) be operators with positive homogeneous symbols for $|\xi| \geq C$. Then*

$$\begin{aligned} &a(x,D)b(x,D) = c(x,D) + R, \\ &c(x,\xi) = a(x,\xi)b(x,\xi) \in C_b^0 S_{cl}^1, \\ &\|Ru\|_{L^2} \leq C\pi^j_{N,C_b^0}(a)\pi^{1-j}_{N,C_b^1}(b)\,\|u\|_{L^2}\,. \end{aligned}$$

Proof. We have

$$\begin{aligned} &a(x,\xi) = \sum_{l,m} a_{lm}(x)Y_{lm}(\xi)\langle\xi\rangle^j + r_a(x,\xi), \\ &\operatorname{supp} r_a(x,\xi) \subset M \times \{|\xi| \leq C\}, \quad r_a(x,\xi) \in C_b^0 S_{1,0}^{-\infty}, \\ &b(x,\xi) = \sum_{l,m} b_{lm}(x)Y_{lm}(\xi)\langle\xi\rangle^{1-j} + r_b(x,\xi), \\ &\operatorname{supp} r_b(x,\xi) \subset M \times \{|\xi| \leq C\}, \quad r_b(x,\xi) \in C_b^1 S_{1,0}^{-\infty}. \end{aligned}$$

Using this decomposition we can write

$$\begin{aligned} a(x,D)b(x,D) &= \sum_{l,m,p,q} a_{lm}(x)b_{pq}(x)Y_{lm}(D)\langle D\rangle^j Y_{pq}(D)\langle D\rangle^{1-j} \\ &+ \sum_{l,m,p,q} a_{lm}(x)\left[Y_{lm}(D)\langle D\rangle^j, b_{pq}(x)\right] Y_{pq}(D)\langle D\rangle^{1-j} \\ &+ \sum_{l,m} a_{lm}(x)Y_{lm}(D)\langle D\rangle^j r_b(x,D) \\ &+ \sum_{p,q} r_a(x,D)b_{pq}(x)Y_{pq}(D)\langle D\rangle^{1-j} + r_a(x,D)r_b(x,D). \end{aligned}$$

The symbol of the first sum is $c(x,\xi)$ for $|\xi| \geq C$. If we add a suitable symbol $r \in C_b^0 S_{1,0}^{-\infty}$ with support in $M \times \{|\xi| \leq C\}$, then we get $c(x,\xi)$ for all $\xi \in \mathbb{R}^n$, i.e.,

$$\begin{aligned} a(x,D)b(x,D) &= c(x,D) \\ &+ \sum_{l,m,p,q} a_{lm}(x)\left[Y_{lm}(D)\langle D\rangle^j, b_{pq}(x)\right] Y_{pq}(D)\langle D\rangle^{1-j} \end{aligned}$$

$$+ \sum_{l,m} a_{lm}(x) Y_{lm}(D) \langle D \rangle^j r_b(x, D)$$
$$+ \sum_{p,q} r_a(x, D) b_{pq}(x) Y_{pq}(D) \langle D \rangle^{1-j} + r_a(x, D) r_b(x, D) - r(x, D).$$

The first sum can be estimated by the aid of

$$\begin{aligned}
&\left\| a_{lm}(x) \left[Y_{lm}(D) \langle D \rangle^j, b_{pq}(x) \right] Y_{pq}(D) \langle D \rangle^{1-j} u \right\|_{L^2} \\
&\leq C \left\| a_{lm} \right\|_{C_b^0} \left\| \left[Y_{lm}(D) \langle D \rangle^j, b_{pq}(x) \right] Y_{pq}(D) \langle D \rangle^{1-j} u \right\|_{L^2} \\
&\leq C \left\| a_{lm} \right\|_{C_b^0} \left\| b_{pq} \right\|_{Lip^1} \left\| u \right\|_{L^2},
\end{aligned}$$

see (2.4.1) for $j = 1$ and Proposition 2.4.1 for $j = 0$. It is trivial to estimate the remaining terms, see Proposition 2.2.8. □

Corollary 2.6.2. *Let* $A(x, D) \in OPC_b^0 S_{cl}^j$, $B(x, D) \in OPC_b^1 S_{cl}^{1-j}$ *(*$j = 0$ *or* $j = 1$*) be pseudodifferential matrix operators with positive homogeneous symbols for* $|\xi| \geq C$*. Then*

$$\begin{aligned}
&A(x, D) B(x, D) = C(x, D) + R, \\
&C(x, \xi) = A(x, \xi) B(x, \xi) \in C_b^0 S_{cl}^1, \\
&\left\| RU \right\|_{L^2} \leq C \pi^j_{N, C_b^0}(A) \pi^{1-j}_{N, C_b^1}(B) \left\| U \right\|_{L^2}.
\end{aligned}$$

Chapter 3

Weakly Hyperbolic Cauchy Problems with Spatial Degeneracy

3.1 Introduction

Let us consider the following quasilinear weakly hyperbolic Cauchy problem:

$$
\begin{aligned}
&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{\tilde{c}_{k,\beta}(x,t) D_x^\beta D_t^k u\}) \sigma(x)^{|\alpha|} D_x^\alpha D_t^j u \\
&\quad = f(x,t,\{\tilde{c}_{k,\beta}(x,t) D_x^\beta D_t^k u\}), \quad k+|\beta| \le m-1, \quad m \ge 2, \\
&u(x,t_0) = \varphi_0(x), \ldots, D_t^{m-1} u(x,t_0) = \varphi_{m-1}(x).
\end{aligned}
$$

The functions $\tilde{c}_{k,\beta} = \tilde{c}_{k,\beta}(x,t)$ are weight functions that have to satisfy additional conditions, the so–called *Levi conditions*. Examples of such weight functions are $\tilde{c}_{k,\beta}(x,t) = \sigma(x)^{|\beta|}$. More examples can be found in Section 4.5. However, it has advantages to transform the differential equation into another one. By the Leibniz formula we have

$$
\sigma(x)^{|\alpha|} D_x^\alpha u(x,t) = D_x^\alpha(\sigma(x)^{|\alpha|} u(x,t)) + \sum_{\gamma<\alpha} d_\gamma(x) D_x^\gamma(\sigma(x)^{|\gamma|} u(x,t)).
$$

Similar relations hold for $\tilde{c}_{k,\beta}(x,t) D_x^\beta D_t^k u$. We shift all lower order terms to the right–hand side. Thus, we arrive at the Cauchy problem

$$
\begin{aligned}
&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k u\}) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) \\
&\quad = f(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k u\}), \quad k+|\beta| \le m-1, \quad m \ge 2, \qquad (3.1.1) \\
&u(x,t_0) = \varphi_0(x), \dots, D_t^{m-1} u(x,t_0) = \varphi_{m-1}(x).
\end{aligned}
$$

The linearised form of this new equation is

$$
\begin{aligned}
&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) = f(x,t), \qquad (3.1.2) \\
&u(x,t_0) = \varphi_0(x), \dots, D_t^{m-1} u(x,t_0) = \varphi_{m-1}(x).
\end{aligned}
$$

The chapter is organised as follows.
The following results will be given in the second section:

- The Cauchy problem (3.1.2) will be transformed into a system of first order. After that this system will be regularised by inserting a smoothing operator (see Section 2.3). An energy estimate and an existence result will be proved for this regularised system, see Proposition 3.2.2, (a).
- An energy estimate for the corresponding non-regularised system will be shown in Proposition 3.2.2, (b).

The question of *existence* of a solution to (3.1.2) will be studied after we have considered quasilinear equations. The third section contains the following results:

- a local existence result for the Cauchy problem (3.1.1), see Theorem 3.3.1,
- a blow–up criterion: if the C_*^1 norm of certain weighted derivatives of u up to the order $m-1$ is bounded, then a blow–up of the solution in the H^s norm is impossible,
- the global existence of solutions in Sobolev spaces of the *linear* problem (3.1.2), see Corollary 3.3.6,
- a local existence result in C^∞, see Theorem 3.3.7. This is an immediate consequence of the blow–up criterion.

Finally, in the fourth section the above results will be applied to

- prove the continuous dependence of the solution to (3.1.1) from coefficients and data, cf. Theorem 3.4.1,

- give a sharp lower estimate of the life–span, see Corollary 3.4.2.

These three sections make a strong use of ideas from [Tay91]. In this book strictly hyperbolic equations and systems have been studied. We are able to extend the results won in [Tay91] to the weakly hyperbolic case.
It might be a bit surprising that it is not necessary to study the existence of solutions to linear equations *before* the existence of solutions to quasilinear ones. The reason is that we are able to prove the local existence for quasilinear equations in a direct way without an existence result for linear equations. Then, linear equations can be regarded as a special case of quasilinear ones. Let us sketch the proof:
We transform the quasilinear weakly hyperbolic Cauchy problem into a system of first order,

$$d_t U^* = K^*(x,t,U^*,D)(\sigma U^*) + B^*(x,t,U^*,D)U^* + F^*(x,t,U^*),$$

see (3.3.13). The vector U^* consists of all (weighted) derivatives of u up to the order $m-1$, K^* is a *strictly* hyperbolic matrix pseudodifferential operator of order 1, B^* is some operator of order 0 and F^* contains the right–hand side and some other terms. If $U^* \in H^s$, then the right–hand side of this above system belongs to H^{s-1}, because K^* has order 1. Let us insert a regularising operator J_ε (see Definition 2.3.1):

$$d_t U^*_\varepsilon = J_\varepsilon K^*(x,t,U^*_\varepsilon,D)(\sigma U^*_\varepsilon) + B^*(x,t,U^*_\varepsilon,D)U^*_\varepsilon + F^*(x,t,U^*_\varepsilon).$$

Then the right–hand side maps $U^*_\varepsilon \in H^s$ to some function from H^s. Hence, we can regard this system as an ODE for some function U^*_ε with values in the Banach space H^s. The Theorem of Picard–Lindelöf immediately gives the local existence of U^*_ε.
This approach has the following advantage: we are able to derive an estimate of the form

$$\begin{aligned} d_t & \left(R^*\langle D\rangle^s U^*_\varepsilon, \langle D\rangle^s U^*_\varepsilon\right) \\ &\leq C(\|K^*\|_{C^1_b}, \|B^*\|_{C^1_b}) \|U^*_\varepsilon\|^2_{H^s} \\ &\quad + C(\|K^*\|_{H^s} + \|B^*\|_{H^s}) \|U^*_\varepsilon\|_{C^1_{\sharp,K_0}} \|U^*_\varepsilon\|_{H^s} + C\|F^*_\varepsilon\|^2_{H^s}. \end{aligned}$$

The operator R^* is a generalised (since U^*_ε contains lower order terms) symmetrizer. From Moser's inequality we get

$$d_t \left(R^*\langle D\rangle^s U^*_\varepsilon, \langle D\rangle^s U^*_\varepsilon\right) \leq C(\|U^*_\varepsilon\|_{C^1_{\sharp,K_0}} + 1)(\|U^*_\varepsilon\|^2_{H^s} + 1). \qquad (3.1.3)$$

This leads us (after some calculations) to an interesting blow–up criterion:

A blow–up of U^* in the H^s norm is impossible as long as the Zygmund norm $\|U^*\|_{C^1_*}$ is bounded.
This characterisation seems to require the approach via the theorem of Picard–Lindelöf. The usual iteration technique (constructing a sequence of functions that are solutions of linear Cauchy problems) will not lead to an estimate of the type (3.1.3).
This blow–up criterion immediately yields the local existence in C^∞, see Theorem 3.3.7.
Finally, the following results are proved in the fourth section: let us consider a quasilinear weakly hyperbolic Cauchy problem with spatial degeneracy and its solution which is assumed to exist in the interval $[0,T]$. We perturb the data, the coefficients, the right–hand side and the weight functions of this Cauchy problem. It will be shown that the solution of this perturbed Cauchy problem exists up to T and differs by an arbitrary small value from the solution of the unperturbed problem (in appropriate norms), if the perturbation is small. This result includes a life–span estimate.

3.2 The Linear Case

First, we list the assumptions.

Condition 1. *We assume that the roots $\tau_j(x,t,\xi)$ of*

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\xi^\alpha\tau^j = 0$$

are real and distinct,

$$|\tau_j(x,t,\xi) - \tau_i(x,t,\xi)| \geq c|\xi|, \quad c>0, \quad i \neq j, \quad \forall(x,t,\xi).$$

Let $M = \mathbb{T}^n$ be an n–dimensional torus. We suppose that

$$\sigma \in C^\infty(M), \tag{3.2.1}$$

$$a_{j,\alpha} \in C^1([t_0,T], H^{s_0}(M)), \quad s_0 > \frac{n}{2}+1, \tag{3.2.2}$$

$$\varphi_j \in H^{s_1+m-1-j}, \quad s_1 \geq 0, \tag{3.2.3}$$

$$f \in C([t_0,T], H^{s_1}(M)). \tag{3.2.4}$$

As already mentioned in the introduction to this chapter, the linear problem will be transformed into an equivalent system of first order. Then energy estimates of solutions to this system and to a regularised system are derived.

3.2.1 Transformation into a 1^{st} Order System

We define the vector $U = (U_1, \dots, U_m)^T$,

$$\begin{aligned}
&U_1 := \langle D\rangle^{m-1}\left(\sigma^{m-1}u\right),\\
&U_2 := \langle D\rangle^{m-2}\left(\sigma^{m-2}D_t u\right),\\
&\dots,\\
&U_m := D_t^{m-1}u
\end{aligned} \tag{3.2.5}$$

and get the system

$$\begin{aligned}
\partial_t U_1 =& i\langle D\rangle(\sigma U_2) + i\langle D\rangle\left[\langle D\rangle^{m-2},\sigma\right]\langle D\rangle^{2-m}U_2,\\
\partial_t U_2 =& i\langle D\rangle(\sigma U_3) + i\langle D\rangle\left[\langle D\rangle^{m-3},\sigma\right]\langle D\rangle^{3-m}U_3,\\
&\dots,\\
\partial_t U_{m-1} =& i\langle D\rangle(\sigma D_t^{m-1}u) = i\langle D\rangle(\sigma U_m),\\
\partial_t U_m =& iD_t^m u = -i\sum_{j+|\alpha|=m, j<m} a_{j,\alpha}D_x^\alpha D_t^j\left(\sigma^{|\alpha|}u\right) + if\\
=& -i\sum_{j+|\alpha|=m, j<m} a_{j,\alpha}P^\alpha\langle D\rangle\langle D\rangle^{|\alpha|-1}\left(\sigma\sigma^{|\alpha|-1}D_t^j u\right) + if\\
=& -i\sum_{j+|\alpha|=m, j<m} a_{j,\alpha}P^\alpha\langle D\rangle(\sigma U_{j+1})\\
&-i\sum_{j+|\alpha|=m, j<m} a_{j,\alpha}P^\alpha\langle D\rangle\left[\langle D\rangle^{|\alpha|-1},\sigma\right]\langle D\rangle^{1-|\alpha|}U_{j+1} + if
\end{aligned}$$

with $P_j = D_{x_j}\langle D\rangle^{-1}$, $P^\alpha = \prod_{j=1}^n P_j^{\alpha_j}$. This gives

$$\partial_t U = K(\sigma U) + BU + F, \quad U(t_0) = \Phi_0, \tag{3.2.6}$$

$$K = K_0\langle D\rangle = i\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ k_0 & k_1 & k_2 & \dots & k_{m-1}\end{pmatrix}\langle D\rangle, \tag{3.2.7}$$

$$k_j = -\sum_{|\alpha|=m-j} a_{j,\alpha}P^\alpha,$$

$$B = i\begin{pmatrix} 0 & b^{(2)} & 0 & \dots & 0\\ 0 & 0 & b^{(3)} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & b^{(m)}\\ b_1 & b_2 & b_3 & \dots & b_m\end{pmatrix}, \tag{3.2.8}$$

$$\begin{aligned}
&b_k = -\sum_{|\alpha|=m+1-k} a_{k-1,\alpha} P^\alpha \langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m} \quad \forall k \le m-1,\\
&b_m = 0,\\
&b^{(j)} = \langle D\rangle \left[\langle D\rangle^{m-j}, \sigma\right] \langle D\rangle^{j-m},\\
&F = (0,0,\dots,0,if)^T, && (3.2.9)\\
&\Phi_0 = (\langle D\rangle^{m-1}(\sigma^{m-1}\varphi_0), \langle D\rangle^{m-2}(\sigma^{m-2}\varphi_1), \dots, \varphi_{m-1})^T. && (3.2.10)
\end{aligned}$$

Obviously, $K \in C_b^1 S_{cl}^1$ with positive homogeneous symbol with respect to ξ for $|\xi| \ge C$.
Let us construct a symmetrizer for K_0, using ideas from [Ler54]. We introduce the notations

$$p_j = \xi_j \langle \xi\rangle^{-1}, \quad p^\alpha = \prod_{j=1}^n p_j^{\alpha_j}.$$

The eigenvalues of $K_0(x,t,p)$ are $i\tau_j(x,t,p)$. Obviously,

$$K_0 \begin{pmatrix} 1 \\ \tau_j(x,t,p) \\ \tau_j(x,t,p)^2 \\ \vdots \\ \tau_j(x,t,p)^{m-1} \end{pmatrix} = i \begin{pmatrix} \tau_j(x,t,p) \\ \tau_j(x,t,p)^2 \\ \tau_j(x,t,p)^3 \\ \vdots \\ \tau_j(x,t,p)^{m} \end{pmatrix} = i\tau_j(x,t,p) \begin{pmatrix} 1 \\ \tau_j(x,t,p) \\ \tau_j(x,t,p)^2 \\ \vdots \\ \tau_j(x,t,p)^{m-1} \end{pmatrix}.$$

Let $S_0 = V(\tau_1(x,t,p), \dots, \tau_m(x,t,p))$ be the Vandermonde–matrix of the numbers $(\tau_1, \dots, \tau_m)$. We have

$$K_0 S_0 = i S_0 \begin{pmatrix} \tau_1 & 0 & \dots & 0 \\ 0 & \tau_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau_m \end{pmatrix} =: i S_0 D.$$

The matrix

$$S := S_0 S_0^T = \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_{m-1} \\ s_1 & s_2 & s_3 & \dots & s_m \\ s_2 & s_3 & s_4 & \dots & s_{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_m & s_{m+1} & \dots & s_{2m-2} \end{pmatrix},$$

$$s_k(x,t,p) = \sum_{j=1}^m \tau_j(x,t,p)^k,$$

is symmetric and positive definite. Vieta's Theorem reveals that the s_k are some polynomials in $a_{j,\alpha}p^\alpha$. The symmetrizer is defined as $R := \det(S)S^{-1}$. Obviously, $R(x,t,p)$ is a symmetric positive definite matrix. It remains to check that RK_0 is symmetric: The matrix K_0S is symmetric since

$$K_0S = K_0S_0S_0^T = iS_0DS_0^T = (iS_0DS_0^T)^T = (K_0S)^T.$$

Further,

$$\begin{aligned} RK_0 &= cS^{-1}K_0 = cS^{-1}(K_0S)S^{-1} = cS^{-1}(K_0S)(S^{-1})^T \\ &= (cS^{-1}(K_0S)(S^{-1})^T)^T = (RK_0)^T \end{aligned}$$

with $c = \det(S)$. This proves that R is a symmetrizer for K_0. The components r_{ij} of R are some polynomials of the $a_{j,\alpha}p^\alpha$, that is,

$$r_{ij}(x,t,p) = \sum_{l\in B_{ij}} c_{ijl}\Bigg(\prod_{(j,\alpha)\in D_{ijl}} a_{j,\alpha}(x,t)\Bigg)\Bigg(\prod_{(j,\alpha)\in D_{ijl}} p^\alpha\Bigg), \tag{3.2.11}$$

with $c_{ijl} \in \mathbb{C}$ and some finite index sets B_{ij} and D_{ijl}. Since the $\tau_k(x,t,p)$ depend on $p_j = \xi_j\langle\xi\rangle^{-1}$, we have $R(t,x,\xi) \in C_b^1S_{cl}^0$. The property of R being a symmetrizer implies

$$C_{R,\sim}^{-1}\|V\|_{L^2}^2 \le (RV,V) \le C_{R,\sim}\|V\|_{L^2}^2 \quad \forall V \in L^2 \tag{3.2.12}$$

with $C_{R,\sim} > 0$, see [Ler54].
Let us characterise the mapping properties of these matrix operators. The product structure of the k_{ij} gives

$$C_K := \max\{\|K\|_{H^1\to L^2}, \|K_0\|_{L^2\to L^2}, \|K\|_{C_\sharp^1\to C_b^0}\} \le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}), \tag{3.2.13}$$

where the term $C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})$ denotes a universal constant which depends on $\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}$ in a nonlinear way. We will use this notation very often in this and the following sections.
The product form of the r_{ij} shows

$$C_R := \|R\|_{L^2\to L^2} \le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}). \tag{3.2.14}$$

Let us devote ourselves to the term K^*R+RK. We will prove that this operator maps L^2 into itself. The adjoint K^* of $K(x,t,D)$ has the representation

$$K^*(x,t,D) = -K^T(x,t,D) + R_1, \quad R_1 : L^2 \to L^2,$$

see Corollary 2.5.2. Then we have

$$\begin{aligned} &R(x,t,D)K(x,t,D) = C_1(x,t,D) + R_2, \\ &C_1(x,t,\xi) = R(x,t,\xi)K(x,t,\xi), \quad R_2 : L^2 \to L^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} &K^*(x,t,D)R(x,t,D) \\ &\quad = -K^T(x,t,D)R(x,t,D) + R_1R(x,t,D) \\ &\quad = C_2(x,t,D) + R_3 + R_1R(x,t,D), \\ &C_2(x,t,\xi) = -K^T(x,t,\xi)R(x,t,\xi), \quad R_3 : L^2 \to L^2. \end{aligned}$$

From $C_1(x,t,\xi) + C_2(x,t,\xi) = 0$ we deduce that

$$\begin{aligned} &R(x,t,D)K(x,t,D) + K^*(x,t,D)R(x,t,D) \\ &\quad = R_2 + R_3 + R_1R(x,t,D) =: R_4, \\ &\|R_4\|_{L^2\to L^2} \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1). \end{aligned} \tag{3.2.15}$$

Finally, mapping properties of the matrix operator B are studied. First, we consider the terms of the secondary diagonal. Since[1] $\sigma \in C^\infty$, the theory of pseudodifferential operators with smooth symbols can be applied to describe the behaviour of $\langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m}$. We get

$$b^{(j)} = \langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m} \in OPS^0_{cl},$$

which gives $\left\|b^{(j)}v\right\|_{H^s} \le C\,\|v\|_{H^s}$ for all $v \in H^s$.
It remains to consider the last row of the matrix B. Two cases are distinguished. In the first case we have $0 < s \le s_0$, in the second $s = 0$. The only reason for this distinction is that Proposition 2.4.5 can not be applied if $s = 0$. Let $0 < s \le s_0$. It remains to examine

$$\left\|a_{k-1,\alpha}(.,t)P^\alpha\langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m}v\right\|_{H^s}, \quad k = 1, \dots, m-1.$$

Utilising the formula (3.1.59) from [Tay91],

$$\|uv\|_{H^{s,p}} \le C\left(\|u\|_{L^\infty} \|v\|_{H^{s,p}} + \|v\|_{L^\infty} \|u\|_{H^{s,p}}\right), \quad s > 0, \quad 1 < p < \infty,$$

we get

$$\begin{aligned} &\left\|a_{k-1,\alpha}(.,t)P^\alpha\langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m}v\right\|_{H^s} \\ &\quad \le C\,\|a_{k-1,\alpha}\|_{L^\infty} \left\|P^\alpha\langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m}v\right\|_{H^s} \\ &\qquad + C\,\|a_{k-1,\alpha}\|_{H^s} \left\|P^\alpha\langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m}v\right\|_{L^\infty}. \end{aligned}$$

[1] This is the only time we use $\sigma \in C^\infty$. Probably it is possible to weaken this assumption.

From $\sigma \in C^\infty$ it may be concluded that

$$P^\alpha \langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m} \in OPS^0_{cl},$$

which gives

$$\begin{aligned} \left\| P^\alpha \langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m} v\right\|_{H^s} &\le C \left\| v\right\|_{H^s}, \\ \left\| P^\alpha \langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m} v\right\|_{L^\infty} &\le C \left\| v\right\|_{C^0_{\sharp,K_0}}. \end{aligned}$$

If $s = 0$, then we immediately arrive at

$$\left\| a_{k-1,\alpha}(.,t) P^\alpha \langle D\rangle \left[\langle D\rangle^{m-k}, \sigma\right] \langle D\rangle^{k-m} v\right\|_{L^2} \le C \left\| a_{k-1,\alpha}\right\|_{L^\infty} \left\| v\right\|_{L^2}.$$

Consequently,

$$\begin{aligned} \left\| BU\right\|_{H^s} &\le C(\max_{j,\alpha} \left\| a_{j,\alpha}\right\|_{C^0_b}) \left\| U\right\|_{H^s} \qquad (3.2.16) \\ &\quad + C \max_{j,\alpha} (\left\| a_{j,\alpha}\right\|_{H^s} + 1) \left\| U\right\|_{C^0_\sharp}, \\ \left\| BU\right\|_{L^2} &\le C(\max_{j,\alpha} \left\| a_{j,\alpha}\right\|_{C^0_b}) \left\| U\right\|_{L^2}. \qquad (3.2.17) \end{aligned}$$

Let us summarise the results:

Proposition 3.2.1. *The linear weakly hyperbolic Cauchy problem (3.1.2) can be transformed into the equivalent system (3.2.6) with U, K, B, F, Φ from (3.2.5), (3.2.7), (3.2.8), (3.2.9) and (3.2.10), respectively.*
The matrix operator K is a strictly *hyperbolic pseudodifferential operator with finite smoothness, $K \in OPC^1S^1_{cl} \cap OPH^{s_0}S^1_{cl}$. Its symbol is positive homogeneous with respect to ξ for $|\xi| \ge C$.*
Furthermore, a symmetrizer R assigned to K exists. This operator R is a zero order pseudodifferential operator with finite smoothness, $R \in OPC^1S^0_{cl} \cap OPH^{s_0}S^0_{cl}$. Its symbol is homogeneous with respect to ξ for $|\xi| \ge C$. The symmetrizer R induces a norm in L^2 which is equivalent to the usual norm, see (3.2.12).
The operators K, R, $K^*R + RK$ and B have the mapping properties given in (3.2.13), (3.2.14), (3.2.15) and (3.2.16), (3.2.17), respectively.

3.2.2 A–priori Estimates

Now we have all tools to show an a–priori estimate of strictly hyperbolic type. The attempt to do this for (3.2.6) leads to 2 problems. First, if $U \in H^s$, then the function $\langle D\rangle^s K\sigma U$ belongs to H^{-1}. Such a function can only be inserted into the scalar product $(\langle D\rangle^s K\sigma U, \langle D\rangle^s U)$, if the other argument $\langle D\rangle^s U$ of this scalar product is from H^1. But in this case this is not true, it is from L^2. Second, the existence of a solution of (3.2.6) is not clear. We overcome these difficulties by considering

$$\partial_t U_\varepsilon = J_\varepsilon K(\sigma U_\varepsilon) + BU_\varepsilon + F, \quad U_\varepsilon(t_0) = \Phi_0. \tag{3.2.18}$$

For the solutions U, U_ε of (3.2.6), (3.2.18) we prove:

Proposition 3.2.2. (a) *The linear system (3.2.18) has a unique global solution $U_\varepsilon \in C^1([t_0,T], H^{\min(s_0,s_1)}(M))$ which satisfies the following estimates for $0 \le s \le \min(s_0, s_1)$:*

$$\begin{aligned}
& d_t\,(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\
&\quad \le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0})\, \|U_\varepsilon\|_{H^s}^2 \\
&\qquad + 2\sqrt{(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon)}\sqrt{(R\langle D\rangle^s F, \langle D\rangle^s F)} \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1)\, \|U_\varepsilon\|_{H^s}^2 \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1)\, \|U_\varepsilon\|_{H^s}\, \|U_\varepsilon\|_{C^1_{\sharp,K_0}}\,.
\end{aligned}$$

(b) *Let $U \in C^1([t_0,T], H^{\min(s_0,s_1)}(M))$ be a solution of (3.2.6) and let $0 \le s \le \min(s_0, s_1) - 1$. Then*

$$\begin{aligned}
& d_t\,(R\langle D\rangle^s U, \langle D\rangle^s U) \\
&\quad \le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0})\, \|U\|_{H^s}^2 \\
&\qquad + 2\sqrt{(R\langle D\rangle^s U, \langle D\rangle^s U)}\sqrt{(R\langle D\rangle^s F, \langle D\rangle^s F)} \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1)\, \|U\|_{H^s}^2 \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1)\, \|U\|_{H^s}\, \|U\|_{C^1_{\sharp,K_0}}\,.
\end{aligned}$$

If $s = 0$, then we can replace the $C^1_{\sharp,K_0}$–norms by L^2–norms in both estimates.

Proof of (a)

The operator J_ε maps H^r into H^{r+k} for any $r, k \in \mathbb{R}$ with norm $O(\varepsilon^{-k})$ for $k > 0$ and $O(1)$ for $k \le 0$. This guarantees $\langle D\rangle^s J_\varepsilon K(\sigma U_\varepsilon) \in L^2$ for $U_\varepsilon \in H^s$. Because (due to (3.2.16)) the right side of (3.2.18) maps H^r continuously into H^r (for $0 \le r \le \min(s_0, s_1)$), the equation (3.2.18) is a linear Banach space ODE which is globally solvable, $U_\varepsilon \in C^1([t_0,T], H^s(M))$, $s \le \min(s_0, s_1)$.
Then it holds

$$\begin{aligned}
& d_t\,(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\
&= (R_t\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) + (R\langle D\rangle^s U_{\varepsilon,t}, \langle D\rangle^s U_\varepsilon) + (R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_{\varepsilon,t}) \\
&= (R_t\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) + (R\langle D\rangle^s (J_\varepsilon K\sigma U_\varepsilon + BU_\varepsilon + F), \langle D\rangle^s U_\varepsilon) \\
&\quad + (R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s (J_\varepsilon K\sigma U_\varepsilon + BU_\varepsilon + F))\,.
\end{aligned}$$

It is easy to estimate the first term on the right:

$$\|R_t\langle D\rangle^s U_\varepsilon\|_{L^2} \le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0}) \, \|U_\varepsilon\|_{H^s} \,.$$

Since $(R.,.)$ is a scalar product of L^2, the Cauchy–Schwarz Inequality results in

$$\begin{aligned} &|\,(R\langle D\rangle^s F, \langle D\rangle^s U_\varepsilon) + (R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s F)\,| \\ &\qquad\qquad \le 2\sqrt{(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon)}\sqrt{(R\langle D\rangle^s F, \langle D\rangle^s F)}. \end{aligned}$$

From the formulas (3.2.16) and (3.2.14) we see that

$$\begin{aligned} &|\,(R\langle D\rangle^s BU_\varepsilon, \langle D\rangle^s U_\varepsilon) + (R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s BU_\varepsilon)\,| \\ &\quad \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \, \|U_\varepsilon\|_{H^s}^2 + C \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \, \|U_\varepsilon\|_{C^0_{\sharp,K_0}} \, \|U_\varepsilon\|_{H^s} \,. \end{aligned}$$

It remains to consider the terms

$$I_1 = (R\langle D\rangle^s J_\varepsilon K\sigma U_\varepsilon, \langle D\rangle^s U_\varepsilon)\,, \quad I_2 = (R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s J_\varepsilon K\sigma U_\varepsilon)\,.$$

The scalar product I_1 can be written in the form

$$\begin{aligned} I_1 =& I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16} + I_{17} + I_{18} + I_{19} \\ =& (RJ_\varepsilon\,[\langle D\rangle^s, K_0]\,\langle D\rangle\sigma U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\ &+ \big(RJ_\varepsilon K_0\,\big[\langle D\rangle^{s+1}, \sigma\big]\,U_\varepsilon, \langle D\rangle^s U_\varepsilon\big) \\ &+ \big(RJ_\varepsilon\,[K_0, \sigma]\,\langle D\rangle^{s+1} U_\varepsilon, \langle D\rangle^s U_\varepsilon\big) \\ &+ (RJ_\varepsilon\sigma\,[K_0, \langle D\rangle]\,\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\ &+ (R\,[J_\varepsilon, \sigma]\,\langle D\rangle K_0\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\ &+ ([R, \sigma]\,\langle D\rangle J_\varepsilon K_0\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\ &+ (\sigma\,[R, J_\varepsilon]\,\langle D\rangle K_0\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\ &+ (\sigma J_\varepsilon R\,[\langle D\rangle, K_0]\,\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \\ &+ (\sigma J_\varepsilon RK\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon)\,. \end{aligned}$$

We estimate now $I_{11}, \ldots, I_{18}$. From (3.2.14), Proposition 2.4.5 and Lemma 2.4.6 it can be deduced that

$$\begin{aligned} |I_{11}| \le& C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \\ &+ C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \, \|\sigma U_\varepsilon\|_{C^1_{\sharp,K_1}} \, \|U_\varepsilon\|_{H^s} \\ \le& C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \\ &+ C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \, \|U_\varepsilon\|_{C^1_{\sharp,K_0}} \, \|U_\varepsilon\|_{H^s} \,. \end{aligned}$$

Exploiting (3.2.14), (3.2.13), (2.4.2) and (2.4.6) shows

$$
\begin{aligned}
|I_{12}| &\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \Big(\|\sigma\|_{Lip^1} \|U_\varepsilon\|_{H^s} + \|\sigma\|_{H^s} \|U_\varepsilon\|_{L^\infty} \Big) \|U_\varepsilon\|_{H^s} \\
&\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s} (\|U_\varepsilon\|_{H^s} + \|U_\varepsilon\|_{C^1_{\sharp,K_0}}).
\end{aligned}
$$

From (3.2.14) and Proposition 2.4.4 ($\alpha = \beta = 0$) it follows that

$$
|I_{13}| \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|\sigma\|_{C_b^1} \|U_\varepsilon\|^2_{H^s} .
$$

By (3.2.14) and Proposition 2.4.4 ($\alpha = 0$, $\beta = 1$) we have

$$
|I_{14}| + |I_{18}| \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|^2_{H^s} .
$$

We use (3.2.14), Lemma 2.4.2, Proposition 2.4.4 ($\alpha = \beta = 0$), (3.2.13) and conclude that

$$
\begin{aligned}
&|I_{15}| + |I_{16}| + |I_{17}| \\
&\quad \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|\langle D\rangle K_0 \langle D\rangle^s U_\varepsilon\|_{H^{-1}} \|U_\varepsilon\|_{H^s} \\
&\quad \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|^2_{H^s} .
\end{aligned}
$$

Summing up shows

$$
\begin{aligned}
|I_1 - I_{19}| \le{}& C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|^2_{H^s} \\
&+ C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \|U_\varepsilon\|_{H^s} \|U_\varepsilon\|_{C^1_{\sharp,K_0}} .
\end{aligned}
$$

The scalar product I_2 satisfies

$$
\begin{aligned}
I_2 &= (R\langle D\rangle^s U_\varepsilon, J_\varepsilon \langle D\rangle^s K\sigma U_\varepsilon) = I_{21} + I_{22} + I_{23} + I_{24} \\
&= (R\langle D\rangle^s U_\varepsilon, J_\varepsilon \left[\langle D\rangle^s, K\right] \sigma U_\varepsilon) + (R\langle D\rangle^s U_\varepsilon, J_\varepsilon K \left[\langle D\rangle^s, \sigma\right] U_\varepsilon) \\
&\quad + (R\langle D\rangle^s U_\varepsilon, \left[J_\varepsilon, K\right] \sigma \langle D\rangle^s U_\varepsilon) + (\sigma J_\varepsilon K^* R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) .
\end{aligned}
$$

Employing (3.2.14), $[\langle D\rangle^s, K] = [\langle D\rangle^s, K^0] \langle D\rangle$, Proposition 2.4.5 and Lemma 2.4.6 we get

$$
\begin{aligned}
|I_{21}| &\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s} \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|\sigma U_\varepsilon\|_{H^s} \\
&\quad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s} \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \|\sigma U_\varepsilon\|_{C^1_{\sharp,K_1}} \\
&\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|^2_{H^s} \\
&\quad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \|U_\varepsilon\|_{C^1_{\sharp,K_0}} \|U_\varepsilon\|_{H^s} .
\end{aligned}
$$

From (3.2.14), (3.2.13), (2.4.3) and (2.4.6) it can be concluded that

$$|I_{22}| \leq C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s} \left(\|\sigma\|_{Lip^1} \|U_\varepsilon\|_{H^s} + \|\sigma\|_{H^{s+1}} \|U_\varepsilon\|_{L^\infty} \right)$$
$$\leq C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s} (\|U_\varepsilon\|_{H^s} + \|U_\varepsilon\|_{C^1_{\sharp,K_0}}).$$

By (3.2.14), Proposition 2.4.4 ($\alpha = 0$, $\beta = 1$) and Lemma 2.4.2 we have

$$|I_{23}| \leq C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s} \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|\sigma\|_{L^\infty} \|U_\varepsilon\|_{H^s} .$$

The above estimates give

$$|I_2 - I_{24}| \leq C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|^2_{H^s}$$
$$+ C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1) \|U_\varepsilon\|_{H^s} \|U_\varepsilon\|_{C^1_{\sharp,K_0}} .$$

Finally, (3.2.15) yields

$$|I_{19} + I_{24}| = |\, (\sigma J_\varepsilon(RK + K^*R)\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \,|$$
$$\leq C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|^2_{H^s} .$$

Summing up we obtain the estimate of **(a)** for $0 < s \leq \min(s_0, s_1)$.
Now we consider the case $s = 0$. The term $\|BU_\varepsilon\|_{L^2}$ can be estimated by

$$C \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^0} + 1) \|U_\varepsilon\|_{L^2} . \tag{3.2.19}$$

We have $I_{11} = I_{21} = I_{22} = 0$. For the estimate of I_{12} we replace Proposition 2.4.5 by (2.4.1). The other items can be estimated in the same manner. We get the sharper inequality

$$d_t\,(RU_\varepsilon, U_\varepsilon) \leq C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|^2_{L^2} + 2\sqrt{(RU_\varepsilon, U_\varepsilon)}\sqrt{(RF, F)}$$
$$+ C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C^1} + 1) \|U_\varepsilon\|^2_{L^2} . \tag{3.2.20}$$

Proof of (b). We prove this estimate in a similar way as the previous one replacing the operators J_ε by the identity operator Id.
The proposition is proved. □

Remark 3.2.3. *The restriction $s \leq \min(s_0, s_1) - 1$ in the part (b) (instead of $s \leq \min(s_0, s_1)$ in the part (a)) has the following reason: The attempt to estimate $d_t E_{s_2} := d_t\,(R\langle D\rangle^{s_2} U, \langle D\rangle^{s_2} U)$ ($s_2 = \min(s_0, s_1)$) leads to a term $(\langle D\rangle^{s_2} K\sigma U, \langle D\rangle^{s_2} U)$ which does in general not exist, if $U(.,t) \in H^{s_2}$. Compare the comments in front of Proposition 3.2.2.*

Or, in other words: it is well–known [Dio62] that the assumptions (3.2.2), (3.2.3), (3.2.4) lead to a solution $U \in C\left([0,T], H^{s_2}\right)$ in the strictly hyperbolic case $\sigma \equiv 1$. Then the energy $E_{s_2}(t)$ is a continuous function of t. However, this energy is in general no C^1 function of t. Hence, one can not expect the estimate from the part (b) to hold for $s = s_2$.

Remark 3.2.4. *The proposition has been proved for the case that M be a torus. If $s = 0$, then an a–priori estimate of U can be proved in the case $M = \mathbb{R}^n$, too. This will be done in Proposition 5.3.3. Such an estimate will be used to study domains of dependence.*
If one is interested in Sobolev solutions to (3.1.1), (3.1.2) in the case $M = \mathbb{R}^n$, then different methods should be applied, e.g. the construction of the parametrix, see [Yag97a].

3.3 The Quasilinear Case

We reflect upon the weakly hyperbolic Cauchy problem (3.1.1). The weight functions $c_{k,\beta} = c_{k,\beta}(x,t)$ are assumed to satisfy

$$c_{k,\beta} \in C^1\left([t_0,T], H^{s_0+|\beta|}\right), \tag{3.3.1}$$

$$\|c_{k,\beta,t}(.,t)v(.)\|_{H^{s+|\beta|}} \le C\,\|c_{k,\beta}(.,t)v(.)\|_{H^{s+|\beta|}}, \quad k+|\beta| \le m-2, \tag{3.3.2}$$

$$\|c_{k,\beta}(.,t)v(.)\|_{H^{s+|\beta|}} \le C\,\|c_{k+1,\beta}(.,t)v(.)\|_{H^{s+|\beta|}}, \quad k+|\beta| \le m-2, \tag{3.3.3}$$

$$c_{k,\beta}(x,t) = \sigma(x)^{|\beta|}, \quad k+|\beta| = m-1, \tag{3.3.4}$$

for $s_0 > 1 + n/2$, $s_0 \ge s \ge 0$. These conditions are called *Levi conditions*. Examples for such functions can be found in Section 4.5.
We suppose that the coefficients and the right–hand side are defined in a suitable neighbourhood K_G of the initial data,

$$K_G := \{(x, \{v_{k,\beta}\}) \in M \times \mathbb{R}^{n_0} : |v_{k,\beta}(x) - D_x^\beta(c_{k,\beta}(x,t_0)\varphi_k(x))| \le G\}. \tag{3.3.5}$$

Further, we assume (3.2.1) and

$$a_{j,\alpha} \in C^1([t_0,T], C^{s_0}(K_G)), \tag{3.3.6}$$

$$\varphi_j \in H^{s_0+m-1-j}(M), \tag{3.3.7}$$

$$f \in C([t_0,T], C^{s_0}(K_G)). \tag{3.3.8}$$

Additionally, let the following condition hold:

Condition 2. *The roots* $\tau_j(t,x,v,\xi)$ *of*

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,v)\xi^\alpha \tau^j = 0$$

are real and distinct,

$$|\tau_j(t,x,v,\xi) - \tau_i(t,x,v,\xi)| \geq c|\xi|, \quad c>0, \quad i \neq j,$$

for all

$$(t,x,v,\xi) \in [t_0,T] \times K_G \times \mathbb{R}^n.$$

The main result of this section is the following theorem:

Theorem 3.3.1. *Under the above assumptions, the Cauchy problem (3.1.1) has a uniquely determined solution* u *with*

$$\langle D\rangle^k(\sigma^k D_t^{m-1-k}u) \in C([t_0,T_0], H^{s_0}(M)) \cap C^1([t_0,T_0], H^{s_0-1}(M))$$

for $0 \leq k \leq m-1$. *This solution exists as long as*

$$(x, \{D_x^\beta c_{k,\beta}(x,t) D_t^k u(x,t)\}) \in K_G \quad \forall x$$

and

$$\left\| \langle D\rangle^k(\sigma^k D_t^{m-k-1}u)\right\|_{C_*^1} < \infty, \quad 0 \leq k \leq m-1.$$

The space C_*^1 *is a Zygmund space from Example 2.2.2.*

The proof consists of several propositions. At first, we construct a hyperbolic first order system for the vector of all weighted derivatives up to the order $m-1$. Then a smoothing operator J_ε is inserted in the same way as in the previous section. Employing the ideas from Section 3.2 we prove the existence and estimates of the solution U_ε^* to the perturbed system. Then we show that the length of the existence interval of the U_ε^* does not tend to zero as ε approaches zero. This allows to prove the convergence of the U_ε^* in certain spaces for $\varepsilon \to 0$. The limit is a solution with the asserted smoothness. Finally, the characterisation of the existence interval will be proved. This immediately leads to an existence result in C^∞, see Theorem 3.3.7.

3.3.1 Transformation into a 1^{st} Order System

We define

$$U_{k,\beta}(x,t) := D_x^\beta(c_{k,\beta}(x,t)D_t^k u(x,t)), \quad k+|\beta| \le m-1. \tag{3.3.9}$$

If $k+|\beta| \le m-2$, then we have

$$\partial_t U_{k,\beta}(x,t) = D_x^\beta(c_{k,\beta,t}(x,t)D_t^k u(x,t)) + iD_x^\beta(c_{k,\beta}(x,t)D_t^{k+1}u(x,t)).$$

The right–hand side can be estimated by

$$\begin{aligned} &\left\|c_{k,\beta,t}(.,t)D_t^k u(.,t)\right\|_{H^{s+|\beta|}} + \left\|c_{k,\beta}(.,t)D_t^{k+1}u(.,t)\right\|_{H^{s+|\beta|}} \\ &\quad \le C\left\|c_{k,\beta}(.,t)D_t^k u(.,t)\right\|_{H^{s+|\beta|}} + C\left\|c_{k+1,\beta}(.,t)D_t^{k+1}u(.,t)\right\|_{H^{s+|\beta|}} \\ &\quad \le C\left\|U_{k,\beta}(.,t)\right\|_{H^s} + C\left\|U_{k+1,\beta}(.,t)\right\|_{H^s}, \quad k+|\beta| \le m-3, \end{aligned} \tag{3.3.10}$$

$$\begin{aligned} &\left\|c_{k,\beta,t}(.,t)D_t^k u(.,t)\right\|_{H^{s+|\beta|}} + \left\|c_{k,\beta}(.,t)D_t^{k+1}u(.,t)\right\|_{H^{s+|\beta|}} \\ &\quad \le C\left\|c_{k,\beta}(.,t)D_t^k u(.,t)\right\|_{H^{s+|\beta|}} + C\left\|\sigma(.)^{m-k-2}D_t^{k+1}u(.,t)\right\|_{H^{s+|\beta|}} \\ &\quad \le C\left\|U_{k,\beta}(.,t)\right\|_{H^s} + C\left\|U_{k+2}(.,t)\right\|_{H^s}, \quad k+|\beta| = m-2, \end{aligned} \tag{3.3.11}$$

see (3.3.1)–(3.3.4). We define the vector

$$U^* = (\{U_{k,\beta}\},\ U^T)^T \tag{3.3.12}$$

and obtain

$$d_tU^* = \begin{pmatrix} 0 & 0 \\ 0 & K(x,t,U^*,D) \end{pmatrix}(\sigma U^*) \quad + \begin{pmatrix} 0 & 0 \\ 0 & B(x,t,U^*,D) \end{pmatrix}U^* + \begin{pmatrix} G(x,t,U^*) \\ F(x,t,U^*) \end{pmatrix}$$

where G is bounded as mapping from H^s into H^s, see (3.3.10), (3.3.11). This system can be rewritten as

$$\begin{aligned} d_tU^* &= K^*(x,t,U^*,D)(\sigma U^*) + B^*(x,t,U^*,D)U^* + F^*(x,t,U^*), \\ U^*(t_0) &= \Phi^*. \end{aligned} \tag{3.3.13}$$

The matrix

$$R^*(x,t,U^*,D) = \begin{pmatrix} E & 0 \\ 0 & R(x,t,U^*,D) \end{pmatrix}$$

is a symmetrizer for K^*, where R is the symmetrizer from Section 3.2. From now on, we consider a regularised version of (3.3.13),

$$\begin{aligned} d_tU_\varepsilon^* &= J_\varepsilon K^*(x,t,U_\varepsilon^*,D)(\sigma U_\varepsilon^*) + B^*(x,t,U_\varepsilon^*,D)U_\varepsilon^* + F^*(x,t,U_\varepsilon^*), \\ U_\varepsilon^*(t_0) &= \Phi^*. \end{aligned} \tag{3.3.14}$$

3.3.2 A–priori Estimates and Common Existence Interval

The system (3.3.14) is a Banach space ODE. Hence, it has a solution

$$U_\varepsilon^* \in C^1\left([t_0, T_\varepsilon], H^s\right)$$

which persists as long as it stays in K_G and as long as $\|U_\varepsilon^*\|_{H^s} < \infty$. Applying Proposition 3.2.2 and $C^{1,\alpha} \subset C^1_{\sharp,K_0}$ we get

$$\begin{aligned}
&d_t\left(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*\right) \\
&\quad \le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0})\, \|U_\varepsilon^*\|_{H^s}^2 \\
&\qquad + 2\sqrt{(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*)}\sqrt{(R^*\langle D\rangle^s F^*, \langle D\rangle^s F^*)} \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1)\, \|U_\varepsilon^*\|_{H^s}^2 \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{H^s} + 1)\, \|U_\varepsilon^*\|_{H^s}\, \|U_\varepsilon^*\|_{C_b^{1,\alpha}}
\end{aligned}$$

for $s_0 \ge s > 1 + n/2 + \alpha$. The Moser–type estimates

$$\begin{aligned}
&\|a_{j,\alpha}\|_{C_b^1} \le C(\|U_\varepsilon^*\|_{C_b^0} + 1)(\|U_\varepsilon^*\|_{C_b^1} + 1), \\
&\|a_{j,\alpha}\|_{H^s} \le C(\|U_\varepsilon^*\|_{L^\infty})(\|U_\varepsilon^*\|_{H^s} + 1)
\end{aligned}$$

and the embedding inequality $\|U_\varepsilon^*\|_{C_b^{1,\alpha}} \le C\,\|U_\varepsilon^*\|_{H^s}$ can be applied on the right. Let us consider the term $C(\max_{j,\alpha}\|\partial_t a_{j,\alpha}\|_{C_b^0})$ which denotes some constant that depends in a nonlinear way on $\max_{j,\alpha}\|\partial_t a_{j,\alpha}\|_{C_b^0}$. The computations which lead to this term show that it has the form

$$C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \cdot \max_{j,\alpha}(\|\partial_t a_{j,\alpha}\|_{C_b^0} + 1)$$

where the first factor stands for some constant that depends in a nonlinear way on $\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}$. It is enough to study the last factor. We see that

$$\begin{aligned}
\|\partial_t a_{j,\alpha}\|_{C_b^0} &\le C(1 + \|\partial_t U_\varepsilon^*\|_{C_b^0}) \\
&\le C(1 + \|J_\varepsilon K^* \sigma U_\varepsilon^*\|_{C_b^0} + \|B^* U_\varepsilon^*\|_{C_b^0} + \|F^*\|_{C_b^0}) \\
&\le C(1 + \|\sigma U_\varepsilon^*\|_{C^1_{\sharp,K_1}} + \|U_\varepsilon^*\|_{C^0_{\sharp,K_0}}) \\
&\le C(1 + \|U_\varepsilon^*\|_{C_b^{1,\alpha}})
\end{aligned}$$

cf. Lemma 2.3.6, (3.2.13), Lemma 2.4.6, (2.4.4) and the proof of (3.2.16). Taking into account all these inequalities we obtain

$$\begin{aligned}
d_t\left(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*\right) &\le C(\|U_\varepsilon^*\|_{L^\infty})(\|U_\varepsilon^*\|_{C_b^{1,\alpha}} + 1)\, \|U_\varepsilon^*\|_{H^s}^2 \\
&\quad + 2\sqrt{(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*)}\sqrt{(R^*\langle D\rangle^s F^*, \langle D\rangle^s F^*)}.
\end{aligned} \tag{3.3.15}$$

In the next step we show that there is a common existence interval of the solutions U_ε^*, that is to say

$$\bigcap_{0<\varepsilon\leq\varepsilon_0} [t_0, T_\varepsilon] \neq \{t_0\}.$$

We define $t_\varepsilon \in (t_0, T_\varepsilon]$ by the inequality

$$\|U_\varepsilon^*(t)\|_{H^s} \leq 2\|\Phi^*\|_{H^s} + 1, \quad t_0 \leq t \leq t_\varepsilon$$

and by the condition that the components of the vector $U_\varepsilon^*(t)$ be in the interior of the domain of definition of the coefficients $a_{j,\alpha}$ and the right–hand side f, if $t_0 \leq t \leq t_\varepsilon$.

To obtain a contradiction, let us assume that for every $\gamma > 0$ an $\varepsilon = \varepsilon(\gamma)$ exists with $t_0 < t_\varepsilon \leq t_0 + \gamma$. Now we study estimates of $\|U_\varepsilon^*\|_{H^s}$ and $\|U_\varepsilon^* - \Phi^*\|_{L^\infty}$. The norms $\|V\|_{L^2}$ and $\sqrt{(R^*V, V)}$ are equivalent as long as R^* is defined (i.e., for $t \leq t_\varepsilon$),

$$C_{R,\sim}^{-1}\|V\|_{L^2}^2 \leq (R^*V, V) \leq C_{R,\sim}\|V\|_{L^2}^2. \tag{3.3.16}$$

From (3.3.15) and the estimate $\|F^*\|_{H^s} \leq C(\|U^*\|_{L^\infty})(\|U^*\|_{H^s}+1)$ we see that

$$d_t(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*) \leq Q((R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*)),$$

where Q is a smooth nonlinear increasing function, independent of ε. Let T_0 be a number with the property that the nonnegative solutions of the differential inequality

$$\begin{aligned} &d_t y(t) \leq Q(y(t)), \\ &y(t_0) = (R^*(x, t_0, \Phi^*, D)\langle D\rangle^s\Phi^*, \langle D\rangle^s\Phi^*) \end{aligned} \tag{3.3.17}$$

satisfy $y(t) \leq 2\|\Phi^*\|_{H^s} + 1$ for $t_0 \leq t \leq T_0$. To estimate $\|U_\varepsilon^* - \Phi^*\|_{L^\infty}$, we write $U_\varepsilon^* = \Phi^* + V_\varepsilon^*$ and get

$$\begin{aligned} d_t V_\varepsilon^* = {} & J_\varepsilon K^*(x, t, U_\varepsilon^*, D)(\sigma V_\varepsilon^*) + B^*(x, t, U_\varepsilon^*, D)U_\varepsilon^* + F^*(x, t, U_\varepsilon^*) \\ & + J_\varepsilon K^*(x, t, U_\varepsilon^*, D)(\sigma\Phi^*), \end{aligned}$$

hence, by Proposition 3.2.2 and Young's inequality,

$$\begin{aligned} &d_t(R^*V_\varepsilon^*, V_\varepsilon^*) \leq C(\|U_\varepsilon^*\|_{L^\infty})((R^*V_\varepsilon^*, V_\varepsilon^*) + (R^*F^*, F^*)), \\ &(R^*V_\varepsilon^*, V_\varepsilon^*)(t_0) = 0. \end{aligned}$$

The norms $\|U_\varepsilon^*\|_{L^\infty}$ are uniformly bounded for $t \leq t_\varepsilon$, see the definition of t_ε. From Gronwall's Lemma it can be concluded that

$$(R^*V_\varepsilon^*, V_\varepsilon^*)(t) \leq g(t)^2, \quad t_0 \leq t \leq \min(T_0, t_\varepsilon),$$

$g(t_0) = 0$, g continuous and increasing. We obtain $\|V_\varepsilon^*(t)\|_{L^2}^2 \le Cg(t)^2$; and the Interpolation Theorem of Nirenberg–Gagliardo gives

$$\begin{aligned}\|V_\varepsilon^*(t)\|_{L^\infty} &\le C\,\|V_\varepsilon^*(t)\|_{H^{s-1}}\\ &\le Cg(t)^\theta\,\|V^*(t)\|_{H^s}^{1-\theta} \le Cg(t)^\theta\left(\|U_\varepsilon^*(t)\|_{H^s} + \|\Phi^*\|_{H^s}\right)^{1-\theta}\\ &\le Cg(t)^\theta\left(3\,\|\Phi^*\|_{H^s}+1\right)^{1-\theta} =: g_1(t) \end{aligned}\tag{3.3.18}$$

with some real number θ between 0 and 1. This proves that t_ε cannot come arbitrary close to t_0, which is a contradiction. Hence, there is a common existence interval.

We have proved:

Lemma 3.3.2. *There is a constant $T_0 > t_0$ with the property that the systems (3.3.14) have unique solutions*

$$U_\varepsilon^* \in C^1\left([t_0, T_0], H^s\right)$$

for $0 < \varepsilon \le \varepsilon_0$ and $s_0 \ge s > 1 + n/2$. It holds

$$\begin{aligned}&\|U_\varepsilon^*(t)\|_{H^s} \le C \quad \forall \varepsilon, t,\\ &\|U_\varepsilon^*(t) - \Phi^*\|_{L^\infty} \le g_1(t)\end{aligned}$$

with some continuous function $g_1(t)$, $g_1(t_0) = 0$.

3.3.3 Convergence and Regularity of the Limit

Let us contemplate on convergence properties for $\varepsilon \to 0$. It holds

$$\begin{aligned}
&d_t(U_\varepsilon^* - U_{\varepsilon'}^*)\\
&\quad= J_\varepsilon K^*(x,t,U_\varepsilon^*,D)(\sigma U_\varepsilon^*) - J_{\varepsilon'} K^*(x,t,U_{\varepsilon'}^*,D)(\sigma U_{\varepsilon'}^*)\\
&\qquad + B^*(x,t,U_\varepsilon^*,D)U_\varepsilon^* - B^*(x,t,U_{\varepsilon'}^*,D)U_{\varepsilon'}^*\\
&\qquad + F^*(x,t,U_\varepsilon^*) - F^*(x,t,U_{\varepsilon'}^*)\\
&\quad= I_1 + I_2 + I_3 + I_4 + I_5 + I_6\\
&\quad= (J_\varepsilon - J_{\varepsilon'})K^*(x,t,U_\varepsilon^*,D)(\sigma U_\varepsilon^*)\\
&\qquad + J_{\varepsilon'}(K^*(x,t,U_\varepsilon^*,D) - K^*(x,t,U_{\varepsilon'}^*,D))(\sigma U_\varepsilon^*)\\
&\qquad + J_{\varepsilon'} K^*(x,t,U_{\varepsilon'}^*,D)\sigma(U_\varepsilon^* - U_{\varepsilon'}^*)\\
&\qquad + (B^*(x,t,U_\varepsilon^*,D) - B^*(x,t,U_{\varepsilon'}^*,D))U_\varepsilon^*\\
&\qquad + B^*(x,t,U_{\varepsilon'}^*,D)(U_\varepsilon^* - U_{\varepsilon'}^*) + F^*(x,t,U_\varepsilon^*) - F^*(x,t,U_{\varepsilon'}^*).
\end{aligned}$$

We obtain from Corollary 2.3.5 and $s > 1 + n/2$

$$\|I_1\|_{L^2} \le C(\varepsilon + \varepsilon')\,\|K^*(x,t,U_\varepsilon^*,D)\sigma U_\varepsilon^*\|_{H^1} \le C(\|U_\varepsilon^*\|_{H^s})(\varepsilon + \varepsilon').$$

By the special structure of K^* (see Subsection 3.2.1 and Subsection 3.2.2), Hadamard's Formula, Lemma 2.4.6 and $H^s \subset C^{1,\alpha} \subset C^1_{\sharp,K_0}$ $(s > \frac{n}{2}+1+\alpha)$, we get

$$\|I_2\|_{L^2} \leq C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{L^2}\,\|\sigma U^*_\varepsilon\|_{C^1_{\sharp,K_1}} \leq C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{L^2}\,.$$

In a similar way it follows that

$$\|I_4\|_{L^2} \leq C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{L^2}\,\|U^*_\varepsilon\|_{C^0_{\sharp,K_0}}\,,$$
$$\|I_5\|_{L^2} \leq C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{L^2}\,,$$

see the proof of (3.2.19). Finally, by Hadamard's Formula,

$$\|I_6\|_{L^2} \leq C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{L^2}\,.$$

Summing up results in

$$\begin{aligned} &d_t(U^*_\varepsilon - U^*_{\varepsilon'}) = G(\varepsilon,\varepsilon',U^*_\varepsilon,U^*_{\varepsilon'}) + J_{\varepsilon'}K^*(x,t,U^*_{\varepsilon'},D)\sigma(U^*_\varepsilon - U^*_\varepsilon),\\ &\|G\|_{L^2} \leq C(\varepsilon+\varepsilon') + C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{L^2}\,. \end{aligned}$$

From Proposition 3.2.2 it can be concluded that

$$\begin{aligned} d_t\left(R^*(U^*_\varepsilon - U^*_{\varepsilon'}), U^*_\varepsilon - U^*_{\varepsilon'}\right) &\leq C\,\|U^*_\varepsilon - U^*_{\varepsilon'}\|^2_{L^2} + C(\varepsilon+\varepsilon')\\ &\leq C\left(R^*(U^*_\varepsilon - U^*_{\varepsilon'}), U^*_\varepsilon - U^*_{\varepsilon'}\right) + C(\varepsilon+\varepsilon'). \end{aligned}$$

By Gronwall's Lemma and $(U^*_\varepsilon - U^*_{\varepsilon'})(t_0) = 0$, we have

$$\left(R^*_\varepsilon(U^*_\varepsilon - U^*_{\varepsilon'}), U^*_\varepsilon - U^*_{\varepsilon'}\right)(t) \leq C\int_{t_0}^t (\varepsilon+\varepsilon')\,dt \leq C(T_0 - t_0)(\varepsilon+\varepsilon').$$

Since the U^*_ε lie in K_G, the inequalities (3.3.16) hold with a constant $C_{R,\sim}$ independent of ε. This yields

$$\|U^*_\varepsilon - U^*_{\varepsilon'}\|^2_{L^2} \leq C(T_0 - t_0)(\varepsilon+\varepsilon')$$

for all $t \in [t_0, T_0]$. By the uniform bound $\|U^*_\varepsilon\|_{H^{s_0}} \leq C$ and interpolation, it follows that

$$\|U^*_\varepsilon - U^*_{\varepsilon'}\|_{H^s} \leq C(\varepsilon+\varepsilon')^\theta, \quad \frac{n}{2}+1+\alpha < s < s_0, \quad \theta = \frac{1}{2}\left(1 - \frac{s}{s_0}\right).$$

Thus, the sequence (U^*_ε) is a Cauchy sequence in $C\left([t_0,T_0],H^s\right)$ and in $C([t_0,T_0],C^{1,\alpha}_b)$. Hence we have proved:

Lemma 3.3.3. *The above sequence* $(U^*_\varepsilon) \subset C^1([t_0, T_0], H^{s_0})$ *converges in*

$$C([t_0, T_0], H^s) \text{ and } C([t_0, T_0], C_b^{1,\alpha})$$

for any s *and* α *with* $1 + n/2 + \alpha < s < s_0$. *The limit* U^* *belongs to* $C^1([t_0, T_0], H^{s-1})$ *and is a solution of (3.3.13).*

It remains to study the regularity of the solution U^*. Here we make use of standard techniques, which can be found e.g. in [Rac92]. The uniform estimate of U^*_ε in H^{s_0} gives

$$U^* \in L^\infty([t_0, T_0], H^{s_0}) \cap Lip^1([t_0, T_0], H^{s_0-1}).$$

We fix $t_0 \le t_1 < T_0$ and consider the forward Cauchy problem (recycling the variable U^*_ε which we do not need anymore)

$$\begin{aligned} d_t U^*_\varepsilon =& J_\varepsilon K^*(x, t, U^*_\varepsilon, D)(\sigma U^*_\varepsilon) + B^*(x, t, U^*_\varepsilon, D)U^*_\varepsilon + F^*(x, t, U^*_\varepsilon), \\ U^*_\varepsilon(t_1) =& U^*(t_1). \end{aligned}$$

From (3.3.15) and (3.3.16) we deduce that

$$\begin{aligned} &d_t \left(R^* \langle D\rangle^{s_0} U^*_\varepsilon(t), \langle D\rangle^{s_0} U^*_\varepsilon(t)\right) \\ &\qquad \le C'(\|U^*_\varepsilon\|_{C_b^{1,\alpha}} + 1) \left(R^* \langle D\rangle^{s_0} U^*_\varepsilon(t), \langle D\rangle^{s_0} U^*_\varepsilon(t)\right) + C \|F^*(t)\|^2_{H^{s_0}}. \end{aligned}$$

Defining the norm

$$\|V\|^2_{H^{s_0},t_1} := (R^*(t_1)\langle D\rangle^{s_0} V, \langle D\rangle^{s_0} V) \tag{3.3.19}$$

we obtain

$$d_t \|U^*_\varepsilon(t)\|^2_{H^{s_0},t} \le C'(\|U^*_\varepsilon\|_{C_b^{1,\alpha}} + 1) \|U^*_\varepsilon(t)\|^2_{H^{s_0},t} + C \|F^*(t)\|^2_{H^{s_0}}.$$

Gronwall's Lemma gives

$$\begin{aligned} &\|U^*_\varepsilon(t)\|^2_{H^{s_0},t} \le \|U^*(t_1)\|^2_{H^{s_0},t_1} e^{C'(t-t_1)} + C \int_{t_1}^t e^{C'(t-\tau)} \|F^*(\tau)\|^2_{H^{s_0}} \, d\tau, \\ &C' = C'(\|U^*_\varepsilon\|_{C_b^{1,\alpha}} + 1). \end{aligned}$$

The right–hand side does not depend on ε. Then the weak compactness of bounded subsets in Hilbert spaces implies ($\varepsilon \to 0$)

$$\|U^*(t)\|^2_{H^{s_0},t} \le \|U^*(t_1)\|^2_{H^{s_0},t_1} e^{C'(t-t_1)} + C \int_{t_1}^t e^{C'(t-\tau)} \|F^*(\tau)\|^2_{H^{s_0}} \, d\tau.$$

The function $\|V\|_{H^{s_0},t}$ defines an equivalent norm in the Hilbert space H^{s_0}, if t is fixed. For proving continuity in t we need a norm which does not depend on t. For this purpose we rewrite the left–hand side:

$$\begin{aligned}(R^*(t)\langle D\rangle^{s_0}U^*(t), \langle D\rangle^{s_0}U^*(t)) &= (R^*(t_1)\langle D\rangle^{s_0}U^*(t), \langle D\rangle^{s_0}U^*(t)) \\ &\quad + ((R^*(t) - R^*(t_1))\langle D\rangle^{s_0}U^*(t), \langle D\rangle^{s_0}U^*(t))\end{aligned}$$

and have the estimate

$$\begin{aligned}&|\,((R^*(t) - R^*(t_1))\langle D\rangle^{s_0}U^*(t), \langle D\rangle^{s_0}U^*(t))\,| \\ &\quad \le C(|t-t_0| + \|U^*(t) - U^*(t_1)\|_{L^\infty})\,\|U^*(t)\|^2_{H^{s_0}} \\ &\quad \le C(1 + \|U^*\|_{C^1([t_0,T_0],H^{s-1})})\,\|U^*(t)\|^2_{H^{s_0}}\,|t-t_1|,\end{aligned}$$

if $s_0 \ge s > 1 + n/2$. This implies

$$\|U^*(t)\|^2_{H^{s_0},t_1} \le \|U^*(t_1)\|^2_{H^{s_0},t_1}\, e^{C'(t-t_1)} + C''|t-t_1|,$$

resulting in

$$\begin{aligned}\limsup_{t\to t_1+0} \|U^*(t)\|^2_{H^{s_0},t_1} &\le \limsup_{t\to t_1+0} \left(\|U^*(t_1)\|^2_{H^{s_0},t_1}\, e^{C'(t-t_1)} + C''|t-t_1|\right) \\ &= \|U^*(t_1)\|^2_{H^{s_0},t_1} \le \liminf_{t\to t_1+0} \|U^*(t)\|^2_{H^{s_0},t_1}\,,\end{aligned}$$

which gives the H^{s_0}–continuity of U^* at t_1 from the right. The following facts from the functional analysis [Heu92] (nr. 27 and nr. 59) have been used here: let H be a Hilbert space. Then

$$f_n \rightharpoonup f \text{ in } H \Longrightarrow \|f\|_H \le \liminf_n \|f_n\|_H\,,$$

$$f_n \rightharpoonup f \text{ in } H, \quad \|f_n\|_H \to \|f\|_H \Longrightarrow f_n \to f.$$

Inverting the time direction the reader can show the continuity from the left. Thus, we have proved:

$$U^* \in C\left([t_0,T_0], H^{s_0}\right) \cap C^1\left([t_0,T_0], H^{s_0-1}\right). \tag{3.3.20}$$

In the last step a criterion for the blow–up is given. The idea of the proof is taken from [Tay91], Proposition 5.1.F.

Proposition 3.3.4. *Let* $U^* \in C\left([t_0,T), H^{s_0}\right) \cap C^1\left([t_0,T), H^{s_0-1}\right)$ *be a solution of (3.3.13) with*

$$\sup_{[t_0,T)} \|U^*(t)\|_{C^1_*} < \infty,$$

$$\inf_{[t_0,T)} \operatorname{dist}((x, \{U_{k,\beta}(x,t)\}), \partial K_G) \ge \delta > 0.$$

Then a constant $T_1 > T$ *exists with*

$$U^* \in C\left([t_0,T_1], H^{s_0}\right) \cap C^1\left([t_0,T_1], H^{s_0-1}\right).$$

Proof. We multiply (3.3.13) with J_ε from the left and estimate the terms on the right as in the proof of Proposition 3.2.2. This gives

$$d_t \left(R^*\langle D\rangle^{s_0} J_\varepsilon U^*, \langle D\rangle^{s_0} J_\varepsilon U^*\right) \\ \leq C(\|U^*\|_{L^\infty})(1 + \|U^*\|_{C_b^1} + \|U^*\|_{C^1_{\sharp,K_0}})\,\|U^*\|^2_{H^{s_0}} + \|F^*\|^2_{H^{s_0}}\,.$$

We suppose that the H^{s_0} norm is arranged in such a way that $\|V^*\|_{C^1_*} \leq \|V^*\|_{H^{s_0}}$ holds for every function $V^* \in H^{s_0}$. Then the inequality

$$\|V^*\|_{C^1_{\sharp,K_0}} \leq C\,\|V^*\|_{C^1_*}\left(1 + \ln\left(\frac{\|V^*\|_{H^{s_0}}}{\|V^*\|_{C^1_*}}\right)\right)$$

can be shown, see [Tay91], (B.2.12). Consequently,

$$\|U^*\|_{C^1_{\sharp,K_0}} \leq C\,\|U^*\|_{C^1_*}(1 + \ln^+\|U^*\|_{H^{s_0}}) + C.$$

From $\|U^*\|_{C_b^1} \leq C\,\|U^*\|_{C^1_{\sharp,K_0}}$ (see Proposition 2.4.6) and $\|F^*\|^2_{H^{s_0}} \leq C(\|U^*\|_{L^\infty})(e + \|U^*\|^2_{H^{s_0}})$ it follows that

$$d_t \left(R^*\langle D\rangle^{s_0} J_\varepsilon U^*, \langle D\rangle^{s_0} J_\varepsilon U^*\right) \\ \leq C(\|U^*\|_{L^\infty})(1 + \|U^*\|_{C^1_*})(1 + \ln^+\|U^*\|^2_{H^{s_0}})(e + \|U^*\|^2_{H^{s_0}}).$$

Using the equivalent norm $\|.\|_{H^{s_0},t}$ from (3.3.19) and $\|U^*\|_{C^1_*} \leq C$ we get

$$d_t\,\|J_\varepsilon U^*(t)\|_{H^{s_0},t} \leq C_0(1 + \ln^+\|U^*(t)\|^2_{H^{s_0},t})(e + \|U^*(t)\|_{H^{s_0},t}).$$

We integrate, let ε tend to 0 and see that

$$\|U^*(t)\|^2_{H^{s_0},t} \leq \|U^*(t_0)\|^2_{H^{s_0},t_0} \\ + C_0\int_{t_0}^t (1 + \ln^+\|U^*(\tau)\|^2_{H^{s_0},\tau})(e + \|U^*(\tau)\|^2_{H^{s_0},\tau})\,d\tau.$$

Introducing $N(t) = e + \|U^*(t)\|^2_{H^{s_0},t}$ we deduce that

$$N(t) \leq N(t_0) + 2C_0\int_{t_0}^t \ln(N(\tau))N(\tau)\,d\tau =: Q(t).$$

Since $N(t)$ is continuous, $Q(t) \in C^1[t_0, T)$ and

$$Q'(t) = 2C_0\ln(N(t))N(t) \leq 2C_0\ln(Q(t))Q(t),$$

hence

$$Q(t) \leq Q(t_0)^{e^{2C_0(t-t_0)}} < C \quad \forall t_0 \leq t < T.$$

Taking into account all these inequalities gives $\|U^*(t)\|_{H^{s_0}} \leq C'$ for $t_0 \leq t < T$. Let us consider the Cauchy problems (3.3.14) with data

$$U^*_\varepsilon(T-\gamma) = U^*(T-\gamma)$$

for some small $\gamma > 0$. The vector functions $U^*_\varepsilon(t)$ persist as long as their H^{s_0} norm remains bounded and as long as each component of these vectors stays in the domain of the functions $a_{j,\alpha}$ and f. The length of the common existence interval of the U^*_ε is determined by (3.3.17) and (3.3.18). But these are autonomous (differential) inequalities (independent of ε), hence the length of the interval only depends on δ and C'. It follows that for small γ the point T is contained in the common existence interval of the U^*_ε, and consequently, of the limit function U. □

This completes the proof of Theorem 3.3.1.

Remark 3.3.5. *The last proposition states a connection between blow–up in the H^{s_0} norm and blow–up in the C^1_* norm. We emphasise that exactly the same connection exists in the* strictly *hyperbolic theory, cf. [Tay91], Proposition 5.1.F and Theorem 5.3.A.*

If the equation is linear, then we have *global* existence:

Corollary 3.3.6. *Let us consider the Cauchy problem (3.1.1). We assume that the coefficients $a_{j,\alpha}$ only depend on (x,t) and that the right–hand side f depends on $\{D^\beta_x c_{k,\beta} D^k_t u\}$ in a linear way,*

$$f(x,t,\{D^\beta_x c_{k,\beta} D^k_t u\}) = f^*(x,t) + \sum_{k+|\beta|\leq m-1} f_{k,\beta}(x,t) D^\beta_x c_{k,\beta} D^k_t u.$$

We suppose that Condition 1 and (3.2.1), (3.2.2), (3.2.3) (with $s_1 > n/2 + 1$), (3.2.4) (with f replaced by f^*) and (3.3.1)–(3.3.4) are true. Then (3.1.1) has a global solution u with

$$\langle D\rangle^k(\sigma^k D^{m-1-k}_t u) \in C([t_0,T], H^{s_2}(M)) \cap C^1([t_0,T], H^{s_2-1}(M))$$

for $0 \leq k \leq m-1$ and $s_2 := \min(s_0, s_1)$.

Proof. Utilising the embedding $H^{s_2} \subset C^1_{\sharp,K_0}$ one can prove the estimate

$$d_t\left(R^*\langle D\rangle^{s_2} J_\varepsilon U^*, \langle D\rangle^{s_2} J_\varepsilon U^*\right) \leq C(\|U^*\|^2_{H^{s_2}} + \|F^*\|^2_{H^{s_2}}),$$

see the proof of Proposition 3.2.2. The constant C does not depend on U^*. Hence, a blow–up is impossible. □

We complete this section with a local existence result in C^∞.

Theorem 3.3.7. *Let us consider the Cauchy problem (3.1.1) on the set* $M \times [t_0, T]$ *with* M *being a torus. We suppose (3.2.1) and*

$$
\begin{aligned}
&c_{k,\beta} \in C^1([t_0, T], C^\infty(M)),\\
&a_{j,\alpha} \in C^1([t_0, T], C^\infty(K_G)),\\
&\varphi_j \in C^\infty(M),\\
&f \in C([t_0, T], C^\infty(K_G)).
\end{aligned}
$$

Furthermore, we assume that (3.3.2)–(3.3.4) hold for all $s \geq 0$ *and that Condition 2 holds. Then the Cauchy problem (3.1.1) has a solution*

$$u \in C^m([t_0, T_1), C^\infty(M)).$$

Proof. We fix some $s_0 > 1 + n/2$. From Theorem 3.3.1 we deduce that a solution u exists with

$$u \in C^{m-1}([t_0, T_{s_0}], H^{s_0}(M)) \cap C^m([t_0, T_{s_0}], H^{s_0-1}(M)).$$

The embedding $H^{s_0} \subset C^1_*$ and the definition of the vector U^*, cf. (3.3.9), (3.3.12), show that $\|U^*(t)\|_{C^1_*} \leq C$ for $t_0 \leq t \leq T_{s_0}$. Let us take an arbitrary $s > s_0$. Then Proposition 3.3.4 reveals that

$$u \in C^{m-1}([t_0, T_{s_0}], H^s(M)) \cap C^m([t_0, T_{s_0}], H^{s-1}(M)) \quad \forall s_0 \leq s < \infty,$$

which results in $u \in C^m([t_0, T_{s_0}], C^\infty(M))$.
We can repeat this procedure with t_0 replaced by T_{s_0}. Following this way we can complete the proof. □

3.4 Stability of Solutions and Life–Span Estimates

The aim of this subsection is to show that the solution of a quasilinear weakly hyperbolic Cauchy problem continuously depends on the data, weight functions, coefficients and right–hand side. This result will be used to derive a life–span estimate.
We consider the Cauchy problem (3.1.1), i.e.,

$$
\begin{aligned}
&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x, t, \{D_x^\beta c_{k,\beta}(x,t) D_t^k u\}) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right)\\
&\quad = f(x, t, \{D_x^\beta c_{k,\beta}(x,t) D_t^k u\}), \quad k + |\beta| \leq m - 1, \qquad (3.4.1)\\
&u(x, t_0) = \varphi_0(x), \ldots, D_t^{m-1} u(x, t_0) = \varphi_{m-1}(x). \qquad (3.4.2)
\end{aligned}
$$

Additionally, let us examine a family of Cauchy problems

$$D_t^m u^\varepsilon + \sum_{j+|\alpha|=m, j<m} a^\varepsilon_{j,\alpha}(x,t,\{D_x^\beta c^\varepsilon_{k,\beta}(x,t)D_t^k u^\varepsilon\}) D_x^\alpha D_t^j \left(\sigma^\varepsilon(x)^{|\alpha|} u^\varepsilon\right)$$
$$= f^\varepsilon(x,t,\{D_x^\beta c^\varepsilon_{k,\beta}(x,t)D_t^k u^\varepsilon\}), \quad k+|\beta| \le m-1, \tag{3.4.3}$$
$$u^\varepsilon(x,t_0) = \varphi_0^\varepsilon(x), \ldots, D_t^{m-1} u^\varepsilon(x,t_0) = \varphi^\varepsilon_{m-1}(x) \tag{3.4.4}$$

with $0 < \varepsilon \le \varepsilon_0$. The given functions from (3.4.1) are assumed to satisfy (3.3.1)–(3.3.8) and Condition 2 with $s_0 > m+1+n/2$, $s_0 \ge s \ge 0$ and $\sigma \in C^\infty(M)$.
We suppose that the functions $c^\varepsilon_{k,\beta}$, σ^ε, $a^\varepsilon_{j,\alpha}$, φ^ε_k and f^ε fulfil (3.3.1)–(3.3.4) with $s_0 > m+1+n/2$, $s_0 \ge s \ge 0$ and Condition 2. The constants C are supposed to be independent of ε. Let us assume that the coefficients $a^\varepsilon_{j,\alpha}$ and the right–hand sides f^ε are defined in a neighbourhood *of the initial data of (3.4.1)*,

$$K_G := \{(x,\{v_{k,\beta}\}) \in M \times \mathbb{R}^{n_0} : |v_{k,\beta}(x) - D_x^\beta(c_{k,\beta}(x,t_0)\varphi_k(x))| \le G\}.$$

Using this K_G, we suppose (3.3.6)–(3.3.8) for the functions $a^\varepsilon_{j,\alpha}$, φ^ε_k and f^ε. Finally, we assume $\sigma^\varepsilon \in C^\infty(M)$. Let the functions $c^\varepsilon_{k,\beta}$, σ^ε, $a^\varepsilon_{j,\alpha}$, φ^ε_k and f^ε be close to $c_{k,\beta}$, σ, $a_{j,\alpha}$, φ_k and f, respectively. More precise,

$$\left\|c^\varepsilon_{k,\beta} - c_{k,\beta}\right\|_{C^1\left([t_0,T],H^{s_0+|\beta|}\right)} \le \varepsilon,$$
$$\|\sigma^\varepsilon - \sigma\|_{C^{s_0+m}} \le \varepsilon,$$
$$\left\|a^\varepsilon_{j,\alpha} - a_{j,\alpha}\right\|_{C^1([t_0,T],C^{s_0}(K_G))} \le \varepsilon,$$
$$\|\varphi^\varepsilon_k - \varphi_k\|_{H^{s_0+m-1-k}} \le \varepsilon,$$
$$\|f^\varepsilon - f\|_{C([t_0,T],C^{s_0}(K_G))} \le \varepsilon.$$

These conditions allow to prove the continuous dependence of the solutions:

Theorem 3.4.1. *We suppose that the above conditions are fulfilled. Let u be a solution of (3.4.1), (3.4.2) with*

$$\langle D\rangle^k(\sigma^k D_t^{m-1-k}u) \in C([t_0,T_0],H^{s_0}(M)) \cap C^1([t_0,T_0],H^{s_0-1}(M))$$

for $0 \le k \le m-1$. Then a number $\varepsilon_0 > 0$ exists with the property that for all $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (3.4.3), (3.4.4) has a solution u^ε with

$$\langle D\rangle^k(\sigma^{\varepsilon,k} D_t^{m-1-k}u^\varepsilon) \in C([t_0,T_0],H^{s_0}(M)) \cap C^1([t_0,T_0],H^{s_0-1}(M))$$

for all $0 \le k \le m-1$. The difference $u - u^\varepsilon$ satisfies

$$\|(U^* - U^{\varepsilon,*})(t)\|_{H^{s_0-m}} \le Ce^{C'(t-t_0)}\varepsilon, \quad t_0 \le t \le T_0.$$

The constants C, C' do not depend on ε.

Proof. Using the function u we define a vector valued function $U^*(x,t)$ of weighted lower order derivatives, see (3.3.12) and (3.3.9). Let us set

$$M_0 := \sup_{[t_0,T_0]} \|U^*(t)\|_{C_b^{1,\alpha}}, \quad M_0' := \sup_{[t_0,T_0]} \|U^*(t)\|_{H^{s_0}}.$$

From Theorem 3.3.1 we conclude that the Cauchy problem (3.4.3), (3.4.4) has a solution u^ε which persists up to some T_ε with $t_0 < T_\varepsilon \le T_0$. There is no loss of generality in assuming that

$$\sup_{[t_0,T_\varepsilon]} \|U^{\varepsilon,*}(t)\|_{C_b^{1,\alpha}} \le 2M_0 + 1, \quad \sup_{[t_0,T_\varepsilon]} \|U^{\varepsilon,*}(t)\|_{H^{s_0}} \le 2M_0' + 1,$$

otherwise we shrink the interval $[t_0, T_\varepsilon]$. The difference $u - u^\varepsilon$ satisfies

$$\begin{aligned}
&D_t^m(u-u^\varepsilon) + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta} D_t^k u\}) D_x^\alpha D_t^j \left(\sigma^{|\alpha|}(u-u^\varepsilon)\right) \\
&\quad = f(x,t,\{D_x^\beta c_{k,\beta} D_t^k u\}) - f^\varepsilon(x,t,\{D_x^\beta c_{k,\beta}^\varepsilon D_t^k u^\varepsilon\}) \\
&\qquad + \sum_{j+|\alpha|=m, j<m} \Big(a_{j,\alpha}^\varepsilon(x,t,\{D_x^\beta c_{k,\beta}^\varepsilon D_t^k u^\varepsilon\}) \\
&\qquad - a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta} D_t^k u\})\Big) D_x^\alpha D_t^j \left(\sigma^{|\alpha|} u^\varepsilon\right) \\
&\qquad + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}^\varepsilon(x,t,\{D_x^\beta c_{k,\beta}^\varepsilon D_t^k u^\varepsilon\}) D_x^\alpha D_t^j \left(\sigma^{\varepsilon,|\alpha|} - \sigma^{|\alpha|}\right) u^\varepsilon.
\end{aligned}$$

The right–hand side can be written in the form

$$h^\varepsilon(x,t) + \sum_{k+|\beta|\le m-1} g_{k,\beta}^\varepsilon(x,t) D_x^\beta c_{k,\beta}(x,t) D_t^k (u-u^\varepsilon)$$

with

$$\begin{aligned}
&\|h^\varepsilon\|_{C\left([t_0,T_\varepsilon],H^{s_0-m}\right)} \le C_1(M_0,M_0')\varepsilon, \\
&\left\|g_{k,\beta}^\varepsilon\right\|_{C\left([t_0,T_\varepsilon],H^{s_0-m}\right)} \le C_2(M_0,M_0').
\end{aligned}$$

The constants $C_1(M_0, M_0')$ and $C_2(M_0, M_0')$ do not depend on ε. The Cauchy problem for $u - u^\varepsilon$ can be transformed into a system:

$$\begin{aligned}
&d_t(U^* - U^{\varepsilon,*}) = K^*(x,t,D)(\sigma(U^* - U^{\varepsilon,*})) + B^*(x,t,D)(U^* - U^{\varepsilon,*}) \\
&\qquad\quad + H^\varepsilon(x,t) + G^\varepsilon(x,t,U^* - U^{\varepsilon,*}), \qquad (3.4.5) \\
&(U^* - U^{\varepsilon,*})(t_0) = \Phi^* - \Phi^{\varepsilon,*},
\end{aligned}$$

compare the calculations which led to (3.3.13). It holds

$$\|H^\varepsilon\|_{C\left([t_0,T_\varepsilon],H^{s_0-m}\right)} \le C_1(M_0,M_0')\varepsilon,$$
$$\|G^\varepsilon(x,t,U^*-U^{\varepsilon,*})\|_{C\left([t_0,T_\varepsilon],H^{s_0-m}\right)}$$
$$\le C_3(M_0,M_0')\,\|U^*-U^{\varepsilon,*}\|_{C\left([t_0,T_\varepsilon],H^{s_0-m}\right)}\,.$$

Multiplying (3.4.5) with J_δ from the left leads to

$$d_t\left(R^*\langle D\rangle^{s_0-m}J_\delta(U^*-U^{\varepsilon,*}),\langle D\rangle^{s_0-m}J_\delta(U^*-U^{\varepsilon,*})\right)$$
$$\le C\,\|U^*-U^{\varepsilon,*}\|^2_{H^{s_0-m}}+C\varepsilon^2.$$

We integrate over $[t_0,t]$, let δ tend to 0 and with (3.3.16) we get

$$\|(U^*-U^{\varepsilon,*})(t)\|^2_{H^{s_0-m}}$$
$$\le C\|(U^*-U^{\varepsilon,*})(t_0)\|^2_{H^{s_0-m}}+C\int_{t_0}^t\|(U^*-U^{\varepsilon,*})(\tau)\|^2_{H^{s_0-m}}\,d\tau+C\varepsilon^2$$
$$\le C_4\int_{t_0}^t\|(U^*-U^{\varepsilon,*})(\tau)\|^2_{H^{s_0-m}}\,d\tau+C_5\varepsilon^2.$$

Writing $v(t)=\int_{t_0}^t\|(U^*-U^{\varepsilon,*})(\tau)\|^2_{H^{s_0-m}}\,d\tau$ we have $v'(t)\le C_4v(t)+C_5\varepsilon^2$ and $v(t_0)=0$, hence, by Gronwall's Lemma,

$$v(t)\le\int_{t_0}^t e^{C_4(t-\tau)}C_5\varepsilon^2d\tau=\frac{C_5\varepsilon^2}{C_4}\left(e^{C_4(t-t_0)}-1\right),$$

which implies

$$\|(U^*-U^{\varepsilon,*})(t)\|^2_{H^{s_0-m}}\le C_5\varepsilon^2e^{C_4(t-t_0)}.$$

The constants C_4 and C_5 only depend on M_0 and M_0'. From $s_0>m+1+n/2$ and the Sobolev Embedding Theorem it follows that the norm $\|U^*-U^{\varepsilon,*}\|_{C^1_*}$ can be made arbitrarily small, if ε is small enough. From this, $\sup_{[t_0,T_0]}\|U^*(t)\|_{C^1_*}<\infty$ and Proposition 3.3.4 we see that $U^{\varepsilon,*}$ persists up to $t=T_0$ for small $\varepsilon>0$. The theorem is proved. □

Corollary 3.4.2 (Life–span). *Let $u\equiv0$ be a global solution of (3.4.1) with vanishing data for $t=t_0$. Let u^ε be the solution of (3.4.1) with data $\varepsilon\varphi_0,\ldots,\varepsilon\varphi_{m-1}$. We assume $\|\varphi_k\|_{H^{s_0+m-1-k}}\le1$. Then the life–span of u^ε, ε small, can be estimated as*

$$T_\varepsilon\ge t_0+\frac{1}{C'}\ln\frac{G}{C\varepsilon} \tag{3.4.6}$$

with the constants G,C,C' from (3.3.5) and Theorem 3.4.1.

Proof. We have $\|U^{\varepsilon,*}(t)\|_{H^{s_0-m}} \le Ce^{C'(t-t_0)}\varepsilon$, see Theorem 3.4.1. The coefficients $a_{j,\alpha}$ and the right–hand side are defined in the set

$$K_G = \{(x, \{v_{k,\beta}\}) \in M \times \mathbb{R}^{n_0} : |v_{k,\beta}| \le G\}$$

see (3.3.5). The solution u^ε exists as long as $\|U^{\varepsilon,*}\|_{C^1_*} < \infty$ and $\|U^{\varepsilon,*}\|_{L^\infty} < G$. This implies

$$\lim_{t \to T_\varepsilon - 0} \|U^{\varepsilon,*}(t)\|_{C^1_*} \ge G,$$

hence $G \le Ce^{C'(T_\varepsilon - t_0)}\varepsilon$, which is equivalent to (3.4.6). □

Remark 3.4.3. *This estimate is in general sharp, up to constants. Namely, let us take functions φ_j, $a_{j,\alpha}$, f that are independent of x and t and let us assume $\sigma \equiv 0$, $c_{m-1,0} \equiv 1$ and $c_{k,\beta} \equiv 0$ otherwise. These choices include the ODE*

$$d_t^m u = \frac{d_t^{m-1} u}{1 - d_t^{m-1} u}, \quad u(t_0) = \cdots = d_t^{m-2} u(t_0) = 0, \quad d_t^{m-1} u(t_0) = \varepsilon.$$

The solution $v(t)$ of the initial value problem $v' = v/(1-v)$, $v(t_0) = \varepsilon$ satisfies $\ln v - v + t_0 + \varepsilon - \ln \varepsilon = t$. This shows $T_\varepsilon = t_0 - 1 + \varepsilon + \ln \frac{1}{\varepsilon}$.

Remark 3.4.4. *For special hyperbolic operators, e.g. $\partial_{tt} - \triangle$ or $\partial_{tt} - \triangle + 1$, and special nonlinear right–hand sides it is possible to prove $T_\varepsilon = \infty$ if $0 < \varepsilon \le \varepsilon_0$. See, for instance, [Kla85] and [LC88] for the case of a wave equation with data from $C_0^\infty(\mathbb{R}^n)$ or [Man94] for the case of a wave equation (in $\mathbb{R}^n \times [0,T]$) whose data are periodic in at most $n-2$ variables. The global existence of the solution to a certain semilinear weakly hyperbolic equation with logarithmic nonlinearity was proved in [D'A94a] and estimates of the life–span of analytic solutions to quasilinear weakly hyperbolic equations were given in [DS91]. The stability of global Gevrey solutions to weakly hyperbolic equations of second order was studied in [RY97]. However, we will not follow these directions, since our goal is the investigation of hyperbolic equations of rather general type.*

Chapter 4

Weakly Hyperbolic Cauchy Problems with Spatial and Time Degeneracy

4.1 Introduction

The theory of local existence for weakly hyperbolic equations with *time* degeneracy presents new difficulties which can not be observed in the case of pure spatial degeneracy. Some of these difficulties are:

Loss of regularity The Example of Qi Min-You [Qi58] shows that the solution can lose Sobolev smoothness in comparison with the initial data. The number of lost derivatives depends (in the linear case) on the L^∞–norm of the coefficients of some lower order terms. At first glance, it is not clear how to show the strictly hyperbolic type property.

Singular coefficients in energy inequalities Let us consider the weakly hyperbolic equation

$$u_{tt} - \lambda(t)^2 u_{xx} = f(x,t), \quad \lambda(0) = 0, \quad \lambda'(t) > 0 \quad (t > 0).$$

If we choose the energy in the usual way, $E(t)^2 = \|u_t\|^2_{L^2} + \|\lambda(t)u_x\|^2_{L^2}$, then we obtain, after some calculations,

$$E'(t) \le \|f(.,t)\|^2_{L^2} + \frac{\lambda'(t)}{\lambda(t)} E(t).$$

The Lemma of Gronwall is not applicable, since the coefficient $\frac{\lambda'(t)}{\lambda(t)}$ becomes unbounded for $t \to 0$. But one can use Nersesyan's Lemma (see Lemma B.0.4) if the initial data vanish and $\|f(.,t)\|_{L^2}$ goes sufficiently fast to 0 as t tends to 0.

At first glance, the loss of regularity seems to make it impossible to prove the local existence of solutions to *quasilinear* equations, because the standard iteration procedure does not work. But in the special case of a right–hand side which is going to zero sufficiently fast as t approaches zero this standard iteration technique works!
Our approach is divided into three steps:

- In Section 4.2 *linear* weakly hyperbolic Cauchy problems with vanishing data and a right–hand side which is going sufficiently fast to 0 as t tends to 0 are studied. We will replace $\lambda(t)$ by $\lambda(t) + \delta$, $\delta > 0$, in such a way that we get a weakly hyperbolic Cauchy problem with pure spatial degeneracy. Then the results of the previous chapter can be applied. We are able to prove the *strictly hyperbolic type property*, see Theorem 4.2.1 and the Chapter Introduction.

- In Section 4.3 we examine *quasilinear* weakly hyperbolic Cauchy problems with vanishing data and a *special right–hand side* with suitable behaviour for $t \to 0$. The strictly hyperbolic type property of the linear problem allows us to apply the usual iteration technique. We obtain the local existence, see Theorem 4.3.5.

- In Section 4.4 we take an *arbitrary quasilinear* weakly hyperbolic Cauchy problem with spatial and time degeneracy and reduce it to another problem which can be handled with the methods described in the section before. The central result of that section is Theorem 4.4.1. The blow–up criterion of Proposition 3.3.4 can be carried over to the case of both degeneracies, since the time degeneracy occurs only for $t = 0$, cf. Corollary 4.4.2.

The last two sections base on ideas taken from [KY98]. The idea of transforming a weakly hyperbolic problem with *general* right–hand side into another weakly hyperbolic problem with *special* right–hand side has been used in [Ole70], [RY93] and [Rei97].

4.2 A Special Linear Case

We study the Cauchy problem

$$D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\lambda(t)^{|\alpha|} D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) = f(x,t), \tag{4.2.1}$$

$$u(x,0) = \cdots = D_t^{m-1}u(x,0) = 0$$

under the assumption

$$\|f(.,t)\|_{H^{s_0}} \leq C_f \lambda(t)^p \lambda'(t). \tag{4.2.2}$$

Later, the number p will be chosen sufficiently large. Then this condition implies that the right–hand side goes fast to zero for $t \to 0$. Additionally, we suppose Condition 1, (3.2.1), (3.2.2) and (3.2.4). For the function $\lambda = \lambda(t)$ we assume:

Condition 3. *Let $\Lambda(t) := \int_0^t \lambda(\tau)\,d\tau$. The function $\lambda(t)$ is nonnegative, strictly monotonically increasing with $\lambda(0) = 0$ and it holds*

$$\lambda \in C^2([0,T]),$$
$$\frac{\lambda(t)}{\Lambda(t)} \leq C_\lambda \frac{\lambda'(t)}{\lambda(t)}, \quad \frac{\lambda'(t)}{\lambda(t)} \leq C'_\lambda \frac{\lambda(t)}{\Lambda(t)}, \quad C_\lambda < \frac{m}{m-1}.$$

The derivative $\lambda'(t)$ is monotonically increasing.

Examples for these weight functions are $\Lambda(t) = t^l$ with $l \geq m+1$ or

$$\Lambda(t) = \exp(-|t|^{-r}), \quad \Lambda(t) = \exp(-\exp(\exp(\exp(|t|^{-r})))), \quad r > 0.$$

The central result of this section is the following theorem:

Theorem 4.2.1. *If the constant p is sufficiently large, then the Cauchy problem (4.2.1) has a solution u with the property that*

$$U := \begin{pmatrix} \langle D\rangle^{m-1}((\lambda\sigma)^{m-1}u) \\ \langle D\rangle^{m-2}((\lambda\sigma)^{m-2}D_t u) \\ \cdots \\ D_t^{m-1}u \end{pmatrix} \in C\left([0,T], H^{s_0}\right) \cap C^1\left([0,T], H^{s_0-1}\right).$$

Let C_R, $C_{R,\sim}$ be the constants from Section 3.2, see (3.2.14) and (3.2.12). Then a constant C_1 (independent of u) exists with the property that the estimate

$$\|U(t)\|_{H^{s_0}} \leq C_{R,\sim}^2 \int_0^t e^{C_1(t-s)} \left(\frac{\lambda(t)}{\lambda(s)}\right)^{C_{R,\sim}C_R(m-1)} \|F(s)\|_{H^{s_0}}\,ds$$

holds. The constant C_1 is given in (4.2.6). The number p of (4.2.2) can be chosen as given in (4.2.3).

Proof. Let C_R, $C_{R,\sim}$ be the constants from Section 3.2, describing some properties of the symmetrizer $R(x,t,D)$. Then we choose some real number p with

$$p > C_{R,\sim}C_R(m-1)+1. \tag{4.2.3}$$

We obtain the system

$$\begin{aligned}
\partial_t U_1 &= (m-1)\frac{\lambda'}{\lambda}U_1 + \lambda i\langle D\rangle(\sigma U_2) + \lambda i \langle D\rangle \left[\langle D\rangle^{m-2},\sigma\right]\langle D\rangle^{2-m}U_2,\\
\partial_t U_2 &= (m-2)\frac{\lambda'}{\lambda}U_2 + \lambda i\langle D\rangle(\sigma U_3) + \lambda i \langle D\rangle \left[\langle D\rangle^{m-3},\sigma\right]\langle D\rangle^{3-m}U_3,\\
&\dots,\\
\partial_t U_{m-1} &= \frac{\lambda'}{\lambda}U_{m-1} + \lambda i\langle D\rangle(\sigma U_m),\\
\partial_t U_m &= iD_t^m u = -i\sum_{j+|\alpha|=m,j<m} a_{j,\alpha}\lambda^{|\alpha|}D_x^\alpha D_t^j\left(\sigma^{|\alpha|}u\right) + if\\
&= -i\lambda\sum_{j+|\alpha|=m,j<m} a_{j,\alpha}P^\alpha\langle D\rangle(\sigma U_{j+1})\\
&\quad - i\lambda\sum_{j+|\alpha|=m,j<m} a_{j,\alpha}P^\alpha\langle D\rangle\left[\langle D\rangle^{|\alpha|-1},\sigma\right]\langle D\rangle^{1-|\alpha|}U_{j+1} + if.
\end{aligned}$$

This leads to

$$\partial_t U = \lambda K(\sigma U) + \lambda BU + H\frac{\lambda'}{\lambda}U + F, \quad U(0)=0 \tag{4.2.4}$$

with K, B, F as in (3.2.7), (3.2.8), (3.2.9) and $H = \operatorname{diag}(m-1, m-2, \dots, 1, 0)$. Considering a regularised version of this system, we substitute λ by $\lambda+\delta$ in suitably chosen places to eliminate the time degeneracy and insert a smoothing operator J_ε. This results in the system

$$\begin{aligned}
\partial_t U_{\delta\varepsilon} &= J_\varepsilon(\lambda+\delta)K(\sigma U_{\delta\varepsilon}) + \lambda BU_{\delta\varepsilon} + H\frac{\lambda'}{\lambda+\delta}U_{\delta\varepsilon} + F, \qquad (4.2.5)\\
U_{\delta\varepsilon}(0) &= 0.
\end{aligned}$$

The function $U_{\delta\varepsilon}$ is a solution of a weakly hyperbolic pseudodifferential system with spatial degeneracy. Therefore it is possible to find an energy estimate for $U_{\delta\varepsilon}$ by the same procedure as in Section 3.2. We can take the same symmetrizer R as in the previous chapter, since the function $\lambda+\delta$ has no influence on the operator K. This operator does not feel the time degeneracy.

If $1 + n/2 < s \le s_0$ or $s = 0$, then the estimate

$$
\begin{aligned}
&d_t\left(R\langle D\rangle^s U_{\delta\varepsilon}, \langle D\rangle^s U_{\delta\varepsilon}\right) \\
&= (R_t\langle D\rangle^s U_{\delta\varepsilon}, \langle D\rangle^s U_{\delta\varepsilon}) + (R\langle D\rangle^s J_\varepsilon(\lambda+\delta)K(\sigma U_{\delta\varepsilon}), \langle D\rangle^s U_{\delta\varepsilon}) \\
&\quad + \left(R\langle D\rangle^s J_\varepsilon(\lambda B U_{\delta\varepsilon} + H\tfrac{\lambda'}{\lambda+\delta}U_{\delta\varepsilon} + F), \langle D\rangle^s U_{\delta\varepsilon}\right) \\
&\quad + (R\langle D\rangle^s U_{\delta\varepsilon}, \langle D\rangle^s J_\varepsilon(\lambda+\delta)K(\sigma U_{\delta\varepsilon})) \\
&\quad + \left(R\langle D\rangle^s U_{\delta\varepsilon}, \langle D\rangle^s(\lambda B U_{\delta\varepsilon} + H\tfrac{\lambda'}{\lambda+\delta}U_{\delta\varepsilon} + F)\right) \\
&\le C(\|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|\partial_t a_{j,\alpha}\|_{C_b^0} + 1) \|U_{\delta\varepsilon}\|_{H^s}^2 \\
&\quad + (\lambda(t)+\delta)C(\|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + \|a_{j,\alpha}\|_{H^s} + 1) \|U_{\delta\varepsilon}\|_{H^s}^2 \\
&\quad + 2C_R(m-1)\tfrac{\lambda'(t)}{\lambda(t)+\delta} \|U_{\delta\varepsilon}\|_{H^s}^2 \\
&\quad + 2\sqrt{(R\langle D\rangle^s U_{\delta\varepsilon}, \langle D\rangle^s U_{\delta\varepsilon})}\sqrt{(R\langle D\rangle^s F, \langle D\rangle^s F)}
\end{aligned}
$$

holds, compare Proposition 3.2.2. We define

$$H_s(U_{\delta\varepsilon}(t)) := \sqrt{(R(t)\langle D\rangle^s U_{\delta\varepsilon}(t), \langle D\rangle^s U_{\delta\varepsilon}(t))}$$

and it follows that

$$
\begin{aligned}
d_t H_s(U_{\delta\varepsilon}(t)) \le C_1 H_s(U_{\delta\varepsilon}(t)) + C_{R,\sim}C_R(m-1)\frac{\lambda'(t)}{\lambda(t)+\delta}H_s(U_{\delta\varepsilon}(t)) \\
+ C_{R,\sim}\|F(t)\|_{H^s}. \qquad (4.2.6)
\end{aligned}
$$

By Gronwall's Lemma we see that

$$
\begin{aligned}
H_s(U_{\delta\varepsilon}(t)) &\le \int_0^t e^{C_{2,\delta}(t-\tau)} C_{R,\sim}\|F(\tau)\|_{H^s} d\tau \\
&\le e^{\frac{CT}{\delta}} C_{R,\sim}C_F \int_0^t \lambda'(\tau)\lambda(\tau)^p d\tau \\
&\le C_\delta \lambda(t)^{p+1}, \\
C_{2,\delta} &= \sup_{[0,T]}\left\{C_1 + C_{R,\sim}C_R(m-1)\tfrac{\lambda'(t)}{\lambda(t)+\delta}\right\} = O(\delta^{-1}).
\end{aligned}
$$

This allows us to apply the Lemma of Nersesyan (see Lemma B.0.4) to (4.2.6). The result is

$$
\begin{aligned}
H_s(U_{\delta\varepsilon}(t)) &\le \int_0^t e^{C_1(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{C_{R,\sim}C_R(m-1)} C_{R,\sim}\|F(\tau)\|_{H^s} d\tau \\
&\le C_{R,\sim}C_F\lambda(t)^{C_{R,\sim}C_R(m-1)} \int_0^t e^{C_1(t-\tau)}\lambda(\tau)^{p-C_{R,\sim}C_R(m-1)}\lambda'(\tau)d\tau \\
&\le \frac{C_{R,\sim}C_F e^{C_1 t}}{p - C_{R,\sim}C_R(m-1)+1}\lambda(t)^{p+1}.
\end{aligned}
$$

We emphasise that this estimate is independent of δ and ε. We know that

$$U_{\delta\varepsilon} \in C\left([0,T], H^{s_0}\right) \cap C^1\left([0,T], H^{s_0-1}\right).$$

Employing the methods from Section 3.3 one can show that

$$\exists \quad \lim_{\varepsilon\to 0} U_{\delta\varepsilon} \in C\left([0,T], H^{s_0}\right) \cap C^1\left([0,T], H^{s_0-1}\right).$$

Using a standard technique one proves that this limit U_δ solves

$$\partial_t U_\delta = (\lambda+\delta)K(\sigma U_\delta) + \lambda B U_\delta + H\frac{\lambda'}{\lambda+\delta}U_\delta + F, \quad U_\delta(0) = 0 \tag{4.2.7}$$

and that the following energy estimate holds for $1 + n/2 < s \le s_0$ and $s = 0$:

$$H_s(U_\delta(t)) \le C_{R,\sim}\int_0^t e^{C_1(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{C_{R,\sim}C_R(m-1)} \|F(\tau)\|_{H^s}\, d\tau. \tag{4.2.8}$$

In the next step we let δ tend to 0 and study the convergence properties of the sequence (U_δ). The difference $U_\delta - U_{\delta'}$ satisfies the equation

$$\begin{aligned} d_t(U_\delta - U_{\delta'}) &= (\lambda+\delta)K(\sigma U_\delta) - (\lambda+\delta')K(\sigma U_{\delta'}) + \lambda B U_\delta - \lambda B U_{\delta'} \\ &\quad + H\left(\frac{\lambda'}{\lambda+\delta}U_\delta - \frac{\lambda'}{\lambda+\delta'}U_{\delta'}\right) \\ &= (\lambda+\delta')K(\sigma(U_\delta - U_{\delta'})) + \lambda B(U_\delta - U_{\delta'}) + H\frac{\lambda'}{\lambda+\delta'}(U_\delta - U_{\delta'}) \\ &\quad + (\delta-\delta')\left(K(\sigma U_\delta) - H\frac{\lambda'}{(\lambda+\delta)(\lambda+\delta')}U_\delta\right). \end{aligned}$$

From (4.2.8) with $s = 0$ it can be concluded that

$$\begin{aligned} H_0((U_\delta - U_{\delta'})(t)) &\le C_{R,\sim}\int_0^t e^{C_1(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{C_{R,\sim}C_R(m-1)} |\delta-\delta'| \\ &\quad \times \left(\|K\sigma U_\delta\|_{L^2} + (m-1)\frac{\lambda'}{(\lambda+\delta)(\lambda+\delta')}\|U_\delta\|_{L^2}\right) d\tau \\ &\le C|\delta-\delta'|e^{C_1 t}\lambda(t)^{C_{R,\sim}C_R(m-1)}\int_0^t \lambda(\tau)^{p-1-C_{R,\sim}C_R(m-1)}\lambda'(\tau)d\tau \\ &\le C|\delta-\delta'|\lambda(t)^p. \end{aligned}$$

So evidence is given that (U_δ) is a Cauchy sequence in the Banach space $C\left([0,T], H^0\right)$. Using (4.2.8), Nirenberg–Gagliardo Interpolation and the differential equation we see that it is also a Cauchy sequence in the spaces

$C([0,T],H^s)$, $C^1([0,T],H^{s-1})$ for $s < s_0$. It is standard to show that the limit U is a solution of (4.2.4). From (4.2.8) we gain a uniform estimate of $U_\delta(t)$ in the Hilbert space H^{s_0}. The weak compactness of bounded subsets in Hilbert spaces and the convergence in spaces of low regularity imply

$$U \in L^\infty([0,T],H^{s_0}) \cap Lip^1([0,T],H^{s_0-1}).$$

It remains to show that

$$U \in C([0,T],H^{s_0}) \cap C^1\left([0,T],H^{s_0-1}\right).$$

This can be done as follows: the function U is a solution of the weakly hyperbolic Cauchy problem with spatial degeneracy

$$\begin{aligned}&\partial_t U = \lambda K(\sigma U) + \lambda B U + H\frac{\lambda'}{\lambda}U + F, \quad t \geq \gamma > 0,\\ &U(.,\gamma) = U(\gamma) \in H^{s_0}.\end{aligned}$$

The techniques for solving such problems lead us to

$$U \in C([\gamma,T],H^{s_0}) \cap C^1\left([\gamma,T],H^{s_0-1}\right) \quad \forall\gamma > 0.$$

On the other hand we have $U(0) = 0$ and $\|U(t)\|_{H^{s_0}} \leq C\lambda(t)^{p+1}$. This gives the continuity for $t = 0$ and the theorem is proved. □

Remark 4.2.2. *The theorem shows that weighted derivatives of the solution of order $m-1$ belong to the same Sobolev space as the right–hand side. This is exactly the* strictly hyperbolic type property*, which will allow us to attack special quasilinear equations with time degeneracy in the next section.*

4.3 A Special Quasilinear Case

A special quasilinear case shall be examined here. We assume that the initial data are zero and that the right–hand side has a suitable asymptotic behaviour. In other words, let us consider the Cauchy problem

$$\begin{aligned}&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\})\lambda(t)^{|\alpha|}D_x^\alpha D_t^j(\sigma(x)^{|\alpha|}u)\\ &\quad = f(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\}), \quad k+|\beta| \leq m-1, \qquad (4.3.1)\\ &u(x,0) = \cdots = D_t^{m-1}u(x,0) = 0\end{aligned}$$

with the following asymptotic behaviour for f:

$$\|f(.,t,0)\|_{H^{s_0}} \leq C_{fp}\lambda(t)^p\lambda'(t), \tag{4.3.2}$$
$$\left\|\frac{\partial f}{\partial g_{k,\beta}}(.,t,(g_{k,\beta}))\right\|_{L^\infty} \leq C_{fk\beta} \tag{4.3.3}$$

for all $t \in [0,T]$ and all $(g_{k,\beta}) \in \mathbb{R}^{n_0}$ from a suitable chosen compact set near zero. Furthermore, we suppose (3.2.1), (3.3.6), (3.3.8) (replace t_0 by 0 and K_G by some compact set near zero) and Condition 2. Concerning the weight functions $c_{k,\beta}$ we assume the Levi conditions

$$c_{k,\beta}(x,t) = \begin{cases} \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}c^0_{k,\beta}(x,t) & : |\beta| > 0, \\ c^0_{k,\beta}(x,t) & : |\beta| = 0, \end{cases} \tag{4.3.4}$$

where $\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau$; and the functions $c^0_{k,\beta}$ fulfil the relations

$$\left\|c^0_{k,\beta,t}(.,t)v(.)\right\|_{H^{s+|\beta|}} \leq C_c \left\|c^0_{k,\beta}(.,t)v(.)\right\|_{H^{s+|\beta|}}, \quad k+|\beta| \leq m-2, \tag{4.3.5}$$
$$\left\|c^0_{k,\beta}(.,t)v(.)\right\|_{H^{s+|\beta|}} \leq C_c \left\|c^0_{k+1,\beta}(.,t)v(.)\right\|_{H^{s+|\beta|}}, \quad k+|\beta| \leq m-2, \tag{4.3.6}$$
$$c^0_{k,\beta}(x,t) = \sigma(x)^{|\beta|}, \quad k+|\beta| = m-1, \tag{4.3.7}$$

for all $0 \leq s \leq s_0$.
Our intention is to show the local existence of a solution to (4.3.1), see Theorem 4.3.5. For this purpose we transform the equation into an equivalent system of first order and study a linearised version of this system. In other words, we introduce $U = (U_1,\dots,U_m)^T$, $V = (V_1,\dots,V_m)^T$, $V^* = (V_{k,\beta},V)$,

$$\begin{aligned}
&U_1 := \langle D\rangle^{m-1}((\lambda\sigma)^{m-1}u), && V_1 := \langle D\rangle^{m-1}((\lambda\sigma)^{m-1}v),\\
&U_2 := \langle D\rangle^{m-2}((\lambda\sigma)^{m-2}D_tu), && V_2 := \langle D\rangle^{m-2}((\lambda\sigma)^{m-2}D_tv),\\
&\dots, && \dots,\\
&U_m := D_t^{m-1}u, && V_m := D_t^{m-1}v,\\
&V_{k,\beta}(x,t) = D_x^\beta(c_{k,\beta}(x,t)D_t^kv(x,t)) && \qquad (4.3.8)
\end{aligned}$$

and get the system

$$\begin{aligned}
\partial_tU(x,t) &= \lambda K(x,t,V^*,D)\sigma U + \lambda B(x,t,V^*,D)U \\
&\quad + F(x,t,V^*) + H\frac{\lambda'}{\lambda}U,\\
U(0) &= 0.
\end{aligned}$$

Utilising the mapping $V \mapsto V^* \mapsto U$ we can construct a sequence $\{V^k\}$ in a standard way, see Subsection 4.3.2. In Subsection 4.3.3 the convergence of this sequence will be proved. Then it is not difficult to show that (4.3.1) has a solution in appropriate spaces, if p from (4.3.2) is large enough. But before this sequence $\{V^k\}$ can be studied, estimates of K, B and F must be found. This will be done now.

4.3.1 Auxiliary Estimates

The following lemma proves the boundedness of the mapping $V \mapsto V^*$ in certain topologies.

Lemma 4.3.1. *Let $T < 1$ and*

$$\|V(t)\|_{H^{s_0}} \leq C_{vp}\lambda(t)^{p+1}.$$

We assume that $p \geq C_\lambda C_c + m$. Then it holds

$$\left\|\langle D\rangle^{|\beta|} c_{k,\beta}(.,t) D_t^k v(.,t)\right\|_{H^{s_0}} \leq C_\lambda C_{vp}\lambda(t)^p \lambda'(t) e^{C_c t}, \tag{4.3.9}$$

$$\|V_{k,\beta}(.,t)\|_{H^{s_0}} \leq C_\lambda C_{vp}\lambda(t)^p \lambda'(t) e^{C_c t}. \tag{4.3.10}$$

Proof. We prove (4.3.9) by induction over $k+|\beta|$. Then the assertion (4.3.10) is an immediate consequence. Let $k+|\beta| = m-1$. If $|\beta| = 0$, then (4.3.9) follows from the definition of V, $V_{k,\beta}$ and [1] $C_\lambda > 1$, see Condition 3. Hence, we may assume $|\beta| > 0$. Then we have

$$\begin{aligned}
\left\|\langle D\rangle^{|\beta|} c_{k,\beta}(.,t) D_t^k v\right\|_{H^{s_0}} &= \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}\left\|\sigma^{|\beta|} D_t^k v\right\|_{H^{s_0+|\beta|}} \\
&= \frac{\lambda(t)}{\Lambda(t)}\left\|\lambda(t)^{|\beta|}\langle D\rangle^{|\beta|}(\sigma^{|\beta|} D_t^k v)\right\|_{H^{s_0}} \\
&\leq C_\lambda \frac{\lambda'(t)}{\lambda(t)}\|V(t)\|_{H^{s_0}} \leq C_\lambda C_{vp}\lambda(t)^p\lambda'(t).
\end{aligned}$$

The case $k+|\beta| = m-1$ is the base of the induction. Let $k+|\beta| \leq m-2$. Then

$$\begin{aligned}
d_t\left\|\langle D\rangle^{|\beta|}(c_{k,\beta} D_t^k v)\right\|_{H^{s_0}}^2 &= 2\left(\langle D\rangle^{|\beta|}(c_{k,\beta,t} D_t^k v), \langle D\rangle^{|\beta|}(c_{k,\beta} D_t^k v)\right)_{H^{s_0}} \\
&\quad + 2\left(\langle D\rangle^{|\beta|}(c_{k,\beta}\partial_t D_t^k v), \langle D\rangle^{|\beta|}(c_{k,\beta} D_t^k v)\right)_{H^{s_0}},
\end{aligned}$$

which leads to

$$\begin{aligned}
&d_t\left\|\langle D\rangle^{|\beta|}(c_{k,\beta} D_t^k v)\right\|_{H^{s_0}} \\
&\qquad \leq S_1 + S_2 := \left\|c_{k,\beta,t} D_t^k v\right\|_{H^{s_0+|\beta|}} + \left\|c_{k,\beta} D_t^{k+1} v\right\|_{H^{s_0+|\beta|}}.
\end{aligned}$$

[1] The inequality $C_\lambda > 1$ can be proved in a similar way as (6.3.3), see also (6.1.6).

If $|\beta| > 0$, then using (4.3.4) and (4.3.5) we obtain

$$\begin{aligned} S_1 &\le \left((m-k)\frac{\lambda'}{\lambda} + (k+|\beta|-m)\frac{\lambda}{\Lambda}\right)\lambda^{m-k}\Lambda^{k+|\beta|-m}\left\|c^0_{k,\beta}D_t^k v\right\|_{H^{s_0+|\beta|}} \\ &\quad + \lambda^{m-k}\Lambda^{k+|\beta|-m}\left\|c^0_{k,\beta,t}D_t^k v\right\|_{H^{s_0+|\beta|}} \\ &\le \left((m-k)\frac{\lambda'}{\lambda} + C_c\right)\left\|c_{k,\beta}D_t^k v\right\|_{H^{s_0+|\beta|}}. \end{aligned}$$

In the case $|\beta| = 0$ the condition (4.3.5) yields

$$S_1 \le C_c \left\|c_{k,\beta}D_t^k v\right\|_{H^{s_0+|\beta|}}.$$

To estimate S_2 for $|\beta| = 0$, we exploit $\lambda(t) \le t\lambda'(t) \le \lambda'(t)$, $C_\lambda > 1$, the induction hypothesis and conclude that

$$\begin{aligned} S_2 &= \left\|c_{k,\beta}D_t^{k+1}v\right\|_{H^{s_0+|\beta|}} = \left\|c^0_{k,\beta}D_t^{k+1}v\right\|_{H^{s_0+|\beta|}} \\ &\le C_c\left\|c^0_{k+1,\beta}D_t^{k+1}v\right\|_{H^{s_0+|\beta|}} \le C_c C_\lambda C_{vp}\lambda(t)^p\lambda'(t)e^{C_c t} \\ &\le C_c C_\lambda^2 C_{vp}(\lambda'(t))^2\lambda(t)^{p-1}e^{C_c t}. \end{aligned}$$

If $|\beta| > 0$, then

$$S_2 = \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}\left\|c^0_{k,\beta}D_t^{k+1}v\right\|_{H^{s_0+|\beta|}}.$$

Employing (4.3.6), (4.3.4), Condition 3 and the induction hypothesis we get

$$\begin{aligned} S_2 &\le \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}C_c\left\|c^0_{k+1,\beta}D_t^{k+1}v\right\|_{H^{s_0+|\beta|}} \\ &= \frac{\lambda(t)}{\Lambda(t)}C_c\left\|\langle D\rangle^{|\beta|}(c_{k+1,\beta}D_t^{k+1}v)\right\|_{H^{s_0}} \\ &\le \frac{\lambda'(t)}{\lambda(t)}C_\lambda^2 C_c C_{vp}\lambda(t)^p\lambda'(t)e^{C_c t} \\ &\le C_c C_{vp} C_\lambda^2(\lambda'(t))^2\lambda(t)^{p-1}e^{C_c t}. \end{aligned}$$

Summing up results in

$$\begin{aligned} d_t\left\|\langle D\rangle^{|\beta|}(c_{k,\beta}D_t^k v)\right\|_{H^{s_0}} &\le \left(m\frac{\lambda'(t)}{\lambda(t)} + C_c\right)\left\|\langle D\rangle^{|\beta|}(c_{k,\beta}D_t^k v)\right\|_{H^{s_0}} \\ &\quad + C_c C_{vp} C_\lambda^2(\lambda'(t))^2\lambda(t)^{p-1}e^{C_c t}. \end{aligned}$$

In order to apply the Lemma of Nersesyan, we have to ensure that

$$\left\|\langle D\rangle^{|\beta|}(c_{k,\beta}D_t^k v)\right\|_{H^{s_0}} = o(\lambda(t)^m).$$

This is true for $|\beta| = 0$, cf. the estimate of S_2 in the case $|\beta| = 0$. If $|\beta| > 0$, then we know from the induction hypothesis that

$$\left\| c_{k+1,\beta} D_t^{k+1} v \right\|_{H^{s_0+|\beta|}} \leq C\lambda(t)^p \lambda'(t),$$

C is a universal constant. Making use of $\lambda' \sim \lambda^2 \Lambda^{-1}$ (see Condition 3) we obtain

$$\begin{aligned}\left\| c^0_{k+1,\beta} D_t^{k+1} v \right\|_{H^{s_0+|\beta|}} &\leq C\lambda(t)^{p+k+1-m} \lambda'(t) \Lambda(t)^{m-k-|\beta|-1} \\ &\leq C\lambda(t)^{p+k+3-m} \Lambda(t)^{m-k-|\beta|-2}.\end{aligned}$$

By (4.3.5), (4.3.6) and the product formula for the differentiation, we get

$$\begin{aligned}&d_t \left\| c^0_{k,\beta} D_t^k v \right\|_{H^{s_0+|\beta|}} \\ &\quad\leq C\lambda(t)^{p+k+3-m} \Lambda(t)^{m-k-|\beta|-2} + C \left\| c^0_{k,\beta,t} D_t^k v \right\|_{H^{s_0+|\beta|}} \\ &\quad\leq C\lambda(t)^{p+k+3-m} \Lambda(t)^{m-k-|\beta|-2} + C \left\| c^0_{k,\beta} D_t^k v \right\|_{H^{s_0+|\beta|}}.\end{aligned}$$

Gronwall's Lemma and Condition 3 give

$$\begin{aligned}\left\| c^0_{k,\beta} D_t^k v \right\|_{H^{s_0+|\beta|}} &\leq C \int_0^t \lambda(\tau)^{p+k+3-m} \Lambda(\tau)^{m-k-|\beta|-2}\, d\tau \\ &\leq C\lambda(t)^{p+k+2-m} \Lambda(t)^{m-k-|\beta|-1} \\ &\leq C\lambda(t)^{p+k-m} \Lambda(t)^{m-k-|\beta|} \lambda'(t).\end{aligned}$$

This proves $\left\| c_{k,\beta} D_t^k v \right\|_{H^{s_0+|\beta|}} \leq C_0 \lambda(t)^p \lambda'(t) = o(\lambda(t)^m)$. In order to describe the constant C_0 more precisely we will apply the Lemma of Nersesyan. Utilising the monotonicity of λ' and $p - m \geq C_\lambda C_c$ we deduce that

$$\begin{aligned}&\left\| \langle D \rangle^{|\beta|} (c_{k,\beta} D_t^k v) \right\|_{H^{s_0}} \\ &\quad\leq \int_0^t \exp\left(\int_s^t m \frac{\lambda'(\tau)}{\lambda(\tau)} + C_c\, d\tau \right) C_c C_{vp} C_\lambda^2 (\lambda'(s))^2 \lambda(s)^{p-1} e^{C_c s}\, ds \\ &\quad\leq C_c C_{vp} C_\lambda^2 \lambda'(t) \int_0^t e^{C_c t} \left(\frac{\lambda(t)}{\lambda(s)} \right)^m \lambda(s)^{p-1} \lambda'(s)\, ds \\ &\quad= C_c C_{vp} C_\lambda^2 \lambda'(t) e^{C_c t} \lambda(t)^m \frac{1}{p-m} \lambda(t)^{p-m} \\ &\quad\leq C_\lambda C_{vp} \lambda(t)^p \lambda'(t) e^{C_c t}.\end{aligned}$$

The lemma is proved. □

Remark 4.3.2. *The conclusion of this lemma can be sharpened in the following way. If $0 \leq s \leq s_0$, then the estimates*

$$\|V_{k,\beta}\|_{H^s} \leq \left\| \langle D \rangle^{|\beta|} (c_{k,\beta} D_t^k v) \right\|_{H^s} \leq \lambda(t)^p \lambda'(t) e^{C_c t} C_\lambda \sup_{[0,t]} \frac{\|V(\tau)\|_{H^s}}{\lambda(\tau)^{p+1}}$$

hold for $k + |\beta| = m - 1$. *And for* $k + |\beta| \le m - 2$ *we have*

$$\begin{aligned}\|V_{k,\beta}\|_{H^s} &\le \left\| \langle D\rangle^{|\beta|}(c_{k,\beta} D_t^k v)\right\|_{H^s} \\ &\le \lambda(t)^p \lambda'(t) e^{C_c t} \sup_{[0,t]} \frac{\left\|\langle D\rangle^{|\beta|}(c_{k+1,\beta} D_t^{k+1} v)\right\|_{H^s}}{\lambda(\tau)^p \lambda'(\tau) e^{C_c \tau}}.\end{aligned}$$

This lemma gives us an estimate for V^* if a bound of V is known. We will also employ this lemma to derive an estimate for U^* from a bound of U. Such an estimate of U in the terms of the right–hand side is given by Theorem 4.2.1. The next lemma will be useful to find an estimate of F in terms of V^*.

Lemma 4.3.3. *Let* $K \subset \mathbb{R}^{n_0}$ *be a compact set and* M *be an* n*-dimensional smooth closed compact manifold. Let* $f \in C^N(M \times K)$ *with sufficiently large* N*. We assume* $v_i \in H^N(M)$ *with*

$$(x, v_1(x), \dots, v_{n_0}(x)) \in M \times K \quad \forall x \in M.$$

Then

$$\begin{aligned}&\|f(., v_1(.), \dots, v_{n_0}(.))\|_{H^N} \\ &\quad \le \varphi_N(\|v_1\|_{L^\infty}, \dots, \|v_{n_0}\|_{L^\infty})(\|v_1\|_{H^N} + \dots + \|v_{n_0}\|_{H^N} + 1).\end{aligned}$$

More precisely, it can be proved: if N *is sufficiently large and* $0 \le m < n_0$*, then a constant* $N_1 < N$ *(independent of* N*) exists with*

$$\begin{aligned}&\|f(., v_1(.), \dots, v_{n_0}(.))\|'_{H^N} \\ &\quad \le \varphi_N(\|v_1\|_{C_b^{N_1}}, \dots, \|v_{n_0}\|_{C_b^{N_1}}) \\ &\qquad \times (\|v_1\|_{H^N} + \dots + \|v_m\|_{H^N} + \|v_{m+1}\|_{H^{N-1}} + \dots + \|v_{n_0}\|_{H^{N-1}}) \\ &\qquad + \sum_{j=m+1}^{n_0} \left\| \frac{\partial f}{\partial v_j}(., v_1(.), \dots, v_{n_0}(.))\right\|_{L^\infty} \sum_{|\alpha|=N} \|\partial_x^\alpha v_j\|_{L^2} \\ &\qquad + \sum_{|\alpha| \le N} \left\| f^{(\alpha,0,\dots,0)}(., v_1(.), \dots, v_{n_0}(.))\right\|_{L^2}.\end{aligned}$$

Here we used the norm $\|w\|'_{H^N} = \sum_{|\alpha|\le N} \|\partial_x^\alpha w\|_{L^2}$*, which is equivalent to* $\|.\|_{H^N}$*,*

$$C_N^{-1} \|w\|'_{H^N} \le \|w\|_{H^N} \le C_N \|w\|'_{H^N}. \tag{4.3.11}$$

This lemma is a generalisation of Remark A.1 in [DR98b] and gives a precise description of the dependence of $\|f(., v_1, \dots, v_n)\|'_{H^N}$ on the highest orders of some v_j (see the terms $\|\partial_x^\alpha v_j\|_{L^2}$). The proof is omitted. We will use this lemma

to determine the loss of Sobolev regularity (it depends on the terms $\left\|f_{v_j}\right\|_{L^\infty}$), or, in other words, to determine the space in which the solution lives.
From now on we assume $s_0 = N \in \mathbb{N}$ and set $s_1 := N_1$. Lemma 4.3.3 gives

$$\begin{aligned}
&\|f(x,t,\{V_{k,\beta}\})\|_{H^{s_0}} \le C_{s_0} \|f(x,t,\{V_{k,\beta}\})\|'_{H^{s_0}} \\
&\quad \le C_{s_0}\varphi_{s_0}(\|\{V_{k,\beta}\}\|_{C_b^{s_1}}) \|\{V_{k,\beta}\}\|_{H^{s_0-1}} + C_{s_0}\sum_{|\alpha|\le s_0} \left\|\frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\})\right\|_{L^2} \\
&\qquad + C_{s_0} \sum_{k+|\beta|\le m-1} \left\|\frac{\partial f}{\partial V_{k,\beta}}\right\|_{L^\infty} \sum_{|\alpha|=s_0} \|\partial_x^\alpha V_{k,\beta}\|_{L^2}. \qquad (4.3.12)
\end{aligned}$$

We estimate the terms on the right. Repeated application of Remark 4.3.2 reveals

$$\begin{aligned}
&\|\{V_{k,\beta}\}\|_{H^{s_0-1}} = \max_{\beta,k}\{\|V_{k,\beta}\|_{H^{s_0-1}}\} \\
&\quad \le \max\left\{ \max_{\beta\ne 0} \lambda(t)^p\lambda'(t)e^{C_c t} \sup_{[0,t]} \frac{\left\|c_{m-1-|\beta|,\beta} D_t^{m-1-|\beta|} v\right\|_{H^{s_0+|\beta|-1}}}{\lambda(\tau)^p\lambda'(\tau)e^{C_c\tau}}, \right. \\
&\qquad\qquad \left. \max_{k=0,\dots,m-1} \|V_{0k}\|_{H^{s_0-1}} \right\}.
\end{aligned}$$

In the case $|\beta| > 0$ we get

$$\begin{aligned}
I_1 &:= \lambda(t)^p\lambda'(t)e^{C_c t} \sup_{[0,t]} \frac{\lambda(\tau)^{|\beta|+1-p}}{\Lambda(\tau)\lambda'(\tau)e^{C_c\tau}} \left\|\sigma^{|\beta|} D_t^{m-1-|\beta|} v\right\|_{H^{s_0+|\beta|-1}} \\
&\le C_\lambda C_\sigma \lambda(t)^p\lambda'(t)e^{C_c t} \\
&\quad \times \sup_{[0,t]} \lambda(\tau)^{-p} \left\|\langle D\rangle^{|\beta|-1}(\lambda(\tau)^{|\beta|-1}\sigma^{|\beta|-1} D_t^{m-1-|\beta|} v)\right\|_{H^{s_0}}. \qquad (4.3.13)
\end{aligned}$$

Here Condition 3 has been applied and a constant C_σ has been introduced. By the definition of V, it holds

$$\begin{aligned}
&d_\tau \left\|\langle D\rangle^{|\beta|-1}(\lambda(\tau)^{|\beta|-1}\sigma^{|\beta|-1} D_t^{m-1-|\beta|} v)\right\|_{H^{s_0}} \\
&\quad \le (|\beta|-1)\frac{\lambda'(\tau)}{\lambda(\tau)} \left\|\langle D\rangle^{|\beta|-1}(\lambda(\tau)^{|\beta|-1}\sigma^{|\beta|-1} D_t^{m-1-|\beta|} v)\right\|_{H^{s_0}} \\
&\qquad + \left\|\langle D\rangle^{|\beta|-1}(\lambda(\tau)^{|\beta|-1}\sigma^{|\beta|-1} D_t^{m-|\beta|} v)\right\|_{H^{s_0}} \\
&\quad \le (|\beta|-1)\frac{\lambda'(\tau)}{\lambda(\tau)} \left\|\langle D\rangle^{|\beta|-1}(\lambda(\tau)^{|\beta|-1}\sigma^{|\beta|-1} D_t^{m-1-|\beta|} v)\right\|_{H^{s_0}} \\
&\qquad + \sup_{[0,\tau]} \frac{\|V(s)\|_{H^{s_0}}}{\lambda(s)^{p+1}} \lambda(\tau)^{p+1}.
\end{aligned}$$

Then Nersesyan's Lemma yields

$$\left\| \langle D \rangle^{|\beta|-1} (\lambda(t)^{|\beta|-1} \sigma^{|\beta|-1} D_t^{m-1-|\beta|} v) \right\|_{H^{s_0}} \le t\lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}.$$

Making use of the constant C_σ of (4.3.13) we deduce that

$$\begin{aligned} I_1 &\le C_\lambda C_\sigma \lambda(t)^p \lambda'(t) e^{C_c t} \sup_{[0,t]} \tau\lambda(\tau) \sup_{[0,\tau]} \frac{\|V(s)\|_{H^{s_0}}}{\lambda(s)^{p+1}} \\ &\le C_\lambda C_\sigma t \lambda(t)^{p+1} \lambda'(t) e^{C_c t} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}. \end{aligned}$$

Now let us devote ourselves to the case $\beta = 0$. It holds the estimate

$$\max_{k=0,\dots,m-1} \left\| c_{k,0}^0(.,t) D_t^k v \right\|_{H^{s_0-1}} \le \max_{k=0,\dots,m-1} C_c^{m-1-k} \left\| D_t^k v \right\|_{H^{s_0-1}},$$

see (4.3.6) and (4.3.7). For $k = m-1$ we get

$$\left\| D_t^{m-1} v(.,t) \right\|_{H^{s_0-1}} \le \left\| D_t^{m-1} v(.,t) \right\|_{H^{s_0}} \le \lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}.$$

Assuming $tC_c \le 1$ we obtain

$$\begin{aligned} C_c \left\| D_t^{m-2} v(.,t) \right\|_{H^{s_0-1}} &\le C_c \int_0^t \left\| D_t^{m-1} v(.,\tau) \right\|_{H^{s_0-1}} d\tau \\ &\le C_c \int_0^t \lambda(\tau)^{p+1} d\tau \cdot \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} \le \lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}. \end{aligned}$$

Following this way we see that

$$\max_{k=0,\dots,m-1} \left\| c_{k,0}^0(.,t) D_t^k v \right\|_{H^{s_0-1}} \le \lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}.$$

Hence it can be concluded that

$$\begin{aligned} \|\{V_{k,\beta}\}\|_{H^{s_0-1}} &\le \max(1, C_\lambda C_\sigma t \lambda'(t) e^{C_c t}) \lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} \\ &\le \lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}, \end{aligned}$$

if t is sufficiently small. From this, (4.3.12) and (4.3.3) it follows that

$$\begin{aligned} \|f(.,t,\{V_{k,\beta}\})\|_{H^{s_0}} &\le C_{s_0} \varphi_{s_0}(\|V^*\|_{C_b^{s_1}}) \lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} \\ &+ C_{s_0} \sum_{k+|\beta| \le m-1} C_{fk\beta} \sum_{|\alpha|=s_0} \|\partial_x^\alpha V_{k,\beta}\|_{L^2} \\ &+ C_{s_0} \sum_{|\alpha| \le s_0} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\}) \right\|_{L^2}. \end{aligned}$$

We employ Remark 4.3.2 and (4.3.11) to estimate $\|\partial_x^\alpha V_{k,\beta}\|_{L^2}$ and see that

$$\sum_{|\alpha|=s_0} \|\partial_x^\alpha V_{k,\beta}\|_{L^2} \leq C_{s_0} \|V_{k,\beta}\|_{H^{s_0}} \leq C_{s_0} C_\lambda \lambda(t)^p \lambda'(t) e^{C_c t} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}}.$$

Finally, from Hadamard's Formula and (4.3.2) it can be deduced that

$$\begin{aligned}
&\sum_{|\alpha|\leq s_0} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\}) \right\|_{L^2} \\
&\leq \sum_{|\alpha|\leq s_0} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\}) - \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,0) \right\|_{L^2} + \sum_{|\alpha|\leq s_0} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,0) \right\|_{L^2} \\
&\leq \sum_{|\alpha|\leq s_0} \left\| \sum_{k,\beta} g_{\alpha\beta k}(x,t,\{V_{k,\beta}\}) V_{k,\beta} \right\|_{L^2} + C_{s_0} C_{fp} \lambda(t)^p \lambda'(t) \\
&\leq C \|V^*(t)\|_{L^2} + C_{s_0} C_{fp} \lambda(t)^p \lambda'(t) \\
&\leq C \|V^*(t)\|_{H^{s_0-1}} + C_{s_0} C_{fp} \lambda(t)^p \lambda'(t).
\end{aligned}$$

Summing up and introducing the notation

$$C_f := \sum_{k+|\beta|\leq m-1} C_{fk\beta} \tag{4.3.14}$$

we get the following estimate, if t is small:

$$\begin{aligned}
&\|f(.,t,\{V_{k,\beta}\})\|_{H^{s_0}} \\
&\leq C_{s_0}(1+\varphi_{s_0}(\|\{V_{k,\beta}\}\|_{C_b^{s_1}}))\lambda(t)^{p+1} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} \\
&\quad + C_{s_0}^2 C_\lambda C_f \lambda(t)^p \lambda'(t) e^{C_c t} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} + C_{s_0}^2 C_{fp} \lambda(t)^p \lambda'(t) \\
&\leq \left((1+\varphi_{s_0}(\|\{V_{k,\beta}\}\|_{C_b^{s_1}})) \frac{\lambda(t)}{\lambda'(t)} + C_{s_0} C_\lambda C_f \right) \\
&\qquad \times C_{s_0} \lambda(t)^p \lambda'(t) e^{C_c t} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} + C_{s_0}^2 C_{fp} \lambda(t)^p \lambda'(t) \\
&\leq 2C_{s_0}^2 C_\lambda C_f \lambda(t)^p \lambda'(t) e^{C_c t} \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} + C_{s_0}^2 C_{fp} \lambda(t)^p \lambda'(t) \\
&\leq C_{s_0}^2 \lambda(t)^p \lambda'(t) e^{C_c t} \left(2C_\lambda C_f \sup_{[0,t]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} + C_{fp} \right).
\end{aligned} \tag{4.3.15}$$

4.3.2 Iteration

Now we have all tools to find a bound for the mapping $V \mapsto V^* \mapsto U$.

Lemma 4.3.4. *We assume that* p *is so large and that* T^* *is so small that*

$$\begin{aligned}
&p > 2C_{R,\sim}^2 C_{s_0}^2 (2C_\lambda C_f + C_{fp}) + C_{R,\sim} C_R (m-1),\\
&p \geq C_c C_\lambda + m,\\
&T^* \leq \min(C_c^{-1}, 1),\\
&C_\lambda C_\sigma T^* \lambda'(T^*) e^{C_c T^*} \leq 1,\\
&(1 + \varphi_0(\|\{V_{k,\beta}\}\|_{C_b^{s_1}})) \frac{\lambda(T^*)}{\lambda'(T^*)} \leq C_{s_0} C_\lambda C_f \quad (\mathit{for}\ \|V(t)\|_{H^{s_0}} \leq \lambda(t)^{p+1}),\\
&e^{(C_1 + C_c)T^*} \leq 2.
\end{aligned}$$

For the definitions of the constants in these conditions we refer the reader to the formulas given in the following table:

$C_{R,\sim}$	*(3.2.12)*	C_{s_0}	*(4.3.11)*	C_λ	*Condition 3*
C_f	*(4.3.14)*	C_{fp}	*(4.3.2)*	C_R	*(3.2.14)*
C_c	*(4.3.5), (4.3.6)*	C_1	*(4.2.6)*	C_σ	*(4.3.13)*

We suppose

$$\sup_{[0,T^*]} \frac{\|V(t)\|_{H^{s_0}}}{\lambda(t)^{p+1}} \leq 1.$$

Then it holds

$$\sup_{[0,T^*]} \frac{\|U(t)\|_{H^{s_0}}}{\lambda(t)^{p+1}} \leq 1.$$

Proof. Due to Theorem 4.2.1 and (4.3.15) we have

$$\begin{aligned}
\|U(t)\|_{H^{s_0}} &\leq C_{R,\sim}^2 \int_0^t e^{C_1(t-s)} \left(\frac{\lambda(t)}{\lambda(s)}\right)^{C_{R,\sim} C_R (m-1)} \|f(s)\|_{H^{s_0}}\, ds\\
&\leq C_{R,\sim}^2 e^{C_1 t} \int_0^t \left(\frac{\lambda(t)}{\lambda(s)}\right)^{C_R C_{R,\sim}(m-1)} C_{s_0}^2 \lambda(s)^p \lambda'(s) e^{C_c s}\\
&\qquad \times \left(2C_\lambda C_f \sup_{[0,s]} \frac{\|V(\tau)\|_{H^{s_0}}}{\lambda(\tau)^{p+1}} + C_{fp}\right) ds\\
&\leq C_{R,\sim}^2 C_{s_0}^2 (2C_\lambda C_f + C_{fp}) e^{(C_1 + C_c)t} \frac{\lambda(t)^{p+1}}{p - C_{R,\sim} C_R (m-1) + 1}\ .
\end{aligned}$$

The assumption on p implies

$$\frac{C_{R,\sim}^2 C_{s_0}^2 (2C_\lambda C_f + C_{fp})}{p - C_{R,\sim} C_R (m-1) + 1} \le \frac{1}{2} \ .$$

Exploiting $e^{(C_1+C_c)T^*} \le 2$ we obtain $\|U(t)\|_{H^{s_0}} \le \lambda(t)^{p+1}$. □

Now we restrict the constant T^* in such a manner, that

$$\|V(t)\|_{H^{s_0}} \le \lambda(t)^{p+1} \quad \forall t \in [0, T^*]$$

implies

$$(x, (V_{k,\beta}(x,t))) \in K_G \quad \forall (x,t) \in M \times [0, T^*].$$

All these results allow us to define a sequence

$$(V^l) \subset C\left([0,T^*], H^{s_0}\right) \cap C^1\left([0,T^*], H^{s_0-1}\right)$$

by $V^0(t) \equiv 0$ and

$$\begin{aligned} &V^l(0) = 0 \quad \forall l, \\ &\partial_t V^l(t) = \lambda(t) K(x,t,V^{*,l-1},D)\sigma V^l + \lambda(t) B(x,t,V^{*,l-1},D) V^l \\ &\quad + F(x,t,V^{*,l-1}) + H\frac{\lambda'(t)}{\lambda(t)} V^l \quad \forall l \ge 1. \end{aligned}$$

Due to Lemma 4.3.4 and Remark 4.3.2 the functions V^l fulfil

$$\begin{aligned} &\left\|V^l(t)\right\|_{H^{s_0}} \le \lambda(t)^{p+1} \quad \forall t \in [0,T^*], \\ &\left\|V^l_{k,\beta}(t)\right\|_{H^{s_0}} \le C_\lambda e^{C_c t} \lambda(t)^p \lambda'(t) \quad \forall t \in [0,T^*]. \end{aligned}$$

4.3.3 Convergence

Now we want to give evidence for the convergence of the sequence (V^l) and cogitate upon the regularity of the limit. Due to Hadamard's Formula we have

$$\begin{aligned} &\partial_t (V^{l+1} - V^l)(t) = \lambda(t) K(x,t,V^{*,l},D)\sigma(V^{l+1} - V^l) \\ &\quad + \lambda(t)(K(x,t,V^{*,l},D) - K(x,t,V^{*,l-1},D))\sigma V^l \\ &\quad + \lambda(t) B(x,t,V^{*,l},D)(V^{l+1} - V^l) + \\ &\quad + \lambda(t)(B(x,t,V^{*,l},D) - B(x,t,V^{*,l-1},D)) V^l \\ &\quad + F(x,t,V^{*,l}) - F(x,t,V^{*,l-1}) + H\frac{\lambda'(t)}{\lambda(t)}(V^{l+1} - V^l) \\ &= \lambda(t) K(x,t,V^{*,l},D)\sigma(V^{l+1} - V^l) + \lambda(t) B(x,t,V^{*,l},D)(V^{l+1} - V^l) \\ &\quad + H\frac{\lambda'(t)}{\lambda(t)}(V^{l+1} - V^l) + G(x,t,V^{*,l},V^{*,l-1},D)(V^{l,*} - V^{*,l-1}) \end{aligned}$$

with

$$\left\| G(.,t,V^{*,l},V^{*,l-1},D)(V^{l,*}-V^{*,l-1}) \right\|_{L^2} \le C_3 \left\| V^{*,l}-V^{*,l-1} \right\|_{L^2}. \tag{4.3.16}$$

The difference $W^l := V^{l+1} - V^l$ satisfies the differential equation

$$\begin{aligned} \partial_t W^l(t) = {} & \lambda(t)K(x,t,D)\sigma W^l + \lambda(t)B(x,t,D)W^l \\ & + H\frac{\lambda'(t)}{\lambda(t)}W^l + G(x,t)W^{*,l-1}. \end{aligned}$$

Theorem 4.2.1 and Remark 4.3.2 give

$$\begin{aligned} \left\| W^l(t) \right\|_{L^2} & \le C_{R,\sim}^2 \int_0^t e^{C_1(t-s)} \left(\frac{\lambda(t)}{\lambda(s)} \right)^{C_{R,\sim}C_R(m-1)} \left\| G(.,s)W^{*,l-1} \right\|_{L^2} ds \\ & \le C_{R,\sim}^2 C_\lambda C_3 e^{(C_1+C_c)t} \lambda(t)^{C_{R,\sim}C_R(m-1)} \\ & \quad \times \int_0^t \lambda(s)^{p-C_{R,\sim}C_R(m-1)} \sup_{[0,s]} \frac{\left\| W^{l-1} \right\|_{L^2}}{\lambda(\tau)^{p+1}} \lambda'(s) ds \\ & \le \frac{C_{R,\sim}^2 C_\lambda C_3 e^{(C_1+C_c)t}}{p - C_{R,\sim}C_R(m-1)+1} \lambda(t)^{p+1} \sup_{[0,t]} \frac{\left\| W^{l-1} \right\|_{L^2}}{\lambda(s)^{p+1}}. \end{aligned}$$

If p satisfies

$$\frac{2C_{R,\sim}^2 C_\lambda C_3}{p - C_{R,\sim}C_R(m-1)+1} \le \frac{1}{2}, \tag{4.3.17}$$

then we get

$$\sup_{[0,T^*]} \frac{\left\| W^l(t) \right\|_{L^2}}{\lambda(t)^{p+1}} \le \frac{1}{2} \sup_{[0,T^*]} \frac{\left\| W^{l-1}(t) \right\|_{L^2}}{\lambda(t)^{p+1}}.$$

This proves that the sequence (V^l) is a Cauchy sequence in $C([0,T^*],H^0)$. By the Interpolation Theorem of Nirenberg–Gagliardo we see that (V^l) is a Cauchy sequence in $C([0,T^*],H^{s_0-1})$, too. We denote the limit by U and prove in a standard way that U is a solution of (4.3.1). Exploiting the arguments which gave (3.3.20) we get $U \in C([0,T^*],H^{s_0})$. Thus, we have proved:

Theorem 4.3.5 (Existence). *Let the conditions mentioned at the beginning of this section be fulfilled. Let $s_0 \in \mathbb{N}$ be sufficiently large and let p be a constant satisfying the conditions from Lemma 4.3.4 and (4.3.17). We assume that T^* fulfils the conditions given in Lemma 4.3.4. Then the Cauchy problem (4.3.1) has a solution u with*

$$U \in C([0,T^*],H^{s_0}) \cap C^1\left([0,T^*],H^{s_0-1}\right).$$

Remark 4.3.6. *One assumption of this theorem was*

$$f \in C([0,T], C^{s_0}(K_G)).$$

It is possible to weaken this assumption. Namely, it suffices to assume that f depends on x with smoothness H^{s_0} and on $\{V_{k,\beta}\}$ with smoothness C^{s_0}. Then weighted derivatives of order $m-1$ of u will belong to the same Sobolev space as the right–hand side. This is the strictly hyperbolic type property.

4.4 Reduction of a General Quasilinear Equation to a Quasilinear Equation with Special Right–Hand Side

In this section we reflect upon a general quasilinear weakly hyperbolic Cauchy problem and prove the existence of a solution using the technique of the previous section. Namely, we will transform the Cauchy problem

$$\begin{aligned} &D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k u\}) \lambda(t)^{|\alpha|} D_x^\alpha D_t^j (\sigma(x)^{|\alpha|} u) \\ &\quad = f(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k u\}), \quad k+|\beta| \le m-1, \\ &u(x,0) = \varphi_0(x), \ldots, D_t^{m-1} u(x,0) = \varphi_{m-1}(x) \end{aligned} \tag{4.4.1}$$

into another Cauchy problem

$$\begin{aligned} &D_t^m v + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha,p'}(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k v\}) \lambda(t)^{|\alpha|} D_x^\alpha D_t^j (\sigma(x)^{|\alpha|} v) \\ &\quad = f_{p'}(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k v\}), \\ &v(x,0) = \cdots = D_t^{m-1} v(x,0) = 0, \end{aligned} \tag{4.4.2}$$

whose right–hand side $f_{p'}$ fulfils the relations (4.3.2) and (4.3.3) with $p = p(p')$. It will be shown that these two Cauchy problems are equivalent in the sense that functions u_1, u_2, … , $u_{p'}$ exist with

$$u = u_1 + u_2 + \cdots + u_{p'} + v.$$

The functions u_1, u_2, … , $u_{p'}$ are solutions of ODEs in t with parameter x. If p is large enough, then we can apply Theorem 4.3.5 to (4.4.2) and see that there is a solution v of (4.4.2). This proves the existence of a solution u of the

general Cauchy problem (4.4.1). This idea has been used in [KY98], [RY93] and [Rei97].
The Example of Qi Min–You [Qi58] shows that a loss of Sobolev regularity (in comparison to the data) must be expected. This phenomenon can be observed in our computations, too. Namely, the smoothness of the u_j decreases by m, as j increases by 1.
We list the assumptions.
The functions $c_{k,\beta}$ are assumed to satisfy (4.3.4)–(4.3.7). The functions $a_{j,\alpha}$ and the right–hand side f are defined in the set K_G; see (3.3.5). Finally, we suppose (3.2.1), (3.3.6), (3.3.7), (3.3.8), Condition 2 and Condition 3.
We will prove:

Theorem 4.4.1 (Existence). *If $s_0 \in \mathbb{N}$ is large enough, then some number $T^* \in (0,T]$ and some $\gamma > 0$ (independent of s_0) exist with the property that there is a solution u of (4.4.1) with*

$$U := \begin{pmatrix} \langle D\rangle^{m-1}((\lambda\sigma)^{m-1}u) \\ \langle D\rangle^{m-2}((\lambda\sigma)^{m-2}D_t u) \\ \dots \\ D_t^{m-1}u \end{pmatrix} \in \bigcap_{j=0}^{1} C^j\left([0,T^*], H^{s_0-\gamma-j}\right).$$

Proof. We define

$$\varepsilon_{l,i,\beta} = \begin{cases} 1 & : |\beta| = 0 \text{ or } l > i, \\ 0 & : |\beta| > 0 \text{ and } l = i \end{cases}$$

and consider the system of ODEs in t with parameter x

$$\begin{aligned} &D_t^m u_1(x,t) = f(x,t,\{D_x^\beta c_{k,\beta}(x,t) D_t^k u_1(x,t)\varepsilon_{1,1,\beta}\}), \\ &u_1(x,0) = \varphi_0(x), \dots, D_t^{m-1}u_1(x,0) = \varphi_{m-1}(x), \\ &D_t^m u_l(x,t) = g_l(x,t,u_l(x,t),\dots,D_t^{m-1}u_l(x,t)) \\ &\quad := f\Big(x,t,\Big\{D_x^\beta c_{k,\beta}(x,t) D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i(x,t)\Big\}\Big) \\ &\qquad - f\Big(x,t,\Big\{D_x^\beta c_{k,\beta}(x,t) D_t^k \sum_{i=1}^{l-1} \varepsilon_{l-1,i,\beta} u_i(x,t)\Big\}\Big) - \\ &\qquad - \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l}(x,t) \sum_{i=1}^{l-1} \lambda(t)^{|\alpha|} D_x^\alpha D_t^j(\sigma(x)^{|\alpha|} u_i(x,t)) \\ &\qquad + \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l-1}(x,t) \sum_{i=1}^{l-2} \lambda(t)^{|\alpha|} D_x^\alpha D_t^j(\sigma(x)^{|\alpha|} u_i(x,t)) \\ &u_l(x,0) = \dots = D_t^{m-1}u_l(x,0) = 0, \quad l = 2,\dots,p. \end{aligned}$$

The following notations have been used here:

$$\tilde{a}_{j,\alpha,l}(x,t) := a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta}(x,t)D_t^k\sum_{i=1}^{l}u_i(x,t)\varepsilon_{l,i,\beta}\Big\}\Big), \quad \sum_{i=1}^{0} = 0.$$

This system can be solved step by step and possesses solutions

$$u_l \in C^m\left([0,T_l], H^{s_0-m(l-1)}\right), \quad s_0 - ml > \frac{n}{2} + 1.$$

To see this, one may apply Theorem 3.3.1 with $\sigma \equiv 0$.
The functions u_l have a special asymptotical behaviour for $t \to 0$. Obviously,

$$\sum_{j=0}^{m-1} \left\|D_t^j u_1(.,t)\right\|_{H^{s_0}} \le C_1\lambda(t)^0.$$

Let ε be a positive number with $\varepsilon < m-(m-1)C_\lambda$, C_λ from Condition 3. We assume

$$\sum_{j=0}^{m-1} \left\|D_t^j u_l(.,t)\right\|_{H^{s_0-m(l-1)}} \le C_l\lambda(t)^{\varepsilon(l-1)}$$

with $s_0 - ml > 1 + n/2$ and prove

$$\sum_{j=0}^{m-1} \left\|D_t^j u_{l+1}(.,t)\right\|_{H^{s_0-ml}} \le C_{l+1}\lambda(t)^{\varepsilon l}.$$

For this purpose we consider $\|g_{l+1}(t)\|_{H^{s_0-ml}}$. Hadamard's Formula yields

$$\begin{aligned} f\Big(x,t,\Big\{D_x^\beta c_{k,\beta}D_t^k\sum_{i=1}^{l+1}\varepsilon_{l+1,i,\beta}u_i\Big\}\Big) - f\Big(x,t,\Big\{D_x^\beta c_{k,\beta}D_t^k\sum_{i=1}^{l}\varepsilon_{l,i,\beta}u_i\Big\}\Big) \\ = \sum_k d_{1lk}(x,t)c_{k,0}D_t^k u_{l+1} + \sum_{|\beta|>0,k} d_{2lk\beta}(x,t)D_x^\beta c_{k,\beta}D_t^k u_l. \end{aligned}$$

We drop the arguments of $c_{k,\beta}(x,t)$, $\lambda(t)$, $\sigma(x)$, $u_j(x,t)$ and $\tilde{a}_{j,\alpha,l}(x,t)$ from now on. Similarly we get

$$\begin{aligned} \sum_{\substack{j+|\alpha|=m,\\ j<m}} \left(\tilde{a}_{j,\alpha,l+1}\sum_{i=1}^{l}\lambda^{|\alpha|}D_x^\alpha D_t^j(\sigma^{|\alpha|}u_i) - \tilde{a}_{j,\alpha,l}\sum_{i=1}^{l-1}\lambda^{|\alpha|}D_x^\alpha D_t^j(\sigma^{|\alpha|}u_i)\right) \\ = S_1 + S_2, \end{aligned}$$

$$S_1 = \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l+1} \lambda^{|\alpha|} D_x^{|\alpha|} D_t^j (\sigma^{|\alpha|} u_l),$$

$$\begin{aligned} S_2 &= \sum_{j+|\alpha|=m, j<m} (\tilde{a}_{j,\alpha,l+1} - \tilde{a}_{j,\alpha,l}) \sum_{i=1}^{l-1} \lambda^{|\alpha|} D_x^{\alpha} D_t^j (\sigma^{|\alpha|} u_i) \\ &= \sum_{j,\alpha,k} d_{3\alpha lk}(x,t) \left(c_{k,0} D_t^k u_{l+1} \right) \sum_{i=1}^{l-1} \lambda^{|\alpha|} D_x^{\alpha} D_t^j (\sigma^{|\alpha|} u_i) \\ &\quad + \sum_{j,\alpha,|\beta|>0,k} d_{4\alpha lk\beta}(x,t) \left(D_x^{\beta} c_{k,\beta} D_t^k u_l \right) \sum_{i=1}^{l-1} \lambda^{|\alpha|} D_x^{\alpha} D_t^j (\sigma^{|\alpha|} u_i). \end{aligned}$$

The H^{s_0-ml}–norms of the functions d_{1lk}, $d_{2lk\beta}$, $d_{3\alpha lk}$ and $d_{4\alpha lk}$ are bounded since

$$\begin{aligned} &\sum_{j=0}^{m-1} \left\| D_t^j u_i(.,t) \right\|_{H^{s_0-m(l-1)}} \le C_i \lambda(t)^{\varepsilon(i-1)}, \quad i \le l, \\ &\sum_{j=0}^{m-1} \left\| D_t^j u_{l+1}(.,t) \right\|_{H^{s_0-ml}} \le C. \end{aligned}$$

The function $\lambda(t)^{m-\varepsilon} \Lambda(t)^{1-m}$ is monotonically increasing due to the choice of ε, see Condition 3. This implies

$$\|c_{k,\beta}(.,t)\|_{H^s} \le C_s \lambda(t)^{\varepsilon} \quad \forall s \in \mathbb{R}^+, \quad \forall |\beta| > 0.$$

We conclude that u_{l+1} is a solution of

$$D_t^m u_{l+1} + \sum_{k<m} d_{5k}(x,t) D_t^k u_{l+1} = h_{l+1}(x,t)$$

with

$$\|h_{l+1}(.,t)\|_{H^{s_0-ml}} \le C \lambda(t)^{\varepsilon l}, \quad \|d_{5k}(.,t)\|_{H^{s_0-ml}} \le C. \tag{4.4.3}$$

Utilising a standard technique one shows

$$\sum_{j=0}^{m-1} \left\| D_t^j u_{l+1}(.,t) \right\|_{H^{s_0-ml}} \le C \int_0^t \|h_{l+1}(.,\tau)\|_{H^{s_0-ml}} \, d\tau \le C_{l+1} \lambda(t)^{\varepsilon l}.$$

This is the desired estimate. Summing up the differential equations for u_1,

$\dots$, u_l we deduce that

$$\begin{aligned}
D_t^m(u_1+\dots+u_l) &= f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big)\\
&\quad - \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l} \sum_{i=1}^{l-1} \lambda^{|\alpha|} D_x^\alpha D_t^j(\sigma^{|\alpha|} u_i),\\
D_t^j(u_1+\dots+u_l)(x,0) &= \varphi_j(x), \quad j=0,\dots,m-1.
\end{aligned}$$

If a function v has homogeneous initial data and satisfies

$$\begin{aligned}
&D_t^m v\\
&= f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big) - f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big)\\
&- \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big)\\
&\qquad \times \lambda^{|\alpha|} D_x^\alpha D_t^j \Big(\sigma^{|\alpha|}\Big(\sum_{i=1}^{l} u_i + v\Big)\Big)\\
&+ \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big) \lambda^{|\alpha|} D_x^\alpha D_t^j \Big(\sigma^{|\alpha|} \sum_{i=1}^{l-1} u_i\Big),
\end{aligned}$$

then the function $u := \sum_{i=1}^{l} u_i + v$ solves (4.4.1). We define

$$\begin{aligned}
&a_{j,\alpha,l}(x,t,\{D_x^\beta c_{k,\beta} D_t^k v\}) := a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big),\\
&f_l(x,t,\{D_x^\beta c_{k,\beta} D_t^k v\})\\
&:= f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big) - f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big)\\
&- \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big)\\
&\qquad \times \lambda^{|\alpha|} D_x^\alpha D_t^j \Big(\sigma^{|\alpha|} \sum_{i=1}^{l} u_i\Big)\\
&+ \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big) \lambda^{|\alpha|} D_x^\alpha D_t^j \Big(\sigma^{|\alpha|} \sum_{i=1}^{l-1} u_i\Big)
\end{aligned}$$

and obtain (4.4.2) with $p' := l$. It remains to verify the conditions (4.3.2) and (4.3.3) for this function f_l: We fix $\delta > 0$ and restrict all the intervals $[0, T_l]$ in such a way that

$$\left\| D_t^j \sum_{i=1}^{l} u_i(.,t) - \varphi_j(.) \right\|_{H^{s_0-ml}} \leq \delta, \quad 0 \leq t \leq T_l, \quad 0 \leq j \leq m-1.$$

The condition (4.3.3) is obviously satisfied for $t \leq T_l$ and the constants $C_{fk\beta}$ only depend on δ, but not on l. This argument reveals more, namely that the constant C_3 from (4.3.16) and (4.3.17) only depends on δ and does not depend on p and l. In a similar way as in the proof of (4.4.3) one shows

$$\|f_l(.,t,0)\|_{H^{s_0-ml}} \leq C_l \lambda(t)^{\varepsilon l},$$

which proves (4.3.2) with, e.g., $p := \varepsilon l - 1$. It has to be remarked that the meaning of s_0 has changed. One has to replace s_0 by $s_0 - ml$.
Theorem 4.3.5 gives the existence of a solution v to (4.4.2) which satisfies $\langle D \rangle^k ((\lambda\sigma)^k v) \in C\left([0, T^*], H^{s_0-ml}\right)$, if p and $s_0 - ml$ are large enough.
The proof is complete. □

Proposition 3.3.4 leads to the following criterion for the blow–up of the solution:

Corollary 4.4.2 (Blow–up). *Let the solution U from Theorem 4.4.1, which belongs to $C([0,T), H^{s_0-\gamma}) \cap C^1([0,T), H^{s_0-\gamma-1})$, satisfy*

$$\sup_{[0,T)} \|U^*(t)\|_{C_*^1} < \infty,$$
$$\inf_{[0,T)} \operatorname{dist}((x, \{U_{k,\beta}(x,t)\}), \partial K_G) \geq \delta > 0.$$

The vector U^* is defined in a similar way to V^*, cf. (4.3.8). If $s_0 - \gamma > 1 + n/2$, then a constant $T_1 > T$ exists with the property that

$$U^* \in C\left([0, T_1], H^{s_0-\gamma}\right) \cap C^1\left([0, T_1], H^{s_0-\gamma-1}\right)$$

is a solution.

Proof. Theorem 4.4.1 shows the existence of a solution u in the interval $[0, t_0]$. We consider the Cauchy problem with data for $t = t_0$. Then Proposition 3.3.4 yields the assertion. □

In other words, a blow–up of the solution in the $H^{s_0-\gamma}$ norm can only happen if the solution blows up in the C_*^1 norm or if some argument of the coefficients or the right–hand side leaves the domain of definition.

4.5 Examples

In this section we give some examples for the weight functions $c_{k,\beta}$, $c^0_{k,\beta}$.

Example 4.5.1. *We may choose* $c^0_{k,\beta}(x,t) = \sigma(x)^{|\beta|}$.

Example 4.5.2. *We may take functions* $d^0_{k,\beta} \in C^1([0,T], C^\infty(M))$ *and define*

$$c^0_{k,\beta}(x,t) = d^0_{k,\beta}(x,t)\sigma(x)^{|\beta|}.$$

However, this choice can be reduced to the first example by modifying the definition of the functions $a_{j,\alpha}$, f.

Example 4.5.3. *Let* M *be the unit circle. Functions on* M *can be regarded as* 2π*–periodic functions over* $\mathbb{R}$. *We define*

$$\sigma(x) := \exp\left(-(\sin(x))^{-2}\right),$$
$$c^0_{k,\beta}(x,t) = \begin{cases} (t+1)\sigma(x)^{|\beta|} & : x \in [2k\pi, (2k+1)\pi), \quad k \in \mathbb{Z}, \\ (t+2)\sigma(x)^{|\beta|} & : x \in [(2k+1)\pi, (2k+2)\pi), \quad k \in \mathbb{Z}. \end{cases}$$

The functions $c^0_{k,\beta}(x,t)$ *belong to* $C^\infty([0,T] \times M)$ *and we can write*

$$c^0_{k,\beta}(x,t) = d(x,t)\sigma(x)^{|\beta|}$$

with some function $d(x,t) \in C^\infty([0,T], L^\infty(M))$. *Since the function* $d(x,t)$ *is not smooth enough it is not possible to reduce this example to the first one. Otherwise one would get coefficients* $\tilde{a}_{j,\alpha}$ *and a right–hand side* f *which are not differentiable with respect to* x.

The last example shows that the usage of the weight functions $c_{k,\beta}(x,t)$ with (4.3.4)–(4.3.7) gives more generality than the natural choice

$$c_{k,\beta}(x,t) = \begin{cases} \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}\sigma(x)^{|\beta|} & : |\beta| > 0, \\ 1 & : |\beta| = 0 \end{cases}$$

that is motivated by the Levi conditions.

Chapter 5

Domains of Dependence

5.1 Introduction

One of the fundamental properties of solutions to strictly hyperbolic Cauchy problems is the finite propagation speed, or, in other words, the existence of a cone of dependence. We will see how to extend these properties to weakly hyperbolic equations. In [Kaj83] the cone of dependence for solutions of fully nonlinear weakly hyperbolic systems (including Leray–Volevich effects) was proved using the theory of local solutions in Gevrey spaces. In this Ph.D. thesis solutions in Sobolev spaces will be studied. Since the existence of a cone of dependence implies uniqueness of solutions, and because the uniqueness of solutions to hyperbolic equations with time degeneracy could not be proved in spaces of low Sobolev regularity up to now, we study equations with spatial degeneracy only.
We will construct so-called *domains of dependence*. It turns out that our definition generalises the definition from [AM84] from the strictly hyperbolic case to the weakly hyperbolic case. These domains can be exhausted with hypersurfaces, and the Cauchy problem is weakly hyperbolic in the normal direction at each point of each hypersurface, see Definition 5.2.1. One example of such domains is a cone, whose slope does not exceed some critical value, see Example 5.2.2. This concept will be applied to prove some results of uniqueness, finite propagation speed and regularity:

Global uniqueness for linear equations: The solution of a linear Cauchy problem is unique in any domain of dependence, see Theorem 5.3.1.

Local uniqueness for quasilinear equations: For every ball in the initial plane one can find a cone (with suitably small slope) over this ball with the property that the solution is unique in this cone, see Theorem 5.4.4.

Local existence for quasilinear equations: For every rectangle in the initial plane one can find a rectangular parallelepipedon over the rectangle with the property that a Sobolev solution of the quasilinear equation exists in this parallelepipedon, cf. Theorem 5.4.1. This solution exists in the whole domain of dependence if the equation is linear, cf. Corollary 5.4.5.

Extension property for solutions of quasilinear equations: Let Ω_0 be a domain in the initial plane and let u_1, u_2 be two solutions of a quasilinear Cauchy problem which are defined in the domains Ω_1, Ω_2 over Ω_0. Let Ω_1 and Ω_2 be domains of dependence. Then $u_1 \equiv u_2$ in $\Omega_1 \cap \Omega_2$, see Theorem 5.4.6.

C^∞ regularity: We consider a quasilinear Cauchy problem, whose coefficients, right-hand side, weight functions and initial data are C^∞. Let us be given a Sobolev solution in some domain of dependence. Then this solution is C^∞ in this domain, cf. Theorem 5.5.2.

How to extend the results presented in this chapter is a challenging question. Let us consider two examples. First, one may think about the *analytic regularity*: we study a quasilinear Cauchy problem, whose coefficients, right-hand side, weight functions and initial data are analytic. Let us be given a Sobolev (or C^∞) solution in some domain of dependence. Then this solution is analytic in this domain. It is planned to devote a forthcoming publication to this subject. The problem of analytic regularity was studied in [AM84] for fully nonlinear strictly hyperbolic equations and in [CZ97] for a class of weakly hyperbolic equations with characteristic roots of constant multiplicity. In [CZ95] it was shown that solutions are analytic in domains of dependence, if the quasilinear equation satisfies some Gevrey type Levi conditions and the solution is from some Gevrey space.

A second example is the *Gevrey regularity*: we consider a given quasilinear Cauchy problem, whose coefficients, right-hand side, weight functions and initial data are from some Gevrey space G^s, $1 < s < \infty$. Then every Sobolev solution is a Gevrey function defined in the aforementioned domain of dependence. An important tool to prove the Gevrey regularity is a local existence result in Gevrey spaces. Such local existence results can be found, for instance, in [Kaj83] or [GR]. Gevrey regularity results for second order equations have been proved in [MT96] (for the domain $\mathbb{R}^n \times [0, T)$).

5.2 Definition of Domains of Dependence

We come to the definition of a domain of dependence. Here we cite ideas from [AM84].
We contemplate the Cauchy problem

$$D_t^m u + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) = f(x,t), \tag{5.2.1}$$
$$u(x,0) = \varphi_0(x), \dots, D_t^{m-1} u(x,0) = \varphi_{m-1}(x)$$

for $(x,t) \in \Omega_0 \times [0,T] \subset \mathbb{R}_x^n \times \mathbb{R}_t^1$; Ω_0 is an open and bounded domain with smooth boundary.
We suppose Condition 1 and

$$\sigma \in C_b^\infty(\Omega_0), \tag{5.2.2}$$
$$a_{j,\alpha} \in \begin{cases} C_b^1(\overline{\Omega_0} \times [0,T]) & : j+|\alpha| = m, \\ C_b^0(\overline{\Omega_0} \times [0,T]) & : j+|\alpha| < m. \end{cases} \tag{5.2.3}$$

The principal part of the operator from the left is

$$P_{m,\sigma}(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j \sigma(x)^{|\alpha|}.$$

To this operator we assign the strictly hyperbolic operator

$$P_{m,1}(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j.$$

The domain of dependence Ω over a bounded domain $\Omega_0 \subset \mathbb{R}^n$ has to satisfy the following conditions. At first,

$$\Omega = \Omega' \cap \{(x,t) : t \ge 0\}, \quad \Omega' \Subset \mathbb{R}^{n+1}, \quad \Omega' \text{ open }, \tag{5.2.4}$$
$$\Omega_0 = \Omega' \cap \{(x,t) : t = 0\}. \tag{5.2.5}$$

Next, the projections $\pi : (x,t) \mapsto x$ of the level sets $\Omega_{t_0} := \Omega' \cap \{(x,t) : t = t_0\}$ shall become "smaller" for increasing t_0,

$$\pi\Omega_{t_1} \Subset \pi\Omega_{t_0} \quad \forall 0 \le t_0 < t_1 \le T. \tag{5.2.6}$$

The set Ω can be exhausted with hyper-surfaces S_r,

$$\Omega = \bigcup_{0\le r<r^*} S_r := \bigcup_{0\le r<r^*} \{(x,t) : g(x,t) = r\}. \tag{5.2.7}$$

We suppose $g \in C_b^\infty(\Omega_0 \times [0,T])$ and

$$\frac{\partial g}{\partial t} > 0 \quad \text{in } \Omega. \tag{5.2.8}$$

Furthermore, we assume that each hypersurface S_r intersects[1] with the initial domain Ω_0 and

$$\left(\Omega_0 \cap \bigcup_{0\le r\le r_0} S_r\right) \Subset \left(\Omega_0 \cap \bigcup_{0\le r\le r_1} S_r\right) \quad \forall 0 \le r_0 < r_1 < r^*. \tag{5.2.9}$$

Finally, we will need a connection between the slope of the normal vector to S_r at the point (x,t) and the largest characteristic root of $P_{m,\sigma}$ at the point (x,t):

$$\begin{aligned} &\lambda_{max,\sigma}(x,t)\left|\frac{\nabla_x g(x,t)}{g_t(x,t)}\right| < 1, \qquad (5.2.10)\\ &\lambda_{max,\sigma}(x,t) = \sup\{|\tau| : P_{m,\sigma}(x,t,\tau,\xi) = 0, |\xi| = 1\},\\ &P_{m,\sigma}(x,t,\tau,\xi) = \tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\sigma(x)^{|\alpha|}\xi^\alpha\tau^j. \end{aligned}$$

This condition can be interpreted in the way that the polynomial $P_{m,\sigma}$ is weakly hyperbolic at the point (x,t) in the normal direction of S_r. Later we will use the equivalent formula

$$\begin{aligned} &\left|\frac{\sigma(x)\nabla_x g(x,t)}{g_t(x,t)}\right| < \frac{1}{\lambda_{max,1}(x,t)},\\ &\lambda_{max,1}(x,t) = \sup\{|\tau| : P_{m,1}(x,t,\tau,\xi) = 0, |\xi| = 1\},\\ &P_{m,1}(x,t,\tau,\xi) = \tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\xi^\alpha\tau^j. \end{aligned}$$

For technical reasons we assume the following condition:

$$\text{The set } \Omega_0 \text{ is strongly star–shaped with centre } x^*. \tag{5.2.11}$$

"Strongly star–shaped with centre x^*" means that a point $x^* \in \Omega_0$ exists with the property that each ray starting at x^* intersects $\partial\Omega_0$ at exactly one point. This restriction excludes the case that some line segment of some ray starting from x^* belongs to $\partial\Omega_0$. The mapping $S^{n-1} \to \partial\Omega_0$ which maps a ray to its intersection point with $\partial\Omega_0$ is assumed to be C^∞.

[1] This condition differs from the definition of [AM84]. There it was assumed that $S_r \cap \Omega_0 = \emptyset$ and $S_r \cap \overline{\Omega_0} = \partial\Omega_0$ for each r.

If (5.2.11) holds, then there is a well-known radial extension of functions defined in $\Omega \times [0,T]$ to functions defined in $\mathbb{R}^n \times [0,T]$. It is possible to relax Condition (5.2.11); for instance, it may be assumed that Ω_0 is C^∞ diffeomorphic to a strongly star–shaped domain.

Definition 5.2.1. *A set Ω is called a* domain of dependence over Ω_0 for the operator $P_{m,\sigma}$ *if the conditions (5.2.4)–(5.2.11) are satisfied.*

Example 5.2.2 (Characteristic cone). *The characteristic cone $K(B)$ for the ball $B = B(x^*, d)$ in the initial plane is defined by*

$$K(B) = \left\{ (x,t) : |x - x^*| < d - \lambda'_{max,\sigma} t, \quad 0 \le t < \frac{d}{\lambda'_{max,\sigma}} \right\}$$

with

$$\lambda'_{max,\sigma} = \lambda'_{max,1} \, \|\sigma\|_{L^\infty(B)} \,,$$
$$\lambda'_{max,1} = \sup \left\{ |\tau| : P_{m,1}(x,t,\tau,\xi) = 0, \quad (x,t) \in \overline{B} \times [0,T], \quad |\xi| = 1 \right\}.$$

Figure 5.1 shows this cone and one of the exhausting surfaces S_r.

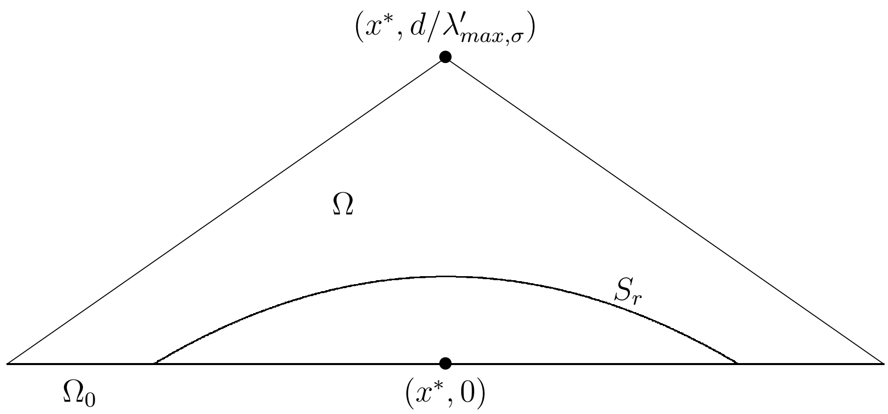

Figure 5.1: Characteristic cone of dependence

In this section we will also examine the quasilinear Cauchy problem

$$\begin{aligned} &D_t^m u + \sum_{j+|\alpha| \le m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta \sigma(x)^{|\beta|} D_t^k u\}) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) \\ &\quad = f(x,t,\{D_x^\beta \sigma(x)^{|\beta|} D_t^k u\}), \quad k + |\beta| \le m-1, \qquad (5.2.12) \\ &u(x,0) = \varphi_0(x), \ldots, D_t^{m-1} u(x,0) = \varphi_{m-1}(x). \end{aligned}$$

This equation will be written as $P^{(u)}_{m,\sigma}u = f^{(u)}$.
If u is a solution of this Cauchy problem, then one can define a domain of dependence $\Omega^{(u)}$, which itself depends on u, since the coefficients of the principal part depend on u. For this Cauchy problem we will assume almost the same conditions as in the case of the torus:

$$a_{j,\alpha} \in C^1([0,T], C^{s_0}(\Omega_0 \times \mathbb{R}^{n_0})), \quad s_0 > \frac{n}{2} + 1, \tag{5.2.13}$$

$$\varphi_j \in H^{s_0+m-1-j}(\Omega_0), \tag{5.2.14}$$

$$f \in C([0,T], C^{s_0}(\Omega_0 \times \mathbb{R}^{n_0})). \tag{5.2.15}$$

And we replace Condition 2 by Condition 4:

Condition 4. *The roots* $\tau_j(t,x,v,\xi)$ *of*

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,v)\xi^\alpha\tau^j = 0$$

are real and distinct,

$$|\tau_j(t,x,v,\xi) - \tau_i(t,x,v,\xi)| \geq c|\xi|, \quad c > 0, \quad i \neq j$$

for all

$$(t,x,v,\xi) \in [0,T] \times \Omega_0 \times \mathbb{R}^{n_0} \times \mathbb{R}^n.$$

5.3 Uniqueness for Linear Equations

There is another way to define domains of dependence. A set $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^n$ is called *domain of dependence over* $\Omega_0 \subset \mathbb{R}^n$ for some hyperbolic operator if the vanishing of data in Ω_0 and the vanishing of the right–hand side in Ω imply the vanishing of the solution to the Cauchy problem in Ω. The following theorem states that the two definitions are in concordance.

Theorem 5.3.1 (Uniqueness). *We suppose (5.2.2), (5.2.3) and Condition 1. Let* $\Omega \subset \Omega_0 \times [0,T]$ *be a domain of dependence over* Ω_0 *for the operator* $P_{m,\sigma}$. *Let* $\varphi_0 \equiv \cdots \equiv \varphi_{m-1} \equiv 0$ *in* Ω_0 *and* $f \equiv 0$ *in* Ω. *Then* $u \equiv 0$ *in* Ω *for every solution* u *of (5.2.1) with*

$$D_t^{m-|\alpha|}D_x^\alpha\sigma^{|\alpha|}u \in C(\Omega) \quad \forall|\alpha| \leq m.$$

Proof. Let us sketch the structure of the proof. The solution is defined in Ω, the coefficients $a_{j,\alpha}$ are defined in $\overline{\Omega_0}\times[0,T]$. In a first step we extend u and $a_{j,\alpha}$

to the domain $\Omega_0 \times (-\infty, 0]$; the function u vanishes there. This gives a new Cauchy problem with solution u. In a second step we transform the variables. The domain Ω is mapped to some domain $\tilde{\Omega} = \{(y, r) : y \in \Omega_0, g(y, 0) \leq r < r^*\}$, and v, the image of u, vanishes in $\{(y, r) : y \in \Omega_0, 0 \leq r < g(y, 0)\}$. Consequently, this function v has compact support with respect to the spatial variable y if the time variable is frozen. In order to show that v vanishes in $\tilde{\Omega}$, we will apply an energy estimate to be given later. Before we can do this, the coefficients $a_{j,\alpha}$ and the weight function σ must be extended from $\Omega_0 \times [0, T]$ to $\mathbb{R}^n \times [0, T]$. Then, the energy estimate leads to $v \equiv 0$. This will prove the theorem.

The coefficients $a_{j,\alpha}$ satisfy Condition 1, (5.2.3) and are defined in $\Omega_0 \times [0, T]$. For $0 < t \leq \varepsilon$ we set

$$a_{j,\alpha}(x, -t) = 2a_{j,\alpha}(x, 0) - a_{j,\alpha}(x, t), \quad x \in \Omega_0.$$

We obtain $a_{j,\alpha} \in C^1_b(\Omega_0 \times [-\varepsilon, T])$ for $j + |\alpha| = m$ and $a_{j,\alpha} \in C^0_b(\Omega_0 \times [-\varepsilon, T])$ for $j + |\alpha| < m$. If ε is sufficiently small, then the $a_{j,\alpha}$ satisfy Condition 1 on $\Omega_0 \times [-\varepsilon, T]$. This follows from the continuity of $a_{j,\alpha}$. In general this extension procedure is not applicable to the interval $(-\infty, 0]$, since generally Condition 1 will be violated. Therefore we need another procedure to extend from $[-\varepsilon, T]$ to $(-\infty, T]$.

The coefficients $a_{j,\alpha}$ will be extended by $a_{j,\alpha}(x, t) = a_{j,\alpha}(x, -\varepsilon)$ for $t \leq -\varepsilon$. But before this can be done, we have to guarantee that

$$D_t a_{j,\alpha}(x, -\varepsilon + 0) = 0 \quad \forall j + |\alpha| = m,$$

because the highest order coefficients must be C^1. For this purpose we take a function $\chi(s) \in C^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$,

$$\chi(s) = \begin{cases} 0 & : s \geq -1, \\ 1 & : s \leq -2 \end{cases}$$

and choose $0 < \delta < \varepsilon/2$. If $-\varepsilon \leq t \leq 0$ and $j + |\alpha| = m$, then we define

$$\begin{aligned} a_{j,\alpha,\delta}(x, t) := a_{j,\alpha}(x, t) \left(1 - \chi\left(\frac{2}{\delta}(t + \varepsilon) - 3\right)\right) \\ + a_{j,\alpha}(x, -\varepsilon) \chi\left(\frac{2}{\delta}(t + \varepsilon) - 3\right). \end{aligned}$$

If $t \geq -\varepsilon + \delta$, then $\chi(\frac{2}{\delta}(t + \varepsilon) - 3) = 0$ and from $t \leq -\varepsilon + \delta/2$ follows that $\chi(\frac{2}{\delta}(t + \varepsilon) - 3) = 1$. We see that $a_{j,\alpha,\delta} \in C^1_b(\Omega_0 \times [-\varepsilon, T])$, $D_t a_{j,\alpha,\delta}(x, -\varepsilon) = 0$ and $\lim_{\delta \to 0} a_{j,\alpha,\delta}(x, t) = a_{j,\alpha}(x, t)$ uniformly. Consequently, Condition 1 is satisfied for small $\delta > 0$. We fix this δ and write $a_{j,\alpha}$ instead of $a_{j,\alpha,\delta}$. Finally, we set

$$a_{j,\alpha}(x,t) = a_{j,\alpha}(x,-\varepsilon) \quad \forall (x,t) \in \Omega_0 \times (-\infty, -\varepsilon]$$

and conclude that $a_{j,\alpha} \in C_b^1(\Omega_0 \times (-\infty, T])$ for $j + |\alpha| = m$ and $a_{j,\alpha} \in C_b^0(\Omega_0 \times (-\infty, T])$ for $j + |\alpha| < m$. The Condition 1 is true in this domain.
The same method can be used to extend the derivative $g_t(x,t)$ of the function $g(x,t)$ to $\Omega_0 \times (-\infty, T]$. The result is

$$g \in C^\infty(\Omega_0 \times (-\infty, T]), \quad g_t \in C_b^\infty(\Omega_0 \times (-\infty, T]),$$
$$g_t(x,t) = g_t(x,-\varepsilon) \quad (t \leq -\varepsilon).$$

We see that the function g takes arbitrarily small values and conclude that

$$\Omega \cup (\Omega_0 \times (-\infty, 0]) = \bigcup_{-\infty < r < r^*} S_r.$$

We extend the solution u by

$$u(x,t) = 0, \quad (x,t) \in \Omega_0 \times (-\infty, 0] =: Z$$

(the letter Z means that the function u is zero). Then u solves (5.2.1) in Z with homogeneous data for $t = -1$, $f \equiv 0$ in Z, and

$$D_t^{m-|\alpha|} D_x^\alpha \sigma^{|\alpha|} u \in C(\Omega \cup Z) \quad \forall |\alpha| \leq m.$$

We apply a Holmgren type transform to change the variables,

$$y := x,$$
$$r := g(x,t),$$
$$\nabla_x = \nabla_y + (\nabla_x g)\partial_r,$$
$$\partial_t = (\partial_t g)\partial_r,$$
$$v(y,r) := u(x,t).$$

The dual variables fulfil

$$\xi = \eta + (\nabla_x g)\varrho =: \eta + \vec{c}(y,r)\varrho,$$
$$\tau = g_t \varrho =: c_0(y,r)\varrho.$$

The domain Ω is mapped to the set (cf. Figure 5.2)

$$\tilde{\Omega} = \{(y,r) : y \in \Omega_0, g(y,0) \leq r < r^*\},$$

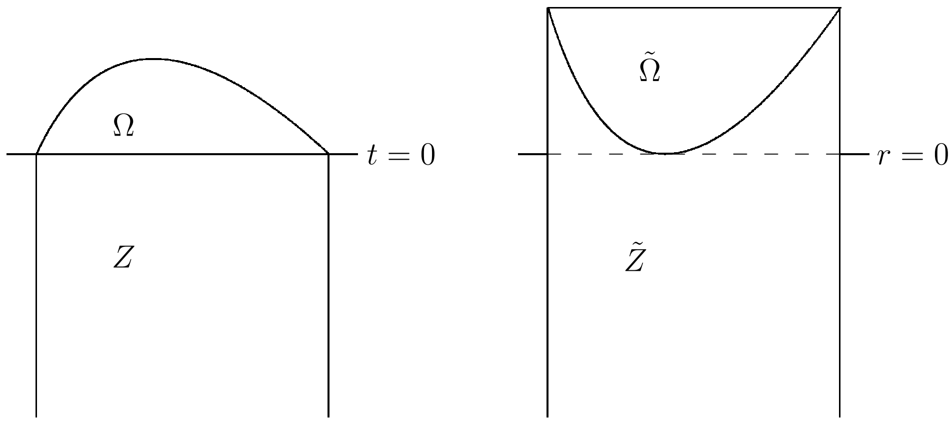

Figure 5.2: Transformation of variables

and $\tilde{Z}$ is the image of Z:

$$\tilde{Z} = \{(y,r) : y \in \Omega_0, -\infty < r \le g(y,0)\}.$$

Let us study the smoothness of the function v. For $|\alpha| \le m$ it holds

$$\begin{aligned} &D_r^{m-|\alpha|} D_y^\alpha \sigma(y)^{|\alpha|} v(y,r) \\ &\quad = \left(\frac{1}{c_0(x,t)} D_t\right)^{m-|\alpha|} \left(D_x - \frac{\vec{c}(x,t)}{c_0(x,t)} D_t\right)^\alpha \sigma(x)^{|\alpha|} u(x,t) \\ &\quad = \sum_{\beta\le\alpha, i\le j} c_{\alpha j \beta i}(x,t) D_x^\beta \sigma(x)^{|\beta|} D_t^i u(x,t) \in C(\Omega \cup Z). \end{aligned}$$

This implies $D_r^{m-|\alpha|} D_y^\alpha \sigma(y)^{|\alpha|} v(y,r) \in C(\tilde{\Omega} \cup \tilde{Z})$ for $|\alpha| \le m$.
The function v is a solution of

$$\begin{aligned} &D_r^m v + \sum_{j+|\alpha|\le m, j<m} \tilde{a}_{j,\alpha}(y,r) D_y^\alpha D_r^j \left(\sigma(y)^{|\alpha|} v\right) = 0, \\ &v(y,0) = \cdots = D_r^{m-1} v(y,0) = 0, \quad (y,r) \in \Omega_0 \times [0, r^*), \end{aligned}$$

$\tilde{a}_{j,\alpha}(y,r)$ are the transformed coefficients. We have to check whether this Cauchy problem is weakly hyperbolic and satisfies the Levi conditions. We remember the definition of (strict) hyperbolicity, see [Miz73]:

Definition 5.3.2. *A differential operator $Q(z, D_z) = \sum_{|\alpha|=m} a_\alpha(z) D_z^\alpha$ is called hyperbolic at the point z_0 in the direction $N \ne 0$ if*

- $Q(z_0, N) \ne 0$,

- *$Q(z_0, \tau N + \zeta) = 0$ has only real roots τ for every $\zeta \neq 0$.*

A differential operator $Q(z, D_z)$ is called strictly hyperbolic at (z_0, N) if it is hyperbolic at (z_0, N) and if $Q(z_0, \tau N + \zeta) = 0$ has m real and distinct roots τ for every $\zeta \perp N$, $\zeta \neq 0$.

By definition, the operator

$$P_{m,\sigma}(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\sigma(x)^{|\alpha|} D_t^j D_x^\alpha$$

is hyperbolic in the direction $N = (1, 0, \ldots, 0) \in \mathbb{R}^{1+n}$ and the operator

$$P_{m,1}(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_t^j D_x^\alpha$$

is strictly hyperbolic in this direction N. The symbol of the principal part $\tilde{P}_{m,\sigma}$ of the transformed operator is

$$P_{m,\sigma}(x,t,\eta + \vec{c}\varrho, c_0\varrho) = \tilde{P}_{m,\sigma}(y,r,\eta,\varrho).$$

That is to say

$$\begin{aligned}
&(c_0\varrho)^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)(\eta + \vec{c}\varrho)^\alpha (c_0\varrho)^j \sigma^{|\alpha|} \\
&\quad = (c_0\varrho)^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(c_0\varrho)^j \sigma^{|\alpha|} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \eta^\beta (\vec{c}\varrho)^{\alpha-\beta} \\
&\quad = (c_0\varrho)^m + \sum_{k+|\beta|=m, k<m} \eta^\beta \varrho^k \sigma^{|\beta|} \sum_{\alpha\geq\beta, j+|\alpha|=m} a_{j,\alpha} c_0^j \sigma^{|\alpha-\beta|} \binom{\alpha}{\beta} \vec{c}^{\alpha-\beta}.
\end{aligned}$$

If $k + |\beta| = m$ and $k < m$, then it holds

$$\tilde{a}_{k,\beta}(y,r) = \sum_{\alpha\geq\beta, j+|\alpha|=m} a_{j,\alpha}(x,t) c_0(x,t)^{-|\alpha|} \sigma(x)^{|\alpha-\beta|} \binom{\alpha}{\beta} \vec{c}(x,t)^{\alpha-\beta}.$$

To be able to apply the results of the Sections 3.2 and 3.3, we have to verify that the operator $\tilde{P}_{m,\sigma}$ is hyperbolic in the direction $\tilde{N}$ and that the operator $\tilde{P}_{m,1}$ is strictly hyperbolic in the direction $\tilde{N}$. Here $\tilde{N} = (1, 0, \ldots, 0) \in \mathbb{R}^1_\varrho \times \mathbb{R}^n_y$ is the normal direction of the hypersurfaces $r = \text{const}$.
At first we show

$$\tilde{P}_{m,\sigma}(y,r,\tilde{N}) = \tilde{P}_{m,\sigma}(y,r,0,1) \neq 0.$$

It can be seen that

$$\tilde{P}_{m,\sigma}(y,r,0,1) = P_{m,\sigma}(x,t,\vec{c},c_0) = |\vec{c}|^m P_{m,\sigma}\left(x,t,\frac{\vec{c}}{|\vec{c}|},\frac{c_0}{|\vec{c}|}\right).$$

From (5.2.10) we obtain

$$\frac{|c_0(x,t)|}{|\vec{c}(x,t)|} > \lambda_{max,\sigma}(x,t).$$

This and the definition of $\lambda_{max,\sigma}$ yield $\tilde{P}_{m,\sigma}(y,r,\tilde{N}) \neq 0$.
In the next step we show that the equation $\tilde{P}_{m,\sigma}(y,r,\eta,\varrho) = 0$ has m real roots $\varrho_1, \ldots, \varrho_m$ for every $\eta \neq 0$. It holds

$$\begin{aligned}\tilde{P}_{m,\sigma}(y,r,\eta,\varrho) &= P_{m,\sigma}(x,t,\eta+\vec{c}\varrho,c_0\varrho) = P_{m,1}(x,t,\sigma(\eta+\vec{c}\varrho),c_0\varrho)\\ &= c_0^m P_{m,1}\left(x,t,\frac{\sigma}{c_0}(\eta+\vec{c}\varrho),\varrho\right).\end{aligned}$$

If $\sigma(x) = 0$, then the only roots are $\varrho_1 = \cdots = \varrho_m = 0$. If $\sigma(x) \neq 0$, then we can write

$$\tilde{P}_{m,\sigma}(y,r,\eta,\varrho) = \sigma^m P_{m,1}\left(x,t,\eta+\vec{c}\varrho,\frac{c_0}{\sigma}\varrho\right).$$

The polynomial $P_{m,1}(x,t,\eta,c_0\varrho/\sigma)$ is strictly hyperbolic in the direction N. From

$$\left|\frac{\sigma(x)\vec{c}(x,t)}{c_0(x,t)}\right| < \frac{1}{\lambda_{max,1}(x,t)},$$

and Proposition B.0.2 it can be deduced that the polynomial $P_{m,1}(x,t,\eta+\vec{c}\varrho,(c_0/\sigma)\varrho)$ is strictly hyperbolic in the direction $N+\sigma\vec{c}/c_0$. We get that $\tilde{P}_{m,\sigma}(y,r,\eta,\varrho) = 0$ has m real roots. They are distinct if $\sigma(y) \neq 0$.
It remains to show that $\tilde{P}_{m,1}$ is strictly hyperbolic. It is easy to check that

$$\begin{aligned}\tilde{P}_{m,1}(y,r,\eta,\varrho) &= (c_0\varrho)^m + c_0^m \sum_{k+|\beta|=m,k<m} \eta^\beta \varrho^k \tilde{a}_{k,\beta}(y,r)\\ &= P_{m,1}(x,t,\eta+\sigma\vec{c}\varrho,c_0\varrho).\end{aligned}$$

Proposition B.0.2 shows that this polynomial is strictly hyperbolic in the direction $N+\sigma\vec{c}/c_0$. Hence we have m real and distinct roots $c_0\varrho_i$. The strict hyperbolicity is proved.
Let us come back to the function v. Our aim is to show that $v \equiv 0$ in $\tilde{\Omega}$. We will do this by the aid of Proposition 5.3.3 and Gronwall's Lemma. But

before the coefficients $\tilde{a}_{j,\alpha}$ and the weight σ have to be extended to the whole $\mathbb{R}^n_x \times [0,T]$, $\mathbb{R}^n_x$, respectively, since they are only defined on $\Omega_0 \times (-\infty, T]$ and Ω_0.
We fix some arbitrary $0 < r_0 < r^*$ and will show $v(y,r) = 0$ in $\Omega_0 \times [0, r_0]$. A small ε with $0 < 2\varepsilon < \operatorname{dist}(\partial\Omega_0, (\Omega_0 \cap S_{r_0}))$ is chosen and we change the values of $\tilde{a}_{j,\alpha}$ in some annular domain $A(r_0, 2\varepsilon)$. The family of these domains is defined by

$$A(r_0,\varepsilon) = \{(y,r) \in \Omega_0 \times [0,r_0] : \operatorname{dist}(y, \partial\Omega_0) < \varepsilon\}.$$

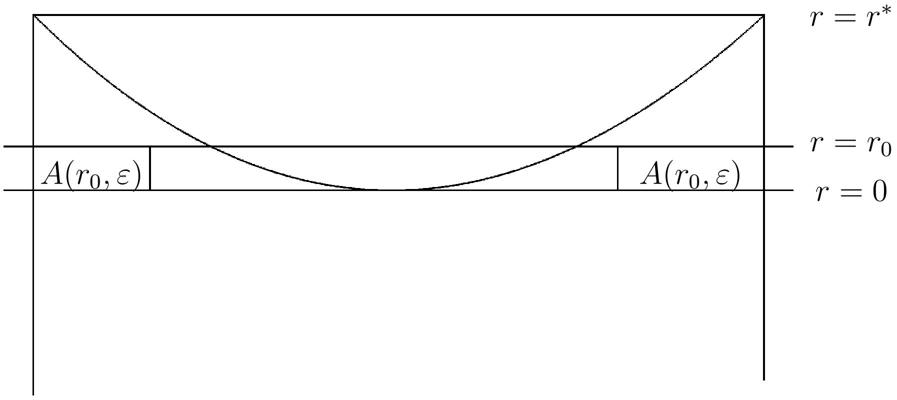

Figure 5.3: The annular domain $A(r_0, \varepsilon)$

Due to (5.2.11) the set Ω_0 is star-shaped with centre $y^* := x^*$. Then for each $y \in \Omega_0$, $y \neq y^*$, we find a unique point $y' \in \partial\Omega_0$, $y' = y'(y)$, lying on the ray starting at y^* and going through y. We replace $\tilde{a}_{j,\alpha}$ by

$$\begin{aligned} a_{j,\alpha,\varepsilon}(y,r) := \tilde{a}_{j,\alpha}(y,r)\left(1 - \chi\left(\frac{1}{\varepsilon}(|y-y'|) - 1\right)\right) \\ + \tilde{a}_{j,\alpha}(y'(y), r)\chi\left(\frac{1}{\varepsilon}(|y-y'|) - 1\right) \end{aligned}$$

with $\chi \in C^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$,

$$\chi(z) = \begin{cases} 1 & : z \leq 0, \\ 0 & : z \geq 1. \end{cases}$$

The difference $|\tilde{a}_{j,\alpha} - a_{j,\alpha,\varepsilon}|$ can be made arbitrarily small if ε is small enough. Hence the coefficients $a_{j,\alpha,\varepsilon}$ satisfy Condition 1. Due to Condition (5.2.11) the mapping $y \mapsto y'$ is C^∞. Then we have $a^*_{j,\alpha} := a_{j,\alpha,\varepsilon} \in C^1_b(\overline{\Omega_0} \times [0, r_0])$ (for $j + |\alpha| = m$) and $a^*_{j,\alpha}(y,r) = \tilde{a}_{j,\alpha}(y,r)$ for

$$\operatorname{dist}(y, \partial\Omega_0) > 2\varepsilon.$$

The use of this procedure is that the derivatives of $a^*_{j,\alpha}$ at $\partial\Omega_0$ in the directions $\tilde{n}$ along rays starting at the star-centre vanish,

$$\frac{\partial a^*_{j,\alpha}}{\partial \tilde{n}}\Big|_{\partial\Omega_0} = 0.$$

This allows us to extend the $a^*_{j,\alpha}$ in a radially constant way[2],

$$a^*_{j,\alpha}(y,r) := a^*_{j,\alpha}(y',r), \quad y \notin \Omega_0, \quad 0 \le r \le r_0.$$

We get coefficients $a^*_{j,\alpha} \in C^1_b(\mathbb{R}^n \times [0,r_0])$ for $j+|\alpha| = m$ and $a^*_{j,\alpha} \in C^0_b(\mathbb{R}^n \times [0,r_0])$ for $j+|\alpha| < m$. The same procedure is applied to the weight function σ. The function v can be extended by zero,

$$v(y,r) = 0, \quad y \notin \Omega_0, \quad 0 \le r \le r_0.$$

We see that v solves

$$D^m_t v + \sum_{j+\alpha=m, j<m} a^*_{j,\alpha}(y,r) D^\alpha_y D^j_r(\sigma(y)^{|\alpha|} v) = 0, \quad (y,r) \in \mathbb{R}^n \times [0,r_0],$$
$$v(y,0) = \cdots = D^{m-1}_r v(y,0) = 0, \quad y \in \mathbb{R}^n.$$

Proposition 5.3.3 and Gronwall's Lemma yield $v(y,r) = 0$ for $y \in \mathbb{R}^n$, $0 \le r \le r_0$. Since $r_0 < r^*$ can be chosen arbitrarily, we have $v = 0$ in $\tilde{\Omega}$, hence $u = 0$ in Ω. The theorem is proved. □

The following proposition giving an $L^2(\mathbb{R}^n)$ estimate for solutions to linear weakly hyperbolic Cauchy problems has been used in the proof of the previous theorem:

Proposition 5.3.3 (Energy estimate). *We consider the Cauchy problem (3.1.1) on the set $\mathbb{R}^n \times [t_0,T]$. We assume that the coefficients $a_{j,\alpha}$ only depend on (x,t) and that the right–hand side f depends on $\{D^\beta_x c_{k,\beta} D^k_t u\}$ in a linear way,*

$$f(x,t,\{D^\beta_x c_{k,\beta} D^k_t u\}) = f^*(x,t) + \sum_{k+|\beta| \le m-1} f_{k,\beta}(x,t) D^\beta_x c_{k,\beta} D^k_t u.$$

We suppose Condition 1, (3.2.1), (3.3.1)–(3.3.4) and

$$a_{j,\alpha} \in C^1_b(\mathbb{R}^n \times [t_0,T]),$$
$$f^*, f_{k,\beta} \in C^0_b(\mathbb{R}^n \times [t_0,T]),$$
$$\varphi_j \in H^{m-1-j}(\mathbb{R}^n).$$

[2] Of course it is possible to apply other methods to extend these functions, as long as the Condition 1 remains valid in $\mathbb{R}^n \times [0,T]$. Then the Condition (5.2.11) could be weakened. For instance, one may assume that the domain Ω_0 is C^∞ diffeomorphic to a strongly star–shaped domain.

Finally, we assume that (3.1.1) has a global solution u with

$$\langle D\rangle^k(\sigma^k D_t^{m-1-k}u) \in C([t_0,T],H^1(\mathbb{R}^n)).$$

for $0 \leq k \leq m-1$. Then it holds the estimate

$$d_t\left(R^*U^*,U^*\right) \leq C(\|U^*\|_{L^2}^2 + \|f^*\|_{L^2}^2).$$

Proof. We follow the lines of the proof of Proposition 3.2.2, part (b). We see that the commutators I_{11} and I_{21}, I_{22} vanish. Then the computations from the proof of Proposition 3.2.2 (with the obvious adaptions to the case of an equation with lower order terms from Section 3.3) give the assertion. □

5.4 Existence and Uniqueness for Quasilinear Equations

The next theorem studies the local existence of Sobolev solutions to quasilinear weakly hyperbolic equations.

Theorem 5.4.1 (Local existence). *We suppose (5.2.13)–(5.2.15) and Condition 4. Let $Q_0 = \Pi_{i=1}^n[a_i,b_i]$ be a rectangular parallelepipedon (RP for short), $Q_0 \Subset \Omega_0$. Then a constant $0 < T_0 \leq T$ and a solution u of (5.2.12) exist with*

$$D_t^j\sigma^{m-1-j}u \in C([0,T_0],H^{s_0+m-1-j}(Q_0)), \quad j = 0,\dots,m-1.$$

Proof. We take a cut–off function $\varphi(x)$ which is supported in a neighbourhood of Q_0 and is identical to 1 on Q_0. Then we replace the functions $\sigma(x)$, $f(x,t,\{V_{k,\beta}\})$, $\varphi_j(x)$ by $\varphi(x)\sigma(x)$, $\varphi(x)f(x,t,\{V_{k,\beta}\})$ and $\varphi(x)\varphi_j(x)$.
We leave it to the reader to check that the conditions of this theorem are still fulfilled. Let Q be an RP with $\operatorname{supp}\varphi \Subset Q$. Lemma B.0.3 is used to extend the coefficients $a_{j,\alpha}$ from $Q \times [0,T] \times \mathbb{R}^{n_0}$ to the larger set $Q' \times [0,T] \times \mathbb{R}^{n_0}$, Q' being an *RP* with twice the edge lengths of Q which can be regarded as a torus. We get a Cauchy problem on Q'. Theorem 3.3.1 shows that a solution u exists with the desired smoothness on the torus Q'. This function is a solution on $Q_0 \times [0,T_0]$, since $\varphi \equiv 1$ on Q_0. □

Remark 5.4.2. *Nothing has been said about the* uniqueness *of this solution u so far. The problem of uniqueness is studied in the next theorem.*

Remark 5.4.3. *The statement of this theorem remains true if the RP Q_0 is replaced by a ball B_0. This can be seen in the following way: Let us take a cube Q_0 with $B_0 \Subset Q_0$. If Q_0 is contained in Ω_0, one can proceed as in the above* *proof. If Q_0 is not entirely contained in Ω_0, the coefficients must be defined in $Q_0 \setminus B_0$ in a suitable way. By the aid of a cut–off function with support near B_0 we can extend the functions σ, f, φ_j from B_0 into $Q_0 \setminus B_0$, see the proof above. Then an idea from the end of the proof of Theorem 5.3.1 is applied to extend the functions $a_{j,\alpha}(.,t,\{V_{k,\beta}\})$ from the star-shaped domain B_0 into $Q_0 \setminus B_0$. This allows to proceed as in the previous proof. In any case we get a solution which is defined in $Q_0 \times [0,T_0]$.*

Theorem 5.4.4 (Local uniqueness). *Let the conditions of Theorem 5.4.1 be satisfied. Let $B_0 \Subset \Omega_0$ be a ball. Then a number $T_0 > 0$ and a cone Ω over B_0 exist with the property that a uniquely determined solution u with*

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T_0], H^{s_0+m-1-j}(\Omega_t)), \quad j = 0,\dots,m-1,$$

exists in Ω. The notation $v \in \mathcal{C}([0,T], H^s(\Omega_t))$ means that

- $\sup_{[0,T]} \|v(.,t)\|_{H^s(\Omega_t)} < \infty$,
- *for all T', Ω'_0 with $\Omega'_0 \times [0,T'] \Subset \Omega$ it holds $v \in C([0,T'], H^s(\Omega'))$.*

Here $H^s(\Omega_t)$ denotes the Sobolev–Slobodeckij space $W_2^s(\Omega_t)$, see [Tri78].

Proof. Remark 5.4.3 and Theorem 5.4.1 show that a small number T_0 and a solution u exist with

$$D_t^j \sigma^{m-1-j} u \in C([0,T_0], H^{s_0+m-1-j}(B_0)), \quad j = 0,\dots,m-1.$$

The function u solves

$$P_{m,\sigma}^{(u)} u = f^{(u)} \text{ in } B_0 \times [0,T_0],$$
$$D_t^j u = \varphi_j \text{ in } B_0.$$

We define the cone $\Omega := K(B_0)$ over B_0 as given in Example 5.2.2 with

$$P_{m,1}(x,t,\tau,\xi) = \tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta \sigma^{|\beta|} D_t^k u\}) \xi^\alpha \tau^j.$$

Then it follows that

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T_0], H^{s_0+m-1-j}(K(B_0)_t)), \quad j = 0,\dots,m-1.$$

Let U^* be another solution in Ω, i.e.,

$$P_{m,\sigma}^{(u^*)} u^* = f^{(u^*)} \text{ in } B_0 \times [0,T_0],$$
$$D_t^j u^* = \varphi_j \text{ in } B_0.$$

Consequently, by Hadamard's Formula,

$$\begin{aligned} P^{(u)}_{m,\sigma}(u-u^*) &= f^{(u)} - f^{(u^*)} + (P^{(u^*)}_{m,\sigma} - P^{(u)}_{m,\sigma})u^* \\ &= \sum_{k+|\beta|\le m-1} g_{k,\beta}(x,t) D^\beta_x(\sigma^{|\beta|} D^k_t(u-u^*)) \end{aligned}$$

with $g_{k,\beta} \in C(\Omega)$. Then Theorem 5.3.1 applied to the homogeneous Cauchy problem reveals $u - u^* \equiv 0$ in Ω. □

If the equation is *linear*, we even have *global* existence:

Corollary 5.4.5 (Global existence). *Let us consider the Cauchy problem (5.2.1). We suppose Condition 1 and*

$$\begin{aligned} &\sigma \in C^\infty(\Omega_0^*), \quad \Omega_0^* \Supset \Omega_0, \\ &a_{j,\alpha} \in \begin{cases} C^1([0,T], H^{s_0}(\Omega_0^*)) & : j+|\alpha| = m, \\ C([0,T], H^{s_0}(\Omega_0^*)) & : j+|\alpha| < m, \end{cases} \quad s_0 > \frac{n}{2}+1, \\ &\varphi_j \in H^{s_0+m-1-j}(\Omega_0^*), \\ &f \in C([0,T], H^{s_0}(\Omega_0^*)). \end{aligned}$$

Let Ω be a domain of dependence for the operator $P_{m,\sigma}$ over the domain Ω_0. Then a unique solution u exists with

$$D^j_t \sigma^{m-1-j} u \in \mathcal{C}([0,T], H^{s_0+m-1-j}(\Omega_t)).$$

Proof. We take a cut–off function $\varphi(x)$ that is identical to 1 in a neighbourhood of Ω_0 and supported in Ω_0^*. Then a cube Q with $\operatorname{supp}\varphi \Subset Q$ is chosen. We replace the functions $\sigma(x)$, $f(x,t)$, $\varphi_j(x)$ by $\varphi(x)\sigma(x)$, $\varphi(x)f(x,t)$ and $\varphi(x)\varphi_j(x)$. The coefficients $a_{j,\alpha}(.,t)$ are extended from Ω_0^* to $Q \setminus \Omega_0^*$ by the aid of the procedure from the end of the proof of Theorem 5.3.1. We gain a linear Cauchy problem on a torus. Corollary 3.3.6 gives us a solution defined in $Q \times [0,T]$ which has the desired smoothness. Theorem 5.3.1 shows that this solution is the only solution in Ω. □

The following Extension Theorem can be seen as a statement about *global* uniqueness for quasilinear Cauchy problems.

Theorem 5.4.6 (Extension of solutions). *Let u_1, u_2 be two solutions of (5.2.12), defined in domains Ω_1, Ω_2 over Ω_0. Let Ω_i be a domain of dependence over Ω_0 for the operator $P^{(u_i)}_{m,\sigma}$, $i = 1, 2$. We assume*

$$D^j_t \sigma^{m-1-j} u_i \in \mathcal{C}([0,T], H^{s_0+m-1-j}(\Omega_{i,t})), \quad 0 \le j \le m-1, \quad i = 1, 2.$$

Then $u_1 \equiv u_2$ in $\Omega_1 \cap \Omega_2$.

Proof. From Theorem 5.4.4 we conclude that $u_1 \equiv u_2$ near the initial domain Ω_0. That is to say, a smooth function $h(x) > 0$ exists with the property that $u_1(x,t) = u_2(x,t)$ in

$$M := \{(x,t) : x \in \Omega_0, 0 \leq t \leq h(x)\}.$$

Let $\{S_r\}_{0 \leq r < r^*}$ be the family of hypersurfaces which exhaust the domain Ω_1. We see that some $r_0 > 0$ exists with

$$M(r_0) := \bigcup_{0 \leq r \leq r_0} S_r \subset M.$$

It follows that $u_1(x,t) = u_2(x,t)$ for $(x,t) \in S_r$, $r \leq r_0$. Furthermore, we know that the solution of (5.2.12) is unique in $M(r_0)$. It will be shown that some $r_1 > r_0$ exists with the property that the solution of (5.2.12) is unique in the larger set $M(r_1)$.

Choose some arbitrary point $(x_0, t_0) \in S_{r_0}$ and a neighbourhood $B_0 = B_0(y_0, t_0)$ on the surface S_{r_0}. We apply Theorem 5.4.4 to some larger set $B^* \Supset B_0$ and find a number $T_0 = T_0(x_0, t_0, B_0, B_0^*)$ with the property that the transformed Cauchy problem

$$\tilde{P}^{(\tilde{u})}_{m,\sigma} \tilde{u} = \tilde{f}^{(\tilde{u})}$$

with data on the surface $r = r_0$ has a unique solution in

$$\{(y,r) : (y, r_0) \in B(x_0, t_0), r_0 \leq r \leq r_0 + T_0\}.$$

It is possible to cover S_{r_0} with $M \cap S_{r_0}$ and a finite number of neighbourhoods $B(x_0, t_0)$. Then the infimum I_0 of the T_0 is positive, we may set $r_1 := r_0 + I_0$. By induction we see that the solution is unique in the whole domain Ω_1. This gives $u_1 \equiv u_2$ in $\Omega_1 \cap \Omega_2$. □

5.5 C^∞ regularity

At first, let us show a *local* regularity result in C^∞.

Lemma 5.5.1 (Local C^∞–regularity). *Let u be a function defined in $\Omega^{(u)}$ which is a domain of dependence over Ω_0 for the operator $P^{(u)}_{m,\sigma}$. Let u with*

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T], H^{s_0+m-1-j}(\Omega_t)), \quad j = 0, \ldots, m-1,$$

be a solution of (5.2.12). The coefficients $a_{j,\alpha}$ and the right–hand side are supposed to be C^∞ with respect to all their arguments. We suppose that

$$u \in C^\infty(\Omega^{(u)} \cap \{(x,t) : t \leq t_0\})$$

for some $t_0 \geq 0$. *Let* $B(x_0, d) \Subset \Omega_{t_0}$ *be a ball. Then a number* $t_1 = t_1(x_0, t_0, d) > t_0$ *exists with*

$$u \in C^\infty(B(x_0, d) \times [t_0, t_1]), \quad B(x_0, d) \times [t_0, t_1] \Subset \Omega^{(u)}.$$

This number t_1 *continuously depends on* t_0 *and* x_0 *in the following sense: let* $(x_0', t_0') \in \Omega^{(u)}$ *be a point with* $t_0' \leq t_0$. *Then a number* $t_1' = t_1'(x_0', t_0', d) > t_0'$ *exists with*

$$u \in C^\infty(B(x_0', d) \times [t_0', t_1'])$$

and it holds $t_1' - t_0' \geq t_1 - t_0$ *if* $|t_0 - t_0'| + |x_0 - x_0'|$ *is sufficiently small.*

Proof. We apply the procedure given in the proof of Theorem 5.4.1 and Remark 5.4.3 to extend the Cauchy problem from $B(x_0, d) \times [t_0, T]$ to the set $Q' \times [t_0, T]$, with Q' being a torus. We get a quasilinear weakly hyperbolic Cauchy problem on a torus with C^∞ coefficients and C^∞ data. Theorem 3.3.7 gives us a local C^∞ solution. Theorem 5.4.4 shows that this solution is unique in some domain of dependence which contains some set $B(x_0, d) \times [t_0, t_1]$ with small $t_1 - t_0$. This implies $u \in C^\infty(B(x_0, d) \times [t_0, t_1])$.
Let (x_0', t_0') be close to (x_0, t_0) with $t_0' \leq t_0$. Then we can apply the same procedure again. We consider the ball $B'(x_0', d) \Subset \Omega_{t_0'}$. Following the path described in the previous paragraph we get a new Cauchy problem on some set $Q'' \times [t_0', T]$. Here Q'' is a torus and congruent to Q'. The set $Q'' \times [t_0', T - (t_0 - t_0')]$ can be mapped onto $Q' \times [t_0, T]$ by the aid of a translation. Let us denote the solution to this translated Cauchy problem by u''. Theorem 3.4.1 shows that u'' is close to u and persists up to t_1 if $|t_0' - t_0| + |x_0' - x_0|$ is small enough. It follows that $t_1' \geq t_1 - (t_0 - t_0')$. □

This lemma is an important tool to prove the following theorem:

Theorem 5.5.2 (Global C^∞ regularity). *Let* u *be a function defined in* $\Omega^{(u)}$ *which is a domain of dependence over* Ω_0 *for the operator* $P_{m,\sigma}^{(u)}$. *We suppose that* u *with*

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0, T], H^{s_0+m-1-j}(\Omega_t^{(u)})), \quad j = 0, \ldots, m-1,$$

is a solution of (5.2.12). We suppose

$$a_{j,\alpha}, f \in C_b^\infty(\Omega_0 \times [0, T] \times \mathbb{R}^{n_0}),$$
$$\varphi_j \in C_b^\infty(\Omega_0).$$

Then $u \in C^\infty(\Omega^{(u)})$.

Proof. If $B(x_0, d) \Subset \Omega_0$ is a ball, Lemma 5.5.1 gives us a number $t_1(x_0, d)$ with

$$u \in C^\infty(B(x_0, d) \times [0, t_1(x_0, d)]).$$

This implies that a smooth function $h = h(x) > 0$ and a set M exist with

$$M = \{(x, t) : x \in \Omega_0, 0 \le t < h(x)\}, \quad u \in C^\infty(M).$$

The domain $\Omega^{(u)}$ can be exhausted with hypersurfaces S_r. We transform the variables in the same way as in the proof of Theorem 5.3.1. This results in a quasilinear weakly hyperbolic initial value problem for the function v,

$$\tilde{P}^{(v)}_{m,\sigma} v = \tilde{f}(y, r, \{D^\beta_y \sigma(y)^{|\beta|} D^k_r v\}), \quad y \in \Omega_0, \quad g(y, 0) \le r < r^*,$$
$$v(y, g(y, 0)) = \psi_0(y), \ldots, D^{m-1}_r v(y, g(y, 0)) = \psi_{m-1}(y).$$

If $\tilde{M}$ stands for the image of M under the transformation of variables, then

$$v \in C^\infty(\tilde{M}).$$

We denote the image of $\Omega^{(u)}$ under the transformation by $\tilde{\Omega}^{(u)}$ and introduce the level sets $\tilde{\Omega}^{(u)}_r$ and the sets $\tilde{\Omega}^{(u)}_{r-}$:

$$\tilde{\Omega}^{(u)}_r := \tilde{\Omega}^{(u)} \cap \{(y, r') : r' = r\} \qquad \tilde{\Omega}^{(u)}_{r-} := \tilde{\Omega}^{(u)} \cap \{(y, r') : r' \le r\}.$$

We see that a small $r_0 > 0$ exists with

$$v \in C^\infty(\tilde{\Omega}^{(u)}_{r_0-})$$

and will prove that

$$v \in C^\infty(\tilde{\Omega}^{(u)}_{r-}) \quad \forall r < r^*.$$

For this purpose two properties must be shown:

- If $v \in C^\infty(\tilde{\Omega}^{(u)}_{r_1-})$, then an r_2 exists with $r_1 < r_2 < r^*$ and $v \in C^\infty(\tilde{\Omega}^{(u)}_{r_2-})$.
- If $v \in C^\infty(\tilde{\Omega}^{(u)}_{r-})$ for every $r < r_2 < r^*$, then $v \in C^\infty(\tilde{\Omega}^{(u)}_{r_2-})$.

Let $v \in C^\infty(\tilde{\Omega}^{(u)}_{r_1-})$ with $0 < r_1 < r^*$. The set $\tilde{\Omega}^{(u)}_{r_1}$ can be covered by $\tilde{M}_{r_1} := \tilde{M} \cap \{(y, r) : r = r_1\}$ and a finite collection of open balls:

$$\tilde{\Omega}^{(u)}_{r_1} = \tilde{M}_{r_1} \cup \bigcup_{k=1}^{l} B(y_k, d_k).$$

Lemma 5.5.1 gives us numbers $r'_{1,k}(r_1, y_k, d_k) > r_1$ with

$$v \in C^\infty(B(y_k, d_k) \times [r_1, r'_{1,k}]).$$

We take the smallest of these numbers,

$$r_2 := \min_{k=1,\ldots,l} r'_{1,k},$$

and obtain

$$v \in C^\infty(\cup_{k=1}^l B(y_k, d_k) \times [r_1, r_2]), \quad v \in C^\infty(\tilde{M}).$$

It may happen that a gap between $C^\infty(\tilde{M}_{r_2})$ and $\cup_{k=1}^l B(y_k, d_k)$ appears. In this case we decrease r_2 until

$$\tilde{\Omega}_r^{(u)} = \tilde{M}_r \cup \bigcup_{k=1}^l B(y_k, d_k), \quad r_1 \leq r \leq r_2, \quad r_1 < r_2.$$

This is possible since the function $h(x)$ continuously depends on x. We get

$$v \in C^\infty(\tilde{\Omega}_{r_2-}^{(u)}).$$

Now we come to the second part.
We take numbers r_1, r_3 with $0 < r_1 < r_2 < r_3 < r^*$ and a collection of balls $B(y_k, d_k)$ $(k = 1, \ldots, l)$ with

$$\tilde{\Omega}_r^{(u)} = \tilde{M}_r \cup \bigcup_{k=1}^l B(y_k, d_k) \quad \forall r_1 \leq r \leq r_3.$$

A number $r_3 \geq r_4 > r_2$ exists with the property that $v(., r_2) \in C_b^\infty(\tilde{\Omega}_{r_2}^{(u)})$ would imply $v \in C^\infty([r_2, r_4], C^\infty(\tilde{\Omega}_r^{(u)}))$, cf. Lemma 5.5.1 and the arguments of the first step. Let us fix a sequence $r_{2,1}$, $r_{2,2}$, $\ldots$ with

$$r_1 \leq r_{2,j} < r_2, \quad r_{2,j} \to r_2.$$

For each $r_{2,j}$ we can choose $r_{4,j}$ with the property that $v(., r_{2,j}) \in C_b^\infty(\tilde{\Omega}_{r_{2,j}}^{(u)})$ implies $v \in C^\infty([r_{2,j}, r_{4,j}], C^\infty(\tilde{\Omega}_r^{(u)}))$. Lemma 5.5.1 reveals that the $r_{4,j}$ can be chosen in such a way that

$$r_{4,j} \to r_4.$$

This shows that some K exists with $r_{4,K} > r_2$. From $v \in C^\infty(\tilde{\Omega}_{r_{2,K}-}^{(u)})$ it follows that

$$v \in C^\infty([r_{2,K}, r_{4,K}], C^\infty(\tilde{\Omega}_r^{(u)})),$$

especially

$$v \in C^\infty(\tilde{\Omega}_{r_2-}^{(u)}).$$

The theorem is proved. □

Chapter 6

Propagation of Singularities for Semilinear Weakly Hyperbolic Equations

6.1 Introduction

Let us recall some results from [Rau79]. We examine the wave equations

$$\Box u = f(u) = \sum_{j=1}^{N} f_j u^j, \quad f_i \in \mathbb{R}, \tag{6.1.1}$$

$$\Box v = 0 \tag{6.1.2}$$

with $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t$ and data

$$u(x,0) = v(x,0) = \varphi(x), \quad u_t(x,0) = v_t(x,0) = \psi(x).$$

We suppose that the data are C^∞ smooth outside the set $B(0,R) := \{|x| = R\}$ for some $R > 0$. Let us assume $u \in H^s_{loc}(\mathbb{R}^n \times \mathbb{R})$ with $s > (n+1)/2$. Since the singularities of the data propagate with speed 1, we have

$$\text{sing–supp}(v) \subset \{(x,t) : B(x,|t|) \cap B(0,R) \neq \emptyset\} =: S_v,$$

$B(x,|t|)$ denoting the ball around $(x,0)$ with radius $|t|$ in the initial surface $\mathbb{R}^n \times \{0\}$. In particular, v is smooth in the lacuna $\mathcal{L} = \{(x,t) : |x| < |t| - R\}$. However, the solution u of the semilinear problem is not smooth in this domain, it holds

$$u \in H^r(\mathcal{L}), \quad \frac{n+1}{2} < s \leq r < 3s - n + 1,$$

see [Rau79], Theorem 3.2 or [Kic96], Theorem 3.13. It can be seen that u has singularities in the lacuna $\mathcal{L}$, however, they are weaker than the singularities on S_v, since $3s - n + 1 > s$.
But the strongest singularities of u and v coincide and are contained in the set S_v. This can be seen as follows: we know

$$\Box(u - v) = f(u), \quad (u - v)(x, 0) = 0, \quad (u - v)_t(x, 0) = 0$$

and $f(u) \in H^s_{loc}(\mathbb{R}^n \times \mathbb{R})$, since this Sobolev space is an algebra. Then it is well–known that $u - v \in H^{s+1}_{loc}(\mathbb{R}^n \times \mathbb{R})$. Choose some arbitrary $0 < \varepsilon < 1$. Then we have $\text{sing–supp}_{H^{s+\varepsilon}}(u - v) = \emptyset$, hence

$$\text{sing–supp}_{H^{s+\varepsilon}}(u) = \text{sing–supp}_{H^{s+\varepsilon}}(v).$$

In other words, the singular support of the solution of the *semilinear* problem coincides with the singular support of the solution of some *suitably linearised* problem.
The aim of this chapter is to prove a similar result for weakly hyperbolic Cauchy problems whose lower order terms satisfy sharp Levi conditions with respect to t. That is to say, we have to take into account the loss of regularity of the solution compared to the initial data. Let us consider the Cauchy problems

$$\begin{aligned} Lu = u_{tt} &+ \sum_{j=1}^n c_j(t)\lambda(t)u_{x_jt} - \sum_{i,j=1}^n a_{ij}(t)\lambda(t)^2 u_{x_ix_j} \\ &+ \sum_{i=1}^n b_i(t)\lambda'(t)u_{x_i} + c_0(t)u_t = f(u), \\ u(x,0) &= \varphi(x), \quad u_t(x,0) = \psi(x), \end{aligned} \tag{6.1.3}$$

$$\begin{aligned} Lv = v_{tt} &+ \sum_{j=1}^n c_j(t)\lambda(t)v_{x_jt} - \sum_{i,j=1}^n a_{ij}(t)\lambda(t)^2 v_{x_ix_j} \\ &+ \sum_{i=1}^n b_i(t)\lambda'(t)v_{x_i} + c_0(t)v_t = 0, \\ v(x,0) &= \varphi(x), \quad v_t(x,0) = \psi(x). \end{aligned} \tag{6.1.4}$$

To present the phenomena, we start with some special case. We take $n = 1$, $c_1 = 0$, $a_{11} = 1$, $\lambda(t) = t$, $b_1(t)\lambda'(t) = -b$, $c_0 = 0$ and $\psi = 0$. Then v solves

$$v_{tt} - t^2 v_{xx} = bv_x, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0.$$

If $b = 4m + 1$, $m \in \mathbb{N}_0$, we have the explicit representation

$$v(x,t) = \sum_{j=0}^m C_j t^{2j} \partial_x^j \varphi\left(x + \frac{1}{2}t^2\right)$$

with some constants C_j, and C_m does not vanish, see [Qi58]. The assumption $\varphi \in H^s$ implies

$$v \in C\left([0,T], H^{s-m}\right).$$

This phenomenon is called *loss of Sobolev regularity* and makes the investigation of such Cauchy problems difficult, though interesting. If $m > s$, then there is *no classical solution* v! The following problems and questions arise:

- Does the solution u of the corresponding semilinear problem

$$u_{tt} - t^2 u_{xx} = b u_x + f(u), \quad u(x,0) = \varphi(x), \quad u_t(x,0) = 0$$

  exist locally (for small time)?

- Does this solution u (if it exists) *belong to the same space* as v? This question seems to be open. We will give a positive answer.

- The explicit representation of v exhibits the surprising phenomenon that propagation of singularities happens only along the characteristic $x + t^2/2 = \text{const}$. Does this special propagation behaviour also happen for the solution u of the *semilinear* problem? This would be an analogue to the above result for solutions of the linear/semilinear Cauchy problems to the wave operator $\Box$. We will give a positive answer by showing that the difference $u - v$ has higher smoothness than v.

These questions will be studied in the context of the rather general equations (6.1.3), (6.1.4). In particular, we will have to deal with an unknown weight function $\lambda(t)$. This gives rise to another question:
To which space do the functions u and v belong ?
If one is interested in propagation of singularities, then a sharp space has to be found. To make this point clear, let us consider (6.1.1), (6.1.2) with data $\varphi \in H^s$, $\psi \in H^{s-1}$. It is a true statement to say that u and v belong to, e.g., $C([0,T], H^{s-5})$. However, it has no sense to investigate singularities in this space, because the singular support is the empty set. The sharp answer to the question for the space is $C([0,T], H^s)$.
Let us come back to the weakly hyperbolic case. Up to now, the sharp spaces have been known only for certain special cases. These special cases are $\lambda(t) = t^l$ (see [TT80] and [Yag97a]) and $\lambda(t) = \partial_t(\exp(-1/t))$ (see [Ale84] and [Yag97a]) and will be presented more precisely below, see Subsection 6.2.2 and Subsection 6.2.3.
It turns out that the sharp spaces for u and v are *no Sobolev spaces* in the second special case! It is necessary to generalise the classes of Sobolev spaces.

We will proceed in the following way: if $w(x,t) \in C([0,T], H^s(\mathbb{R}^n_x))$, then we obviously have $\langle\xi\rangle^s \hat{w}(\xi,t) \in C([0,T], L^2(\mathbb{R}^n_\xi))$. The temperate weight $\langle\xi\rangle^s$ will be replaced by some suitably chosen temperate weight $\vartheta(\xi,t)$, which also depends on some parameters. Thus, we get a scale of spaces which heavily depends on the coefficients of the weakly hyperbolic operator L. The idea to assign a weight $\vartheta(\xi,t)$ to the operator L and to estimate a certain norm of the product $\vartheta(\xi,t)\hat{w}(\xi,t)$ goes back to [RY99].
Using this scale of generalised Sobolev–like spaces we are able to introduce the *framework of optimal spaces* assigned to weakly hyperbolic operators:
We call a framework of function spaces S_φ for φ, S_ψ for ψ, S_f for a right–hand side $f = f(x,t)$ and S_u for the solution u *optimal*, if the following conditions are satisfied:

- There is a general procedure that defines S_φ, S_ψ, S_f, S_u if L is given.
- The assumptions $\varphi \in S_\varphi$, $\psi \in S_\psi$, $f \in S_f$ imply the existence and uniqueness of a solution $u \in S_u$. This solution continuously depends on φ, ψ, f in the topology of the given spaces.
- For certain operators L the spaces S_φ, S_ψ, S_f, S_u coincide with the spaces suggested by explicit representations of the solutions.

Let us list the assumptions on the functions $\lambda(t)$, $c_j(t)$, $a_{ij}(t)$, $b_i(t)$, $c_0(t)$ and $f(u)$:
With $\Lambda(t) := \int_0^t \lambda(\tau)\, d\tau$ we assume that

$$0 = \lambda(0) = \lambda'(0), \quad \lambda'(t) > 0 \quad (t > 0), \tag{6.1.5}$$

$$d_0 \frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1 \frac{\lambda(t)}{\Lambda(t)}, \quad 0 < t \le T, \quad d_0 > \frac{1}{2}, \tag{6.1.6}$$

$$|\lambda''(t)| \le d_2 \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^2, \tag{6.1.7}$$

$$\lambda, c_j, a_{ij}, b_i, c_0 \in C^\infty([0,T]), \tag{6.1.8}$$

$$\alpha_1 |\xi|^2 \ge \left(\sum_{j=1}^n c_j(t)\xi_j\right)^2 + 4\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \ge \alpha_0|\xi|^2 \tag{6.1.9}$$

$$\forall (t,\xi) \in [0,T] \times \mathbb{R}^n, \quad \alpha_0 > 0,$$

$$f(u) = \sum_{j=1}^\infty f_j u^j \quad \forall u \in \mathbb{R}. \tag{6.1.10}$$

The central results are the Theorems 6.6.1, 6.6.2 and 6.6.4 in Section 6.6.

6.2 Examples

In this section we will give explicit representations for the partial Fourier transforms of the solutions to some linear hyperbolic Cauchy problems with a homogeneous right–hand side. These examples will give us some hints how to study weakly hyperbolic Cauchy problems of more general type.

6.2.1 The Strictly Hyperbolic Case

We consider the problem

$$v_{tt} - v_{xx} = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \mathbb{R}.$$

Partial Fourier transform gives

$$\hat{v}_{tt} + \xi^2 \hat{v} = 0, \quad \hat{v}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{v}_t(\xi,0) = \hat{\psi}(\xi).$$

The solution is

$$\hat{v}(\xi,t) = \cos(t\xi)\hat{\varphi}(\xi) + t\frac{\sin(t\xi)}{t\xi}\hat{\psi}(\xi).$$

We fix $t > 0$ and let $|\xi|$ tend to ∞. Then we have asymptotically

$$\hat{v}(\xi,t) = O(1)\hat{\varphi}(\xi) + O(|\xi|^{-1})\hat{\psi}(\xi), \quad |\xi| \to \infty.$$

6.2.2 Weakly Hyperbolic Case with Finite Degeneracy

We study the Cauchy problem

$$v_{tt} - t^{2l}v_{xx} - ht^{l-1}v_x = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \mathbb{R}.$$

The number h is a real constant. In [TT80] and [Yag97a] it is shown how to construct the solution. Partial Fourier transform gives

$$\hat{v}_{tt} + \xi^2 t^{2l}\hat{v} - ih\xi t^{l-1}\hat{v} = 0, \quad \hat{v}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{v}_t(\xi,0) = \hat{\psi}(\xi).$$

The solution is

$$\begin{aligned} \hat{v}(\xi,t) =& e^{-i\Lambda(t)\xi}\,{}_1\mathrm{F}_1\left(\frac{l+h}{2(l+1)}, \frac{l}{l+1}, 2i\Lambda(t)\xi\right)\hat{\varphi}(\xi) \\ &+ te^{-i\Lambda(t)\xi}\,{}_1\mathrm{F}_1\left(\frac{l+2+h}{2(l+1)}, \frac{l+2}{l+1}, 2i\Lambda(t)\xi\right)\hat{\psi}(\xi). \end{aligned}$$

The terms ${}_1\mathrm{F}_1(.,.,z)$ are confluent hypergeometric functions (see [AS84] or [EMO53]),

$$ {}_1\mathrm{F}_1(\alpha,\gamma,z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j}\frac{z^j}{j!} $$

with $(\alpha)_j = \alpha(\alpha+1)\dots(\alpha+j-1)$, $(\alpha)_0 = 1$ and it holds

$$ {}_1\mathrm{F}_1(\alpha,\gamma,0) = 1, \quad \partial_z\, {}_1\mathrm{F}_1(\alpha,\gamma,z) = \frac{\alpha}{\gamma}\, {}_1\mathrm{F}_1(\alpha+1,\gamma+1,z). $$

For $|z| \to \infty$ we have the asymptotic behaviour

$$ \begin{aligned} {}_1\mathrm{F}_1(\alpha,\gamma,z) = {} & \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} e^{\pm i\pi\alpha} z^{-\alpha} + O(|z|^{-\alpha-1}) \\ & + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{z} z^{\alpha-\gamma} + O(|z|^{\alpha-\gamma-1}), \end{aligned} $$

the upper sign being taken if $-\pi/2 < \arg z < 3\pi/2$, the lower sign if $-3\pi/2 < \arg z \le -\pi/2$. We fix $t > 0$ and let $|\xi|$ tend to ∞. Then

$$ \hat{v}(\xi,t) = O\left(|\xi|^{\frac{-l+|h|}{2(l+1)}}\right)\hat{\varphi}(\xi) + O\left(|\xi|^{\frac{-l-2+|h|}{2(l+1)}}\right)\hat{\psi}(\xi), \quad |\xi| \to \infty. $$

The exponents of ξ describe the loss of Sobolev regularity.
We emphasise that the difference of these exponents is not 1 as in the strictly hyperbolic case, but $1/(l+1)$.

6.2.3 Weakly Hyperbolic Case with Infinite Degeneracy

Let $\Lambda(t) = \exp(-\frac{1}{t})$ and $\lambda(t) := \Lambda'(t)$. Then this function λ satisfies all assumptions (6.1.5)–(6.1.7). We reflect upon the Cauchy problem

$$ \begin{aligned} & v_{tt} - \lambda(t)^2 v_{xx} - h\frac{\lambda(t)^2}{\Lambda(t)} v_x = 0, \\ & v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \mathbb{R}. \end{aligned} $$

We note that the coefficient of v_x is not $\lambda'(t)$ times a constant as in the case of finite degeneracy, but $\lambda(t)^2\Lambda(t)^{-1}$ times some constant. However, this difference does not play a large role, since this factor can be bounded from above and below by $\lambda'(t)$ times constants, see (6.1.6). In [Ale84] and [Yag97a] the fundamental solution is constructed; we only list the results. After applying partial Fourier transform we get

$$ \hat{v}_{tt} + \xi^2\lambda(t)^2\hat{v} - ih\xi\frac{\lambda(t)^2}{\Lambda(t)}\hat{v} = 0, \quad \hat{v}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{v}_t(\xi,0) = \hat{\psi}(\xi). $$

The solution $\hat{v}$ has the form

$$\begin{aligned}
\hat{v}(\xi,t) &= \sum_{j=1}^{2} c_j(\xi) t e^{-\beta_j \Lambda(t)\xi} \Psi(\alpha_j, 1, 2\beta_j \Lambda(t)\xi), \qquad (6.2.1)\\
c_j(\xi) &= \frac{\Gamma(\alpha_j)}{\gamma_{3-j}-\gamma_j}(\hat{\varphi}(\xi)(\ln|\xi| + \gamma_{3-j}) + \hat{\psi}(\xi)), \quad j=1,2,\\
\gamma_j &= \ln 2 + \beta_j \operatorname{sign}\xi \frac{\pi}{2} + \tilde{\psi}(\alpha_j) - 2\tilde{\psi}(1), \quad j=1,2,\\
\beta_1 &= i, \quad \beta_2 = -i, \quad \alpha_1 = \frac{1+h}{2}, \quad \alpha_2 = \frac{1-h}{2}.
\end{aligned}$$

The term $\Psi(\alpha, 1, z)$ is a confluent hypergeometric function (logarithmic case, see [AS84] or [EMO53]):

$$\begin{aligned}
\Psi(\alpha, n+1, z) &= \frac{(n-1)!}{\Gamma(\alpha)} \sum_{r=0}^{n-1} \frac{(\alpha-n)_r}{(1-n)_r} \frac{z^{r-n}}{r!}\\
&+ \frac{(-1)^{n+1}}{n!\Gamma(\alpha-n)} \Bigg({}_1\mathrm{F}_1(\alpha, n+1, z) \ln z\\
&\quad + \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(n+1)_r} \frac{z^r}{r!} (\tilde{\psi}(\alpha+r) - \tilde{\psi}(1+r) - \tilde{\psi}(1+n+r)) \Bigg).
\end{aligned}$$

The function $\tilde{\psi}(\zeta)$ is the logarithmic derivative of Euler's Γ–function, $\tilde{\psi}(\zeta) = \partial_\zeta \ln \Gamma(\zeta)$. The confluent hypergeometric function satisfies

$$\begin{aligned}
&\partial_z \Psi(\alpha, \beta, z) = -\alpha \Psi(\alpha+1, \beta+1, z),\\
&\Psi(\alpha, \beta+1, z) = \Psi(\alpha, \beta, z) - \partial_z \Psi(\alpha, \beta, z),\\
&\Psi(\alpha, 1, z) = -\frac{1}{\Gamma(\alpha)}(\ln z + \tilde{\psi}(\alpha) - 2\tilde{\psi}(1)) + O(|z \ln z|), \quad z \to 0,\\
&\Psi(\alpha, 2, z) = \frac{1}{z\Gamma(a)} + O(|\ln z|), \quad z \to 0,\\
&\Psi(\alpha, \beta, z) = \sum_{n=0}^{N} (-1)^n \frac{(\alpha)_n (\alpha-\beta+1)_n}{n!} z^{-\alpha-n} + O(|z|^{-\alpha-N-1}),\\
&\qquad |z| \to \infty, \quad -\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}.
\end{aligned}$$

Utilising these formulas it is possible to verify the explicit representation (6.2.1). For fixed $t > 0$ and large $|\xi|$ we obtain the asymptotic behaviour

$$\hat{v}(\xi,t) = O(|\xi|^{\frac{-1+|h|}{2}} \ln|\xi|)\hat{\varphi}(\xi) + O(|\xi|^{\frac{-1+|h|}{2}})\hat{\psi}(\xi), \quad |\xi| \to \infty.$$

We point out that the coefficients of $\hat{\varphi}$ and $\hat{\psi}$ differ only by a factor $\ln|\xi|$. This observation leads us to the sharp spaces for φ, ψ and v immediately: For φ and ψ we may choose the spaces of all functions Φ, Ψ with

$$\langle\xi\rangle^s\hat{\Phi}(\xi) \in L^2(\mathbb{R}^n_\xi), \quad \langle\xi\rangle^s(\ln\langle\xi\rangle)^{-1}\hat{\Psi}(\xi) \in L^2(\mathbb{R}^n_\xi).$$

The space for v consists of all functions $V = V(x,t)$ with

$$\langle\xi\rangle^{s-(|h|-1)/2}(\ln\langle\xi\rangle)^{-1}\hat{V}(\xi,t) \in L^2(\mathbb{R}^n_\xi) \quad \forall t.$$

6.2.4 Summary and Conclusions

Let us draw some conclusions from the above examples. In the first two cases, the solution can be written as

$$\hat{v}(\xi,t) = G_1(\Lambda(t)\xi)\hat{\varphi}(\xi) + tG_2(\Lambda(t)\xi)\hat{\psi}(\xi)$$

with $G_1(0) = G_2(0) = 0$. And in the third example we have the representation

$$\hat{v}(\xi,t) = tG_1(\Lambda(t)\xi)(\ln|\xi| + C)\hat{\varphi}(\xi) + tG_2(\Lambda(t)\xi)\hat{\psi}(\xi)$$

with the asymptotic behaviour $G_j(s) = O(\ln|s|)$ for $s \to 0$. It can be observed that the sets $\{\Lambda(t)\xi = \text{const}\}$ play a certain role. Furthermore, we have seen that the coefficients G_1 and G_2 have different behaviour for $|\xi| \to \infty$, t fixed. Let us characterise this difference. This characterisation will work in any of the three examples mentioned above.
We fix some large real number $N > 0$ and consider the set $\{(\xi,t) : \Lambda(t)\langle\xi\rangle = N.\}$ Since Λ is strictly monotonically increasing, we can define a mapping $\xi \mapsto t_\xi$ by the formula

$$\Lambda(t_\xi)\langle\xi\rangle = N.$$

In the first example we have $\lambda(t) \equiv 1$, hence $\Lambda(t) = t$ and $t_\xi = C\langle\xi\rangle^{-1}$. In the second example $\Lambda(t) = t^{l+1}/(l+1)$ holds, hence $t_\xi = C\langle\xi\rangle^{-1/(l+1)}$. And in the third example we have $\exp(-1/t_\xi) = N\langle\xi\rangle^{-1}$, which gives us $t_\xi = O((\ln|\xi|)^{-1})$ immediately. We observe that the difference in the asymptotic behaviours of the weights G_1 and G_2 can be described by these t_ξ. For φ we could choose the space $H^s(\mathbb{R}^n)$ and for ψ the space with the temperate weight $\langle\xi\rangle^s t_\xi$.
But what is the *sharp space* for the solution v ? The loss of smoothness is a severe difficulty. If $t = 0$, then v coincides with φ and the temperate weight $\vartheta(\xi,t)$ in the definition of the v–space should behave like $\langle\xi\rangle^s$. If $t > 0$, then the loss of regularity appears and the weight $\vartheta(\xi,t)$ should behave like $\vartheta(\xi,t) = O(\langle\xi\rangle^{s-K})$, $K \in \mathbb{R}$, for large $\langle\xi\rangle$ (at least in the second example).

And of course, the weight $\vartheta(\xi,t)$ should be continuous in ξ and t, even for $t \to 0$.
This difficulty can be overcome by splitting the (ξ,t)–space into two zones, the *pseudodifferential zone* $Z_{pd}(N)$ and the *hyperbolic zone* $Z_{hyp}(N)$:

$$\begin{aligned}
Z_{pd}(N) &= \{(\xi,t) \in \mathbb{R}^n \times [0,T] : \Lambda(t)\langle\xi\rangle \le N\},\\
Z_{hyp}(N) &= \{(\xi,t) \in \mathbb{R}^n \times [0,T] : \Lambda(t)\langle\xi\rangle \ge N\}.
\end{aligned}$$

It is possible to use a hyperbolic type approach in $Z_{hyp}(N)$, since in this zone the influence of the principal symbol is dominating. On the other hand, in the pseudodifferential zone $Z_{pd}(N)$ the influence of the subprincipal symbol becomes important and one has to take a different approach. We will define the temperate weight $\vartheta(\xi,t)$ in both zones in different ways in order to model the loss of regularity. The splitting into two zones allows us to define a *continuous* weight $\vartheta(\xi,t)$ with different growth (for $|\xi| \to \infty$) in the 2 cases $t = 0$ and $t > 0$.

6.3 A–priori Estimates

The main tool in this chapter is an a–priori estimate for solutions of linear weakly hyperbolic Cauchy problems with an inhomogeneous right–hand side. This estimate will be written in terms of spaces with suitably chosen temperate weight for the Fourier transform. Let us sketch the proof. We assume for a moment that the right–hand side does not depend on u. Partial Fourier transform with respect to x results in

$$\begin{aligned}
&\hat{u}_{tt}(\xi,t) + \left(\sum_{i,j=1}^{n} a_{ij}(t)\lambda(t)^2\xi_i\xi_j + i\sum_{j=1}^{n} b_j(t)\lambda'(t)\xi_j\right)\hat{u}(\xi,t)\\
&\qquad + \left(i\sum_{j=1}^{n} c_j(t)\lambda(t)\xi_j + c_0(t)\right)\hat{u}_t(\xi,t) = \hat{f}(\xi,t),\\
&\hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi).
\end{aligned}$$

This is an ODE of second order with parameter $\xi \in \mathbb{R}^n$. The factor of the function $\hat{u}(\xi,t)$ has two terms: $\sum_{i,j=1}^{n} a_{ij}\lambda^2\xi_i\xi_j$, which is the dominating term in the hyperbolic zone, and $i\sum_{j=1}^{n} b_j\lambda'\xi_j$, which dominates in the pseudodifferential zone. We transform this equation into a system of ODEs of first order. The vector W of the unknown functions of this system has two components, $w_2 = D_t\hat{u}$ and $w_1 = G(\xi,t)\hat{u}$. In this case, $G(\xi,t)$ is a weight which is chosen differently in the two zones. We take $G(\xi,t) = \lambda(t)|\xi|$ in the hyperbolic zone and a weight $G(\xi,t)=\varrho(\xi,t)=\sqrt{1+\frac{\lambda(t)^2}{\Lambda(t)}\langle\xi\rangle}$ is chosen in the pseudodifferential zone. We note that λ^2/Λ is equivalent to λ', see (6.1.6).

The idea of splitting the (ξ,t) space into zones can be found, e.g., in [Kg76], [Yos78], [Shi91], [Tar95] and [Yag97a]. Our approach is based on a theory which was used in [RY]. All these steps lead to an estimate for $\hat{u}$ and $D_t\hat{u}$. From this estimate we will learn how to choose the temperate weight.

In a next step it is shown that this weight is a temperate weight in the sense of [Hör69]. This allows us to apply the general theory developed in [Hör69], Part I, Chapter 2.

6.3.1 Preliminaries

In this subsection our intention is to list some properties of the functions $\lambda(t)$, $\Lambda(t)$, t_ξ, $\varrho(\xi,t)$ which will be needed later. The proofs can be found in the Appendix C.

By definition, we have

$$\begin{aligned}
&\lambda(0)=\lambda'(0)=0, \quad \lambda'(t)>0 \quad (t>0),\\
&\Lambda(t)=\int_0^t \lambda(\tau)\,d\tau,\\
&\Lambda(t_\xi)\langle\xi\rangle=N, \quad \langle\xi\rangle=(1+|\xi|^2)^{1/2},\\
&\varrho(\xi,t)=\sqrt{1+\frac{\lambda(t)^2}{\Lambda(t)}\langle\xi\rangle},\\
&d_0\frac{\lambda(t)}{\Lambda(t)}\le\frac{\lambda'(t)}{\lambda(t)}\le d_1\frac{\lambda(t)}{\Lambda(t)}, \quad 0<t\le T, \quad d_0>\frac{1}{2},\\
&|\lambda''(t)|\le d_2\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2.
\end{aligned}$$

With $\Lambda(t_0)\langle 0\rangle=N$ we assume $0<T\le t_0$. Then we have the following proposition:

Proposition 6.3.1. *It holds*

$$\Lambda(t)\le t\lambda(t) \quad \forall t\in[0,T], \tag{6.3.1}$$

$$\left(\frac{\Lambda(t)}{\Lambda(T_0)}\right)^{d_0}\ge\frac{\lambda(t)}{\lambda(T_0)}\ge\left(\frac{\Lambda(t)}{\Lambda(T_0)}\right)^{d_1} \quad \forall 0<t\le T_0\le T, \tag{6.3.2}$$

$$d_0<1, \tag{6.3.3}$$

$$\frac{dt_\xi}{d\langle\xi\rangle} = -\frac{N}{\lambda(t_\xi)\langle\xi\rangle^2} = -\frac{\Lambda(t_\xi)}{\lambda(t_\xi)\langle\xi\rangle} \quad \forall\xi\in\mathbb{R}^n, \tag{6.3.4}$$

$$C_1\langle\xi\rangle^{-d_0} \ge \lambda(t_\xi) \ge C_2\langle\xi\rangle^{-d_1} \quad \forall\xi\in\mathbb{R}^n, \tag{6.3.5}$$

$$p(\langle\xi\rangle) := t_\xi\langle\xi\rangle \text{ is monotonically increasing in } \langle\xi\rangle, \tag{6.3.6}$$

$$C_3\langle\xi\rangle^{d_0-1} \ge t_\xi \ge C_4\langle\xi\rangle^{-1} \quad \forall\xi\in\mathbb{R}^n, \tag{6.3.7}$$

$$\int_0^{t_\xi} \varrho(\xi,t)\,dt \le C \quad \forall\xi\in\mathbb{R}^n, \tag{6.3.8}$$

$$\lambda(t)\langle\xi\rangle \le \sqrt{N}\varrho(\xi,t) \quad \forall(\xi,t)\in Z_{pd}(N), \tag{6.3.9}$$

$$\frac{1}{\sqrt{N}}\lambda(t_\xi)\langle\xi\rangle \le \varrho(\xi,t_\xi) \le \frac{C}{\sqrt{N}}\lambda(t_\xi)\langle\xi\rangle \quad \forall\xi\in\mathbb{R}^n, \tag{6.3.10}$$

$$\int_0^t (t-s)^2\varrho(\xi,s)^2\,ds \le Ct \quad \forall(\xi,t)\in Z_{pd}(N), \tag{6.3.11}$$

$$\partial_t\varrho(\xi,t) > 0 \quad \forall(\xi,t)\in Z_{pd}(N), \tag{6.3.12}$$

$$q(\langle\xi\rangle) := \lambda(t_\xi)\langle\xi\rangle^{d_1} \text{ is monotonically increasing in } \langle\xi\rangle. \tag{6.3.13}$$

Proof. See the Appendix C. □

The proof of the next lemma is left to the reader.

Lemma 6.3.2. *Let $g(t)$ be a continuous, positive and bounded function and define*

$$J(s,t) = \exp\left(\int_s^t \frac{\lambda'(\tau)}{\lambda(\tau)}g(\tau)\,d\tau\right).$$

Then we have

$$J(s,t)J(t,r) = J(s,r) \quad \forall 0 < t,s,r \le T, \tag{6.3.14}$$

$$J(s,t) \text{ is increasing in } t,\ \text{decreasing in } s, \tag{6.3.15}$$

$$1 \le J(s,t) \le \left(\frac{\lambda(t)}{\lambda(s)}\right)^{K_0}, \quad K_0 = \sup_{[0,T]} g(\tau), \quad 0 < s \le t \le T. \tag{6.3.16}$$

6.3.2 A–priori Estimates for Solutions of ODEs

We start with the Cauchy problem

$$u_{tt} + \sum_{j=1}^n c_j(t)\lambda(t)u_{x_jt} - \sum_{i,j=1}^n a_{ij}(t)\lambda(t)^2 u_{x_ix_j} + \sum_{j=1}^n b_j(t)\lambda'(t)u_{x_j} + c_0(t)u_t = f(x,t), \tag{6.3.17}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{6.3.18}$$

Partial Fourier transform with respect to x gives

$$\begin{aligned} &D_{tt}\hat{u}(\xi,t) - \sum_{i,j=1}^{n} a_{ij}(t)\lambda(t)^2\xi_i\xi_j\hat{u}(\xi,t) - i\sum_{j=1}^{n} b_j(t)\lambda'(t)\xi_j\hat{u}(\xi,t) \\ &\quad + \left(\sum_{j=1}^{n} c_j(t)\lambda(t)\xi_j - ic_0(t)\right) D_t\hat{u}(\xi,t) = -\hat{f}(\xi,t), \qquad (6.3.19) \\ &\hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi). \end{aligned}$$

6.3.2.1 The Pseudodifferential Zone

In order to estimate $\hat{u}$ and $D_t\hat{u}$ in the pseudodifferential zone, we define

$$W(\xi,t) = \begin{pmatrix} w_1(\xi,t) \\ w_2(\xi,t) \end{pmatrix} = \begin{pmatrix} \varrho(\xi,t)\hat{u}(\xi,t) \\ D_t\hat{u}(\xi,t) \end{pmatrix}$$

and get $D_tW - AW = F$ with

$$\begin{aligned} A(\xi,t) &= \begin{pmatrix} \frac{D_t\varrho}{\varrho} & \varrho \\ \frac{\sum_{i,j=1}^{n} a_{ij}\lambda^2\xi_i\xi_j + i\sum_{j=1}^{n} b_j\lambda'\xi_j}{\varrho} & -\sum_{j=1}^{n} c_j\lambda\xi_j + ic_0 \end{pmatrix}, \\ F(\xi,t) &= \begin{pmatrix} 0 \\ -\hat{f}(\xi,t) \end{pmatrix}. \end{aligned}$$

Let us estimate the components of $A(\xi,t)$. From (6.3.9) and the definition of $\varrho(\xi,t)$ we deduce that

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(t)\lambda(t)^2\xi_i\xi_j \le \alpha_1\lambda(t)^2\langle\xi\rangle^2 \le \alpha_1 N\varrho(\xi,t)^2, \\ &\sum_{j=1}^{n} |b_j(t)\lambda'(t)\xi_j| \le d_1\frac{\lambda(t)^2}{\Lambda(t)}\sum_{j=1}^{n} |b_j(t)\xi_j| \le C\varrho(\xi,t)^2, \\ &|c_0(t)| + \sum_{j=1}^{n} |c_j(t)\lambda(t)\xi_j| \le C(1+\varrho(\xi,t)) \le C'\varrho(\xi,t). \end{aligned}$$

For the norm of $A(\xi,t)$ (row sum norm or column sum norm) we obtain

$$\|A(\xi,t)\| \le C\varrho(\xi,t) + \frac{\varrho_t(\xi,t)}{\varrho(\xi,t)},$$

compare (6.3.12). Now let us devote ourselves to the differential system for the fundamental matrix $X(t,s,\xi)$:

$$D_t X(t,s,\xi) - A(\xi,t)X(t,s,\xi) = 0, \quad X(s,s,\xi) = I, \quad 0 \le s \le t \le t_\xi.$$

Then W allows the representation

$$W(\xi,t) = \int_0^t X(t,s,\xi)F(\xi,s)\,ds + X(t,0,\xi)W(\xi,0). \tag{6.3.20}$$

The matrix $X(t,s,\xi)$ can be estimated by

$$\|X(t,s,\xi)\| \le \exp\left(\int_s^t \|A(\xi,\tau)\|\,d\tau\right), \quad 0 \le s \le t \le t_\xi,$$

which gives the inequality $\|X(t,s,\xi)\| \le C\frac{\varrho(\xi,t)}{\varrho(\xi,s)}$, see (6.3.8). However, this estimate is not sharp for all components of X. For instance, we have $|X_{12}(s,s,\xi)| \le C\frac{\varrho(\xi,s)}{\varrho(\xi,s)} = C$, but we know that $X_{12}(s,s,\xi) = 0$. For sharper estimates we have to study the differential system more carefully. We introduce the notation

$$A(\xi,t) = \begin{pmatrix} A_{11}(\xi,t) & A_{12}(\xi,t) \\ A_{21}(\xi,t) & A_{22}(\xi,t) \end{pmatrix}, \quad A_{21}(\xi,t) = \frac{A_{21}^0(\xi,t)}{\varrho(\xi,t)}.$$

From the definition of the pseudodifferential zone follows that $|A_{21}^0(\xi,t)| \le C\lambda'(t)\langle\xi\rangle$. We get the differential equations

$$\begin{aligned}
\partial_t X_{11}(t,s,\xi) &= \frac{\partial_t \varrho(\xi,t)}{\varrho(\xi,t)} X_{11}(t,s,\xi) + i\varrho(\xi,t)X_{21}(t,s,\xi), \\
\partial_t X_{21}(t,s,\xi) &= \frac{iA_{21}^0(\xi,t)}{\varrho(\xi,t)} X_{11}(t,s,\xi) - \tilde{c}(\xi,t)X_{21}(t,s,\xi), \\
\partial_t X_{12}(t,s,\xi) &= \frac{\partial_t \varrho(\xi,t)}{\varrho(\xi,t)} X_{12}(t,s,\xi) + i\varrho(\xi,t)X_{22}(t,s,\xi), \\
\partial_t X_{22}(t,s,\xi) &= \frac{iA_{21}^0(\xi,t)}{\varrho(\xi,t)} X_{12}(t,s,\xi) - \tilde{c}(\xi,t)X_{22}(t,s,\xi), \\
\tilde{c}(\xi,t) &= i\sum_{j=1}^n c_j(t)\lambda(t)\xi_j + c_0(t), \\
\begin{pmatrix} X_{11}(s,s,\xi) & X_{12}(s,s,\xi) \\ X_{21}(s,s,\xi) & X_{22}(s,s,\xi) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

From the equation for X_{21} it can be concluded that

$$X_{21}(t,s,\xi) = i\int_s^t \exp\left(-\int_\tau^t \tilde{c}(\xi,\sigma)\,d\sigma\right) \frac{A_{21}^0(\xi,\tau)}{\varrho(\xi,\tau)} X_{11}(\tau,s,\xi)\,d\tau.$$

From $|X_{11}(t,s,\xi)| \le C\frac{\varrho(\xi,t)}{\varrho(\xi,s)}$ and $|\int_0^t \lambda(\tau)\xi_j d\tau| = \Lambda(t)|\xi_j| \le N$ follows that

$$|X_{21}(t,s,\xi)| \le C\int_s^t \frac{|A_{21}^0(\xi,\tau)|}{\varrho(\xi,s)}\,d\tau \le C\frac{(\lambda(t)-\lambda(s))\langle\xi\rangle}{\varrho(\xi,s)},$$

if $0 \le s \le t \le t_\xi$. From the equation for X_{12} it can be deduced that

$$X_{12}(t,s,\xi) = i\varrho(\xi,t)\int_s^t X_{22}(\tau,s,\xi)\,d\tau.$$

We set $f(t,s) = \int_s^t X_{22}(\tau,s,\xi)\,d\tau$ for fixed ξ and have

$$\begin{aligned}&f(s,s)=0,\quad f_t(t,s)=X_{22}(t,s,\xi),\quad f_t(s,s)=1,\\&f_{tt}(t,s)=X_{22,t}(t,s,\xi).\end{aligned}$$

Consequently,

$$f_{tt}(t,s) = -A_{21}^0(\xi,t)f(t,s) - \tilde{c}(\xi,t)f_t(t,s).$$

We define $g(t,s) := f(t,s)\beta(t,s)$ with $\beta(t,s) = \exp(\frac{1}{2}\int_s^t \tilde{c}(\xi,\tau)d\tau)$, resulting in

$$\begin{aligned}&g_{tt}(t,s) = A^0(\xi,t)g(t,s) := \left(-A_{21}^0(\xi,t) + \frac{\tilde{c}(\xi,t)^2}{4} + \frac{\tilde{c}'(\xi,t)}{2}\right)g(t,s),\\&g(s,s)=0,\quad g_t(s,s)=1.\end{aligned}$$

From $0 < C_1^{-1} \le \beta(t,s) \le C_1$ we obtain $|f(t,s)| \le C_1|g(t,s)|$. Furthermore, it holds $|A^0(\xi,t)| \le C_A(1+\lambda'(t)\langle\xi\rangle)$. Let $h(t,s)$ be the solution of

$$h_{tt}(t,s) = C_A(1+\lambda'(t)\langle\xi\rangle)h(t,s),\quad h(s,s)=0,\quad h_t(s,s)=1.$$

Then Lemma B.0.6 shows that $|g(t,s)| \le h(t,s)$. It is easy to see that $h(t,s)$ and $h_t(t,s)$ are positive if $t > s$. Consequently,

$$h_{tt}(t,s) \le C_A((t+\lambda(t)\langle\xi\rangle)h(t,s))_t.$$

Integration from s to t reveals

$$h_t(t,s) - 1 \le C_A(t+\lambda(t)\langle\xi\rangle)h(t,s).$$

By Gronwall's Lemma and the choice of N we conclude that

$$\begin{aligned}h(t,s) &\le \int_s^t \exp\left(C_A\int_\tau^t(\sigma+\lambda(\sigma)\langle\xi\rangle)\,d\sigma\right)d\tau\\&\le (t-s)\exp\left(C_AT^2+C_AN\right) \le C(t-s).\end{aligned}$$

This implies

$$\left|\int_s^t X_{22}(\tau, s, \xi)\, d\tau\right| \le C(t-s).$$

Finally, we deduce that

$$|X_{12}(t, s, \xi)| \le C\varrho(\xi, t)(t-s).$$

The last component $X_{22}(t, s, \xi)$ can be represented by

$$X_{22}(t, s, \xi) - 1 = i\int_s^t \exp\left(-\int_\tau^t \tilde{c}(\xi, \sigma)\, d\sigma\right)\frac{A_{21}^0(\xi, \tau)}{\varrho(\xi, \tau)} X_{12}(\tau, s, \xi)\, d\tau,$$

which results in

$$\begin{aligned}|X_{22}(t, s, \xi) - 1| &\le C\int_s^t \lambda'(\tau)\langle\xi\rangle(\tau - s)\, d\tau\\ &\le C(t-s)(\lambda(t) - \lambda(s))\langle\xi\rangle.\end{aligned}$$

Let us summarise these estimates: If $0 \le s \le t \le t_\xi$, then

$$\begin{aligned}|X_{11}(t, s, \xi)| &\le C\frac{\varrho(\xi, t)}{\varrho(\xi, s)},\\ |X_{12}(t, s, \xi)| &\le C\varrho(\xi, t)(t-s),\\ |X_{21}(t, s, \xi)| &\le C\frac{(\lambda(t) - \lambda(s))\langle\xi\rangle}{\varrho(\xi, s)},\\ |X_{22}(t, s, \xi) - 1| &\le C(t-s)(\lambda(t) - \lambda(s))\langle\xi\rangle.\end{aligned}$$

Using (6.3.20) we can estimate $\varrho\hat{u}$ and $D_t\hat{u}$:

$$|\varrho(\xi, t)\hat{u}(\xi, t)| \le C\varrho(\xi, t)\left(\int_0^t (t-s)|\hat{f}(\xi, s)|\, ds + |\hat{\varphi}(\xi)| + t|\hat{\psi}(\xi)|\right), \tag{6.3.21}$$

$$\begin{aligned}|D_t\hat{u}(\xi, t)| \le C&\int_0^t (1 + (t-s)(\lambda(t) - \lambda(s))\langle\xi\rangle)|\hat{f}(\xi, s)|\, ds\\ &+ C\lambda(t)\langle\xi\rangle|\hat{\varphi}(\xi)| + C(1 + t\lambda(t)\langle\xi\rangle)|\hat{\psi}(\xi)|.\end{aligned} \tag{6.3.22}$$

We immediately get

$$|\hat{u}(\xi, t)| \le C\int_0^t (t-s)|\hat{f}(\xi, s)|\, ds + C|\hat{\varphi}(\xi)| + Ct|\hat{\psi}(\xi)|. \tag{6.3.23}$$

Thus, we have proved:

Proposition 6.3.3 (Estimate in $Z_{pd}(N)$). *Let the function $u = u(\xi, .)$ be a C^2–solution of the ODE (6.3.19) with parameter ξ. Then the estimates (6.3.21), (6.3.22) and (6.3.23) hold in the pseudodifferential zone $Z_{pd}(N)$. Especially, on the border $\{(\xi, t_\xi) : \xi \in \mathbb{R}^n\}$ of $Z_{pd}(N)$ we have the estimates*

$$|\lambda(t_\xi)\langle\xi\rangle\hat{u}(\xi,t_\xi)| \le C\varrho(\xi,t_\xi)\left(\int_0^{t_\xi}(t_\xi - s)|\hat{f}(\xi,s)|\,ds + |\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|\right), \tag{6.3.24}$$

$$|D_t\hat{u}(\xi,t_\xi)| \le C\int_0^{t_\xi}(1 + (t_\xi - s)(\lambda(t_\xi) - \lambda(s))\langle\xi\rangle)|\hat{f}(\xi,s)|\,ds + C\lambda(t_\xi)\langle\xi\rangle(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|), \quad C = C(N). \tag{6.3.25}$$

For the proof we only note $1 \le N = \Lambda(t_\xi)\langle\xi\rangle \le \lambda(t_\xi)t_\xi\langle\xi\rangle$.

Remark 6.3.4. *The estimates (6.3.22) and (6.3.23) are sharp (up to multiplicative constants) in the cases of the Examples 6.2.2 and 6.2.3.*

Proof. In the Example 6.2.2 we could write the solution $\hat{u}$ in the form

$$\hat{u}(\xi,t) = G_1(\Lambda(t)\xi)\hat{\varphi}(\xi) + tG_2(\Lambda(t)\xi)\hat{\psi}(\xi),$$

where $G_1(z)$ and $G_2(z)$ are e^{-iz} times a confluent hypergeometric function with argument $2iz$. The arguments of G_j run between 0 and $\pm N$, if (ξ, t) is in $Z_{pd}(N)$. Hence, the terms $G_j(\Lambda(t)\xi)$ are bounded factors. These factors converge to 1 and $G_j(\pm N)$, if t approaches 0, t_ξ, respectively. This shows that at least in this case (6.3.23) is sharp.
For the first derivative we get

$$\hat{u}_t(\xi,t) = G_1'(\Lambda\xi)\lambda\xi\hat{\varphi}(\xi) + (G_2(\Lambda\xi) + tG_2'(\Lambda\xi)\lambda\xi)\hat{\psi}(\xi)$$

with

$$G_1'(0) = i\frac{h}{l}, \quad G_2'(0) = i\frac{h}{l+2}.$$

We see again that the estimate (6.3.22) is optimal at least in this example.
In Example 6.2.3 the solution is represented by

$$\hat{u}(\xi,t) = \sum_{j=1}^{2} c_j(\xi)te^{-\beta_j\Lambda(t)\xi}\Psi(\alpha_j, 1, 2\beta_j\Lambda(t)\xi),$$

$$c_j(\xi) = \frac{\Gamma(\alpha_j)}{\gamma_{3-j} - \gamma_j}(\hat{\varphi}(\xi)(\ln|\xi| + \gamma_{3-j}) + \hat{\psi}(\xi)).$$

From $t \le t_\xi = O((\ln\langle\xi\rangle)^{-1})$ we deduce that

$$\left| t \ln|\xi| \sum_{j=1}^{2} \frac{\Gamma(\alpha_j)}{\gamma_{3-j}-\gamma_j} \Psi(\alpha_j, 1, 2\beta_j \Lambda(t)\xi) \right| \le C,$$

$$\left| t \sum_{j=1}^{2} \frac{\Gamma(\alpha_j)}{\gamma_{3-j}-\gamma_j} \Psi(\alpha_j, 1, 2\beta_j \Lambda(t)\xi) \right| \le Ct,$$

which gives that (6.3.23) is sharp for $t \to t_\xi$ and $t \to 0$. A more complicated calculation shows the same result for u_t and the estimate (6.3.22). □

6.3.2.2 The Hyperbolic Zone

Our aim is to estimate $\hat{u}$ and $D_t\hat{u}$ in the hyperbolic zone. We define

$$U(\xi, t) = \begin{pmatrix} \lambda(t)|\xi|\hat{u}(\xi, t) \\ D_t\hat{u}(\xi, t) \end{pmatrix}$$

and obtain

$$\begin{aligned} D_tU(\xi, t) =& A(\xi, t)U(\xi, t) + A_0(\xi, t)U(\xi, t) + A_1(\xi, t)U(\xi, t) + F(\xi, t) \\ =& \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \sum_{i,j=1}^n a_{ij}(t)\lambda(t)\frac{\xi_i\xi_j}{|\xi|} & -\sum_{j=1}^n c_j(t)\lambda(t)\xi_j \end{pmatrix} U(\xi, t) \\ &+ \frac{D_t\lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ -\sum_{j=1}^n b_j(t)\frac{\xi_j}{|\xi|} & 0 \end{pmatrix} U(\xi, t) \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & ic_0(t) \end{pmatrix} U(\xi, t) - \begin{pmatrix} 0 \\ \hat{f}(\xi, t) \end{pmatrix}. \end{aligned}$$

The matrix A will be diagonalised. For this purpose we take

$$\begin{aligned} M^{-1}(\xi, t) &= \begin{pmatrix} 1 & -c(\xi, t) - \sqrt{c(\xi, t)^2 + a(\xi, t)} \\ 1 & -c(\xi, t) + \sqrt{c(\xi, t)^2 + a(\xi, t)} \end{pmatrix}^T, \\ M(\xi, t) &= \frac{1}{2\sqrt{c(\xi, t)^2 + a(\xi, t)}} \begin{pmatrix} -c(\xi, t) + \sqrt{c(\xi, t)^2 + a(\xi, t)} & -1 \\ c(\xi, t) + \sqrt{c(\xi, t)^2 + a(\xi, t)} & 1 \end{pmatrix}, \end{aligned}$$

with $a(\xi, t) := \sum_{i,j=1}^n a_{ij}(t)\frac{\xi_i\xi_j}{|\xi|^2}$ and $c(\xi, t) = \frac{1}{2}\sum_{j=1}^n c_j(t)\frac{\xi_j}{|\xi|}$, resulting in

$$\begin{aligned} MAM^{-1}(\xi, t) &= \begin{pmatrix} \tau_1(\xi, t) & 0 \\ 0 & \tau_2(\xi, t) \end{pmatrix} \\ &:= \lambda(t)|\xi| \begin{pmatrix} -c(\xi, t) - \sqrt{c(\xi, t)^2 + a(\xi, t)} & 0 \\ 0 & -c(\xi, t) + \sqrt{c(\xi, t)^2 + a(\xi, t)} \end{pmatrix}. \end{aligned}$$

For the matrix A_0 we get

$$MA_0M^{-1}(\xi,t) = \frac{D_t\lambda(t)}{2\lambda(t)} \begin{pmatrix} 1-\frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} & 1-\frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} \\ 1+\frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} & 1+\frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} \end{pmatrix}$$

with $b(\xi,t) := -\sum_{j=1}^n b_j(t)\frac{\xi_j}{|\xi|}$. Finally, $MA_1M^{-1}(\xi,t)$ has the representation

$$\frac{ic_0(t)}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{ic_0(t)c(\xi,t)}{2\sqrt{c(\xi,t)^2+a(\xi,t)}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Defining $U =: M^{-1}V$, $V = MU$ we obtain

$$\begin{aligned} D_tV &= (D_tM)M^{-1}V + \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V + \frac{ic_0c}{2\sqrt{c^2+a}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} V \\ &\quad + \frac{D_t\lambda}{2\lambda}\begin{pmatrix} 1-\frac{b+c}{\sqrt{c^2+a}} & 1-\frac{b+c}{\sqrt{c^2+a}} \\ 1+\frac{b+c}{\sqrt{c^2+a}} & 1+\frac{b+c}{\sqrt{c^2+a}} \end{pmatrix} V + \frac{ic_0}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} V + MF. \end{aligned}$$

From $D_t(M\cdot M^{-1}) = 0$ we deduce that

$$\begin{aligned} (D_tM)M^{-1} &= -M(D_tM^{-1}) \\ &= -\frac{D_t(c^2+a)}{4(c^2+a)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{D_tc}{2\sqrt{c^2+a}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

The system for V can be rewritten in the form

$$\begin{aligned} & D_tV - DV + BV = MF, \\ & D = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \\ & -B = \frac{D_t\lambda}{2\lambda}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{D_t\lambda}{2\lambda}\frac{b+c}{\sqrt{c^2+a}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \\ & \qquad + \left(\frac{ic_0}{2} - \frac{D_t(c^2+a)}{4(c^2+a)}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{ic_0c - D_tc}{2\sqrt{c^2+a}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \end{aligned} \tag{6.3.26}$$

This is the first step of perfect diagonalisation. We will apply further steps of perfect diagonalisation using a theory which was applied in [RY] and [Yag97a]. It turns out that the standard symbol classes cannot be used anymore, we have to choose classes adapted to the weakly hyperbolic theory. Here we follow the lines of [RY] and define the symbol class $S_N\{m_1, m_2, m_3\}$ as the set of all symbols $a(\xi,t) \in C^\infty(Z_{hyp}(N))$ with

$$|D_t^k D_\xi^\alpha a(\xi,t)| \le C_{k,\alpha}\langle\xi\rangle^{m_1-|\alpha|}\lambda(t)^{m_2}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m_3+k}$$

for all $k \geq 0$, $\alpha \in \mathbb{N}^n$ and all $(\xi, t) \in Z_{hyp}(N)$. The symbols of these classes satisfy

$$S_N\{m_1, m_2, m_3\} \subset S_N\{m_1 + k, m_2 + k, m_3 - k\} \quad \forall k \geq 0, \tag{6.3.27}$$

$$a(\xi, t) \in S_N\{m_1, m_2, m_3\}, \quad b(\xi, t) \in S_N\{k_1, k_2, k_3\} \tag{6.3.28}$$
$$\Longrightarrow a(\xi, t)b(\xi, t) \in S_N\{m_1 + k_1, m_2 + k_2, m_3 + k_3\},$$

$$a(\xi, t) \in S_N\{m_1, m_2, m_3\} \Longrightarrow D_t a(\xi, t) \in S_N\{m_1, m_2, m_3 + 1\}, \tag{6.3.29}$$

$$a(\xi, t) \in S_N\{m_1, m_2, m_3\} \Longrightarrow D_\xi^\alpha a(\xi, t) \in S_N\{m_1 - |\alpha|, m_2, m_3\}. \tag{6.3.30}$$

In our case $D \in S_N\{1, 1, 0\}$ and $B \in S_N\{0, 0, 1\}$. We write

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} + \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix} =: F_0^0 + F_0^1,$$

$$\mathcal{N}^{(1)}(\xi, t) = \begin{pmatrix} 0 & \frac{b_{12}(\xi,t)}{\tau_1(\xi,t) - \tau_2(\xi,t)} \\ \frac{b_{21}(\xi,t)}{\tau_2(\xi,t) - \tau_1(\xi,t)} & 0 \end{pmatrix} \in S_N\{-1, -1, 1\}.$$

It can be seen that $\mathcal{N}^{(1)}(\xi, t) = O(\frac{\lambda'}{\lambda^2 \langle \xi \rangle}) = O(1/(\Lambda \langle \xi \rangle))$. Hence $\left\| \mathcal{N}^{(1)} \right\| \leq 1/2$, if the number N, which was used in the definition of zones, is sufficiently large. Then the matrix $N_1 := I + \mathcal{N}^{(1)}$ is invertible. We observe that

$$DN_1 - N_1 D = F_0^1.$$

Then the following operator equations hold:

$$\begin{aligned}
(D_t - D + B)N_1 &= -i\mathcal{N}_t^{(1)} + N_1 D_t - N_1 D - F_0^1 + B + B\mathcal{N}^{(1)} \\
&= N_1(D_t - D + F_0^0 + N_1^{-1}(-i\mathcal{N}_t^{(1)} - \mathcal{N}^{(1)} F_0^0 + B\mathcal{N}^{(1)})) \\
&=: N_1(D_t - D + F_0^0 + N_1^{-1} B^{(1)}).
\end{aligned}$$

From (6.3.28) and (6.3.29) it follows that $N_1^{-1} B^{(1)} \in S_N\{-1, -1, 2\}$. The next step of perfect diagonalisation is:

$$N_1^{-1} B^{(1)} = \begin{pmatrix} b_{11}^{(1)} & b_{12}^{(1)} \\ b_{21}^{(1)} & b_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} b_{11}^{(1)} & 0 \\ 0 & b_{22}^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & b_{12}^{(1)} \\ b_{21}^{(1)} & 0 \end{pmatrix}$$
$$=: F_1^0 + F_1^1,$$

$$\mathcal{N}^{(2)} = \begin{pmatrix} 0 & \frac{b_{12}^{(1)}(\xi,t)}{\tau_1(\xi,t) - \tau_2(\xi,t)} \\ \frac{b_{21}^{(1)}(\xi,t)}{\tau_2(\xi,t) - \tau_1(\xi,t)} & 0 \end{pmatrix} \in S_N\{-2, -2, 2\},$$

$$N_2 = I + \mathcal{N}^{(2)}.$$

We see again that $DN_2 - N_2D = F_1^1$. Then it can be deduced that

$$\begin{aligned}
&(D_t - D + F_0^0 + N_1^{-1}B^{(1)})N_2 \\
&\quad = -i\mathcal{N}_t^{(2)} + N_2D_t - N_2D - F_1^1 + F_0^0N_2 + (F_1^0 + F_1^1)(I + \mathcal{N}^{(2)}) \\
&\quad = N_2(D_t - D + F_0^0 + F_1^0 \\
&\qquad + N_2^{-1}(-i\mathcal{N}_t^{(2)} - \mathcal{N}^{(2)}(F_0^0 + F_1^0) + (F_0^0 + F_1^0 + F_1^1)\mathcal{N}^{(2)})) \\
&\quad =: N_2(D_t - D + F_0^0 + F_1^0 + N_2^{-1}B^{(2)})
\end{aligned}$$

with $N_2^{-1}B^{(2)} \in S_N\{-2,-2,3\}$, $F_1^0 \in S_N\{-1,-1,2\}$. By induction we get

$$\begin{aligned}
&(D_t - D + F_0^0 + F_1^0 + \cdots + F_{p-2}^0 + N_{p-1}^{-1}B^{(p-1)})N_p \\
&\quad = N_p(D_t - D + F_0^0 + F_1^0 + \cdots + F_{p-1}^0 + N_p^{-1}B^{(p)}), \\
&F_{p-1}^0 \in S_N\{-p+1,-p+1,p\}, \quad N_p^{-1}B^{(p)} \in S_N\{-p,-p,p+1\}.
\end{aligned}$$

Taking into account all operator equations we obtain

$$\begin{aligned}
&(D_t - D + B)N_1N_2\dots N_p \\
&\quad = N_1N_2\dots N_p(D_t - D + F_0^0 + F_1^0 + \cdots + F_{p-1}^0 + N_p^{-1}B^{(p)}).
\end{aligned}$$

For $V =: N_1N_2\dots N_pW$ and with $\tilde{F}_1 := F_1^0 + \cdots + F_{p-1}^0$, $R_p := N_p^{-1}B^{(p)}$ we get

$$\begin{aligned}
&(D_t - D + F_0^0 + \tilde{F}_1 + R_p)W = N_p^{-1}\dots N_1^{-1}MF =: \tilde{F}, \qquad (6.3.31)\\
&D = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad F_0^0 = \operatorname{diag} B, \quad \tilde{F}_1 = \operatorname{diag}\tilde{F}_1 \in S_N\{-1,-1,2\}, \\
&R_p \in S_N\{-p,-p,p+1\}, \\
&\|N_1\dots N_p\| \le C, \quad \left\|N_p^{-1}\dots N_1^{-1}M\right\| \le C.
\end{aligned}$$

The last inequality is valid if the constant N, which was used in the definition of zones, is sufficiently large. Later we will see that the number p depends only on the functions λ, c_j, a_{ij} and b_j. The components of F_0^0 are given in (6.3.26). Let us investigate the fundamental solution $X(t,s,\xi)$ of the system (6.3.31). This matrix function satisfies

$$(D_t - D + F_0^0 + \tilde{F}_1 + R_p)X(t,s,\xi) = 0, \quad X(s,s,\xi) = I.$$

Then we have the representation

$$W(\xi,t) = \int_{t_\xi}^t X(t,s,\xi)\tilde{F}(\xi,s)\,ds + X(t,t_\xi,\xi)W(\xi,t_\xi). \qquad (6.3.32)$$

For the fundamental solution X we make the ansatz

$$
\begin{aligned}
&X(t,s,\xi) = E(t,s,\xi)Q(t,s,\xi),\\
&E(t,s,\xi) = \operatorname{diag}(E_{11}(t,s,\xi), E_{22}(t,s,\xi)),\\
&E_{jj}(t,s,\xi) = \exp\left(i\int_s^t (\tau_j - f^0_{0,jj} - \tilde{f}_{1,jj})(\xi,\sigma)d\sigma \right).
\end{aligned}
$$

The matrix E satisfies $D_tE = (D - F^0_0 - \tilde{F}_1)E$, hence

$$D_tX = (D - F^0_0 - \tilde{F}_1)EQ + ED_tQ = (D - F^0_0 - \tilde{F}_1)EQ - R_pEQ.$$

This gives the initial value problem

$$
\begin{aligned}
&D_tQ(t,s,\xi) + E(t,s,\xi)^{-1}R_p(\xi,t)E(t,s,\xi)Q(t,s,\xi) = 0,\\
&Q(s,s,\xi) = I
\end{aligned}
$$

for the matrix Q. In order to estimate X, we find estimates for E and Q. Since τ_1 and τ_2 are real, it holds

$$\|E(t,s,\xi)\| \le \max_{j=1,2} \exp\left(\left| \int_s^t |f^0_{0,jj}(\xi,\sigma)|d\sigma \right| \right) \exp\left(\left| \int_s^t |\tilde{f}^0_{1,jj}(\xi,\sigma)|d\sigma \right| \right)$$

for all $s,t \in [t_\xi, T]$. For the computation of the first integral, we recall that

$$
\begin{aligned}
-f^0_{0,jj}(\xi,\sigma) = {}& \frac{D_\sigma\lambda(\sigma)}{2\lambda(\sigma)}\left(1 \mp \frac{b(\xi,\sigma)+c(\xi,\sigma)}{\sqrt{c(\xi,\sigma)^2 + a(\xi,\sigma)}} \right)\\
&+ \frac{ic_0(\sigma)}{2} - \frac{D_\sigma(c(\xi,\sigma)^2 + a(\xi,\sigma))}{4(c(\xi,\sigma)^2 + a(\xi,\sigma))} \pm \frac{ic_0(\sigma)c(\xi,\sigma) - D_\sigma c(\xi,\sigma)}{2\sqrt{c(\xi,\sigma)^2 + a(\xi,\sigma)}}.
\end{aligned}
$$

Defining

$$
\begin{aligned}
&K_0 = \frac{1}{2}\sup_{[0,T]\times\mathbb{R}^n}\left(1 + \frac{|b(\xi,t)+c(\xi,t)|}{\sqrt{c(\xi,t)^2 + a(\xi,t)}} \right),\\
&J(s,t) = \exp\left(\int_s^t \sup_\zeta \frac{\lambda'(\tau)}{2\lambda(\tau)} \left| 1 \pm \frac{b(\zeta,\tau)+c(\zeta,\tau)}{\sqrt{c(\zeta,\tau)^2 + a(\zeta,\tau)}} \right| d\tau \right)
\end{aligned}
$$

we observe that

$$\exp\left(\int_s^t |f^0_{0,jj}(\xi,\sigma)|d\sigma \right) \le CJ(s,t) \le C\left(\frac{\lambda(t)}{\lambda(s)} \right)^{K_0}, \quad t_\xi \le s \le t \le T.$$

It remains to estimate the second integral. From $\tilde{F}_1 \in S_N\{-1,-1,2\}$ follows

$$\exp\left(\int_s^t |\tilde{f}^0_{1,jj}(\xi,\sigma)|d\sigma\right) \le C\int_s^t \langle\xi\rangle^{-1}\frac{\lambda(\sigma)}{\Lambda(\sigma)^2}\,d\sigma$$
$$\le C\langle\xi\rangle^{-1}\int_{t_\xi}^T \frac{\lambda(\sigma)}{\Lambda(\sigma)^2}\,d\sigma = C\langle\xi\rangle^{-1}(\Lambda(t_\xi)^{-1}-\Lambda(T)^{-1}) \le \frac{C}{N},$$

which results in $\|E(t,s,\xi)\| \le CJ(s,t) \le C\left(\frac{\lambda(t)}{\lambda(s)}\right)^{K_0}$ for $t_\xi \le s \le t \le T$.
We come to the estimate of $Q(t,s,\xi)$. For simplicity of notation we introduce

$$\tilde{R}_p(t,s,\xi) = E(t,s,\xi)^{-1}R_p(\xi,t)E(t,s,\xi) = E(s,t,\xi)R_p(\xi,t)E(t,s,\xi).$$

Then we have $D_tQ(t,s,\xi)+\tilde{R}_p(t,s,\xi)Q(t,s,\xi)=0$, $Q(s,s,\xi)=I$, which leads to the estimate

$$\|Q(t,s,\xi)\| \le \exp\left(\int_s^t \left\|\tilde{R}_p(\tau,s,\xi)\right\|\,d\tau\right), \quad t_\xi \le s \le t \le T.$$

It is known that

$$\left\|\tilde{R}_p(\tau,s,\xi)\right\| \le C\left(\frac{\lambda(\tau)}{\lambda(s)}\right)^{2K_0}\|R_p(\xi,\tau)\|$$
$$\le C\left(\frac{\lambda(\tau)}{\lambda(s)}\right)^{2K_0}\langle\xi\rangle^{-p}\lambda(\tau)^{-p}\left(\frac{\lambda(\tau)}{\Lambda(\tau)}\right)^{p+1}.$$

In order to compute the integral $I := \int_{t_\xi}^T \lambda(t)^{2K_0}\frac{\lambda(t)}{\Lambda(t)^{p+1}}\,dt$, we employ partial integration and (6.1.6) and obtain

$$I = \lambda(t)^{2K_0}\frac{\Lambda(t)^{-p}}{-p}\Big|_{t_\xi}^T - \int_{t_\xi}^T 2K_0\lambda(t)^{2K_0-1}\lambda'(t)\frac{\Lambda(t)^{-p}}{-p}\,dt$$
$$\le \frac{1}{p}\lambda(t_\xi)^{2K_0}\Lambda(t_\xi)^{-p} + \frac{2K_0d_1}{p}I.$$

If p is greater than $2K_0d_1$, then $I \le C\lambda(t_\xi)^{2K_0}\Lambda(t_\xi)^{-p}$, hence

$$\int_{t_\xi}^T \left\|\tilde{R}_p(\tau,s,\xi)\right\|\,d\tau \le C\lambda(t_\xi)^{-2K_0}\langle\xi\rangle^{-p}\lambda(t_\xi)^{2K_0}\Lambda(t_\xi)^{-p} \le C,$$
$$\|Q(t,s,\xi)\| \le C, \quad t_\xi \le s \le t \le T.$$

Finally, it follows that

$$\|X(t,s,\xi)\| \le CJ(s,t) \le C\left(\frac{\lambda(t)}{\lambda(s)}\right)^{K_0}, \quad t_\xi \le s \le t \le T.$$

Summarising these estimates we have the following proposition:

Proposition 6.3.5 (Estimate in $Z_{hyp}(N)$). *Let $u = u(\xi, .)$ be a C^2–solution of the ODE (6.3.19) with parameter ξ. Then the following estimate holds in the hyperbolic zone:*

$$\begin{aligned}
&|\lambda(t)\xi\hat{u}(\xi,t)| + |D_t\hat{u}(\xi,t)| \\
&\quad\le C\int_{t_\xi}^{t} J(s,t)|\hat{f}(\xi,s)|\,ds + CJ(t_\xi,t)(|\lambda(t_\xi)\xi\hat{u}(\xi,t_\xi)| + |D_t\hat{u}(\xi,t_\xi)|),\\
&J(s,t) = \exp\left(\int_s^t \sup_\zeta \frac{\lambda'(\tau)}{2\lambda(\tau)}\left|1 \pm \frac{b(\zeta,\tau)+c(\zeta,\tau)}{\sqrt{c(\zeta,\tau)^2 + a(\zeta,\tau)}}\right| d\tau\right),\\
&b(\xi,t) := -\sum_{j=1}^{n} b_j(t)\frac{\xi_j}{|\xi|}, \quad c(\xi,t) := \frac{1}{2}\sum_{j=1}^{n} c_j(t)\frac{\xi_j}{|\xi|},\\
&a(\xi,t) := \sum_{i,j=1}^{n} a_{ij}(t)\frac{\xi_i\xi_j}{|\xi|^2}.
\end{aligned}$$

6.3.2.3 Comparison with the Examples

Let us check whether this estimate of the loss of regularity is sharp. We compare the results of the Propositions 6.3.3 and 6.3.5 with the Examples 6.2.2 and 6.2.3. In these examples the loss of regularity is known, since we have an explicit representation of the solution.

We assume that the right–hand side vanishes. The Propositions 6.3.3 and 6.3.5 give the following inequality in the hyperbolic zone:

$$|\lambda(t)\xi\hat{u}(\xi,t)| + |D_t\hat{u}(\xi,t)| \le C\varrho(\xi,t_\xi)J(t_\xi,t)(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|).$$

In the case of the first example we have

$$b(\xi,t) = -\frac{h}{l}\frac{\xi}{|\xi|}, \quad c(\xi,t) \equiv 0, \quad a(\xi,t) \equiv 1, \quad J(s,t) = \left(\frac{\lambda(t)}{\lambda(s)}\right)^{\left(1+\frac{|h|}{l}\right)/2}.$$

Let $t > 0$ be fixed. Then the loss of $\xi\hat{u}$ and $D_t\hat{u}$ in comparison to $|\hat{\varphi}(\xi)| + |t_\xi\hat{\psi}(\xi)|$ is

$$\varrho(\xi,t_\xi)\lambda(t_\xi)^{-\left(1+\frac{|h|}{l}\right)/2} \sim \lambda(t_\xi)\langle\xi\rangle\lambda(t_\xi)^{-\left(1+\frac{|h|}{l}\right)/2} \sim \langle\xi\rangle^{\frac{l}{2(l+1)}\left(-1+\frac{|h|}{l}\right)}\langle\xi\rangle.$$

This shows that (for $\psi \equiv 0$) the loss of u in comparison with φ is $\langle\xi\rangle^{\frac{l}{2(l+1)}\left(-1+\frac{|h|}{l}\right)}$. This is exactly the result from Example 6.2.2. And for the loss of u in comparison with ψ (for $\varphi \equiv 0$) we get

$$t_\xi\langle\xi\rangle^{\frac{l}{2(l+1)}\left(-1+\frac{|h|}{l}\right)} \sim \langle\xi\rangle^{\frac{-l+2+|h|}{2(l+1)}},$$

which coincides with Example 6.2.2.
In the case of the other Example 6.2.3 we have

$$b(\xi,t) = -\frac{h}{\lambda'(t)}\frac{\lambda(t)^2}{\Lambda(t)}\frac{\xi}{|\xi|}, \quad c(\xi,t) \equiv 0, \quad a(\xi,t) \equiv 1,$$
$$J(s,t) = \left(\frac{\lambda(t)}{\lambda(s)}\right)^{1/2}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{|h|/2}.$$

In the hyperbolic zone we get the estimate

$$\begin{aligned}
&|\lambda(t)\xi\hat{u}(\xi,t)| + |D_t\hat{u}(\xi,t)| \\
&\quad \le C\varrho(\xi,t_\xi)\left(\frac{\lambda(t)}{\lambda(t_\xi)}\right)^{1/2}\left(\frac{\Lambda(t)}{\Lambda(t_\xi)}\right)^{|h|/2}(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|) \\
&\quad \le C\lambda(t_\xi)\langle\xi\rangle\left(\frac{\lambda(t)}{\lambda(t_\xi)}\right)^{1/2}(\Lambda(t)\langle\xi\rangle)^{|h|/2}(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|) \\
&\quad \le C(t)\langle\xi\rangle\left(\Lambda(t_\xi)t_\xi^{-2}\right)^{1/2}\langle\xi\rangle^{|h|/2}(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|) \\
&\quad = C(t)\langle\xi\rangle\langle\xi\rangle^{(|h|-1)/2}(t_\xi^{-1}|\hat{\varphi}(\xi)| + |\hat{\psi}(\xi)|).
\end{aligned}$$

Using $t_\xi = O((\ln\langle\xi\rangle)^{-1})$ (for fixed $t > 0$) we regain the estimate from Subsection 6.2.3. This shows that the estimates for $\lambda(t)|\xi\hat{u}|$ and $|D_t\hat{u}|$ are sharp in the cases of the two examples.

6.4 A–priori Estimates in Suitable Spaces

Energy estimates in Sobolev spaces play an important role for the investigation of hyperbolic Cauchy problems. Sobolev norms of a function can be regarded as weighted L^2–norms of the Fourier transform of this function. The aim of this section is to derive estimates of certain *weighted L^2–norms of the Fourier transform* of the solution using the *point-wise estimates of the Fourier transform* derived in the previous section. The structure of these point-wise estimates motivates the following definition.

Definition 6.4.1 (Spaces with special weight). *For $L_1, L_2, M, K_1, K_2 \ge 0$ let $\vartheta_{L_1L_2MK_1K_2}$ be the function*

$$\vartheta_{L_1L_2MK_1K_2}(\xi,t) = \begin{cases} \left(\frac{\varrho(\xi,t_\xi)}{\varrho(\xi,t)}\right)^{L_1}\lambda(t_\xi)^{L_2}J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^{K_1} & : 0 \le t \le t_\xi, \\ \lambda(t)^{L_2}J(t,t_0)\langle\xi\rangle^M t_\xi^{K_2} & : t_\xi \le t \le T, \end{cases}$$
$$J(s,t) = \exp\left(\int_s^t \sup_\zeta \frac{\lambda'(\tau)}{2\lambda(\tau)}\left|1 \pm \frac{b(\zeta,\tau)+c(\zeta,\tau)}{\sqrt{c(\zeta,\tau)^2 + a(\zeta,\tau)}}\right| d\tau\right),$$

$$b(\xi,t) := -\sum_{j=1}^{n} b_j(t)\frac{\xi_j}{|\xi|}, \quad c(\xi,t) := \frac{1}{2}\sum_{j=1}^{n} c_j(t)\frac{\xi_j}{|\xi|},$$
$$a(\xi,t) := \sum_{i,j=1}^{n} a_{ij}(t)\frac{\xi_i\xi_j}{|\xi|^2}.$$

The number t_0 is defined by the formula $\Lambda(t_0)\langle 0\rangle = N$. By $B_{L_1L_2MK_1K_2}$ we denote the space

$$B_{L_1L_2MK_1K_2}$$
$$:= \left\{v \in C([0,T], \mathcal{S}'(\mathbb{R}^n)) : \vartheta_{L_1L_2MK_1K_2}\hat{v} \in C([0,T], L^2(\mathbb{R}^n_\xi))\right\},$$
$$\|v\|_{B_{L_1L_2MK_1K_2}} := \sup_{[0,T]} \|\vartheta_{L_1L_2MK_1K_2}(.,t)\hat{v}(.,t)\|_{L^2(\mathbb{R}^n_\xi)}.$$

We will study the properties of these spaces in the next section. An important special case is given by $L_1 = 1$, $L_2 = 0$. To simplify the notation, we write

$$\vartheta_{MK_1K_2}(\xi,t) := \vartheta_{10MK_1K_2}(\xi,t), \quad B_{MK_1K_2} := B_{10MK_1K_2}.$$

We will even have $K_1 = K_2$ in most applications.
For the initial data we take the following space:

Definition 6.4.2 (Spaces for the data). *Let $C_{L_1L_2MK_1}$ be the space*

$$C_{L_1L_2MK_1} := \left\{v \in \mathcal{S}'(\mathbb{R}^n) : \vartheta_{L_1L_2MK_1K_1}(.,0)\hat{v}(.) \in L^2(\mathbb{R}^n_\xi)\right\},$$
$$\|v\|_{C_{L_1L_2MK_1}} := \|\vartheta_{L_1L_2MK_1K_1}(.,0)\hat{v}(.)\|_{L^2(\mathbb{R}^n_\xi)}.$$

We introduce the abbreviation C_{MK_1} in the special case $L_1 = 1$, $L_2 = 0$:

$$C_{MK_1} := C_{10MK_1}.$$

With these notations, we can now formulate the central energy estimate:

Theorem 6.4.3 (A–priori estimate). *Let $\varphi \in C_{MK}$, $\psi \in C_{M(K+1)}$ and $f \in B_{MKK}$. Then the solution u of (6.3.19) satisfies*

$$Hu \in B_{MKK}, \quad D_t u \in B_{M(K+1)K},$$
$$\|Hu\|_{B_{MKK}} + \|D_t u\|_{B_{M(K+1)K}}$$
$$\le C_{apr}\left(T\|f\|_{B_{MKK}} + \|\varphi\|_{C_{MK}} + \|\psi\|_{C_{M(K+1)}}\right),$$

where $H(D_x,t)$ is a pseudodifferential operator with the symbol

$$h(\xi,t) = \lambda(t)|\xi|\chi\left(\frac{\Lambda(t)|\xi|}{N}\right) + \varrho(\xi,t)\left(1 - \chi\left(\frac{\Lambda(t)|\xi|}{N}\right)\right),$$
$$\chi(s) = 0 \quad (s \le 1/2), \quad \chi(s) = 1 \quad (s \ge 2), \quad \chi \in C^\infty(\mathbb{R}^n).$$

Remark 6.4.4. *If $t > 0$ is fixed, then the operator H acts like $\lambda(t)\langle D_x\rangle$ and the above estimate shows that the first derivative of the solution with respect to x and the right–hand side f are from the same space. In other words, this result is an estimate of* strictly hyperbolic type.

Proof. For fixed $t > 0$, let $R_0(t)$ be the positive real number with

$$\Lambda(t)\langle R_0(t)\rangle = N.$$

In order to estimate $\|Hu\|_{B_{MKK}}$ it is sufficient to show that

$$\begin{aligned}&\|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|_{L^2(|\xi|\le R_0(t))}\\ &\quad+\|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|_{L^2(|\xi|\ge R_0(t))}\\ &\le C\left(T\|f\|_{B_{MKK}}+\|\varphi\|_{C_{MK}}+\|\psi\|_{C_{M(K+1)}}\right).\end{aligned}$$

Here we used the fact that the computations in Section 6.3 remain true, if we replace N by $2N$ or $N/2$ (if N is sufficiently large). Let us start with the first term on the left. Due to (6.3.21) we have

$$|\varrho(\xi,t)\hat{u}(\xi,t)|\le C\varrho(\xi,t)\left(\int_0^t(t-s)|\hat{f}(\xi,s)|\,ds+|\hat{\varphi}(\xi)|+t|\hat{\psi}(\xi)|\right).$$

From (6.3.11) and the Inequality of Cauchy–Schwarz we conclude that

$$\left(\int_0^t(t-s)|\hat{f}(\xi,s)|\,ds\right)^2\le Ct\int_0^t\frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds.$$

It follows that

$$\begin{aligned}&|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|^2\le C\varrho(\xi,t_\xi)^2\langle\xi\rangle^{2M}t_\xi^{2K}J(t_\xi,t_0)^2\\ &\quad\times\left(t\int_0^t\frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds+|\hat{\varphi}(\xi)|^2+t_\xi^2|\hat{\psi}(\xi)|^2\right).\end{aligned}$$

Integration over $|\xi|\le R_0(t)$ and $\varrho(\xi,0)=1$ give

$$\begin{aligned}&\|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|^2_{L^2(|\xi|\le R_0(t))}\\ &\quad\le Ct\int_0^t\int_{|\xi|\le R_0(t)}|\vartheta_{MKK}(\xi,s)\hat{f}(\xi,s)|^2\,d\xi\,ds\\ &\qquad+C\|\hat{\varphi}(\xi)\vartheta_{MKK}(\xi,0)\|^2_{L^2(|\xi|\le R_0(t))}\\ &\qquad+C\left\|\hat{\psi}(\xi)\vartheta_{M(K+1)K}(\xi,0)\right\|^2_{L^2(|\xi|\le R_0(t))}\\ &\quad\le C(T^2\|f\|^2_{B_{MKK}}+\|\varphi\|^2_{C_{MK}}+\|\psi\|^2_{C_{M(K+1)}}).\end{aligned}$$

For the second term we use Proposition 6.3.5 and (6.3.24), (6.3.25):

$$\begin{aligned}|\lambda(t)\xi\hat{u}(\xi,t)| &\le C\int_{t_\xi}^{t} J(s,t)|\hat{f}(\xi,s)|\,ds\\ &+ CJ(t_\xi,t)\int_0^{t_\xi}(1+(t_\xi-s)\varrho(\xi,t_\xi))|\hat{f}(\xi,s)|\,ds\\ &+ CJ(t_\xi,t)\varrho(\xi,t_\xi)(|\hat{\varphi}(\xi)|+t_\xi|\hat{\psi}(\xi)|).\end{aligned}$$

The second integral on the right can be bounded by

$$\begin{aligned}&\int_0^{t_\xi}|\hat{f}(\xi,s)|\,ds+\varrho(\xi,t_\xi)\int_0^{t_\xi}(t_\xi-s)|\hat{f}(\xi,s)|\,ds\\ &\le C\varrho(\xi,t_\xi)\left(t_\xi\int_0^{t_\xi}\frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}ds\right)^{1/2},\end{aligned} \tag{6.4.1}$$

see (6.3.11). As a consequence we obtain

$$\begin{aligned}&|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|^2 \le C\langle\xi\rangle^{2M}t_\xi^{2K}t\int_{t_\xi}^{t}|\hat{f}(\xi,s)|^2J(s,t_0)^2\,ds\\ &+ C\varrho(\xi,t_\xi)^2J(t_\xi,t_0)^2\langle\xi\rangle^{2M}t_\xi^{2K}\\ &\times\left(t_\xi\int_0^{t_\xi}\frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds+|\hat{\varphi}(\xi)|^2+t_\xi^2|\hat{\psi}(\xi)|^2\right).\end{aligned}$$

Integration over $|\xi|\ge R_0(t)$ gives

$$\begin{aligned}&\|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|^2_{L^2(|\xi|\ge R_0(t))}\\ &\le C(T^2\|f\|^2_{B_{MKK}}+\|\varphi\|^2_{C_{MK}}+\|\psi\|^2_{C_{M(K+1)}}).\end{aligned}$$

Then the estimate for $\|Hu\|_{B_{MKK}}$ is proved. It remains to consider $\|D_tu\|_{B_{M(K+1)K}}$. We have for $D_t\hat{u}$ and $\lambda(t)|\xi|\hat{u}$ the same estimate in the hyperbolic zone, see Propositions 6.3.5 and 6.3.3. The weights ϑ_{MKK} and $\vartheta_{M(K+1)K}$ coincide in the hyperbolic zone. Then we immediately get that

$$\begin{aligned}&\left\|D_t\hat{u}(\xi,t)\vartheta_{M(K+1)K}(\xi,t)\right\|^2_{L^2(|\xi|\ge R_0(t))}\\ &\le C(T^2\|f\|^2_{B_{MKK}}+\|\varphi\|^2_{C_{MK}}+\|\psi\|^2_{C_{M(K+1)}}).\end{aligned}$$

So it suffices to study $(D_t\hat{u})\vartheta_{M(K+1)K}$ in the pseudodifferential zone. There the estimate (6.3.22) holds. The additive term 1 in the coefficient $(1+t\lambda(t)\langle\xi\rangle)$ for $|\hat{\psi}|$ causes some difficulties, therefore we choose a higher t_ξ–exponent for

ϑ in the pseudodifferential zone. We estimate the integral on the right in a similar way as in (6.4.1) and get

$$\int_0^t (1+(t-s)(\lambda(t)-\lambda(s))\langle\xi\rangle)|\hat{f}(\xi,s)|\,ds \\ \leq C\varrho(\xi,t)\left(t\int_0^t \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds\right)^{1/2},$$

see (6.3.9) and (6.3.11). Then it follows that

$$|D_t\hat{u}(\xi,t)\vartheta_{M(K+1)K}(\xi,t)|^2 \leq C\varrho(\xi,t_\xi)^2 J(t_\xi,t_0)^2\langle\xi\rangle^{2M}t_\xi^{2K+2} \\ \times\left(t\int_0^t \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds+|\hat{\varphi}(\xi)|^2+|\hat{\psi}(\xi)|^2\right).$$

Integration over $|\xi|\leq R_0(t)$ gives

$$\left\|D_t\hat{u}(\xi,t)\vartheta_{M(K+1)K}(\xi,t)\right\|^2_{L^2(|\xi|\leq R_0(t))} \\ \leq C(T^2\|f\|^2_{B_{M(K+1)K}}+\|\varphi\|^2_{C_{M(K+1)}}+\|\psi\|^2_{C_{M(K+1)}}) \\ \leq C(T^2\|f\|^2_{B_{MKK}}+\|\varphi\|^2_{C_{MK}}+\|\psi\|^2_{C_{M(K+1)}}).$$

The theorem is proved. □

Corollary 6.4.5. *Under the assumptions of the previous theorem, it holds*

$$u\in B_{01(M+1)KK}, \\ \|u\|_{B_{01(M+1)KK}}\leq C(T\|f\|_{B_{MKK}}+\|\varphi\|_{C_{MK}}+\|\psi\|_{C_{M(K+1)}}).$$

Proof. In the pseudodifferential zone, we have

$$|\hat{u}(\xi,t)\vartheta_{01(M+1)KK}(\xi,t)| = \left|\varrho(\xi,t)\hat{u}(\xi,t)\frac{\lambda(t_\xi)\langle\xi\rangle J(t_\xi,t_0)}{\varrho(\xi,t)}\langle\xi\rangle^M t_\xi^K\right| \\ \leq C|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|,$$

see (6.3.10). Let us consider the hyperbolic zone. If T is small enough, then we have the inequality $\langle\xi\rangle\leq 2|\xi|$ for $(\xi,t)\in Z_{hyp}(N)$. It follows that

$$|\hat{u}(\xi,t)\vartheta_{01(M+1)KK}(\xi,t)| = \left|\lambda(t)\langle\xi\rangle\hat{u}(\xi,t)J(t,t_0)\langle\xi\rangle^M t_\xi^K\right| \\ \leq 2|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|.$$

□

6.5 Properties of the Spaces $B_{L_1L_2MK_1K_2}$

In the previous section spaces $B_{L_1L_2MK_1K_2}$ have been introduced. We did not give any property of these spaces. For example, up to now nothing has been said about the question of whether these spaces are Banach spaces. Therefore, we devote this section to the study of spaces $B_{L_1L_2MK_1K_2}$. For this purpose the theory of spaces with temperate weight is applied, which was developed in [Hör69]. In 6.5.1 we show that the restrictions of the spaces $B_{L_1L_2MK_1K_2}$ at the sets $\{t = \text{const}\}$ are spaces with temperate weight (if $K_1 = K_2$). In 6.5.2 we prove that $B_{L_1L_2MK_1K_2}$ is an algebra, if $K_1 = K_2$ and M is large enough. This allows us to study the mapping of superposition operators $u \mapsto f(u)$, when f is an entire analytic function.

6.5.1 Spaces with Temperate Weight

In [Hör69], Part I, Chapter 2, spaces $B_{p,\vartheta}$ were introduced. These spaces consist of all functions u with $\hat{u}(\xi)\vartheta(\xi) \in L^p$. The function ϑ is called a temperate weight function and has to satisfy a condition mentioned below. In the following we recall results about such weight functions and such spaces from [Hör69].

Definition 6.5.1 (Spaces with temperate weight). *A positive function ϑ defined in $\mathbb{R}^n$ will be called a* temperate weight function*, if there exist positive constants C and m such that*

$$\vartheta(\xi+\eta) \le (1 + C|\xi|)^m \vartheta(\eta) \quad \forall \xi, \eta \in \mathbb{R}^n.$$

The set of all such functions will be denoted by $\mathcal{K}$. If $\vartheta \in \mathcal{K}$ and $1 \le p \le \infty$, we denote by $B_{p,\vartheta}$ the set of all distributions $u \in \mathcal{S}'$ such that $\hat{u}$ is a function and

$$\|u\|_{p,\vartheta} := \left((2\pi)^{-n} \int |\vartheta(\xi)\hat{u}(\xi)|^p d\xi \right)^{1/p} < \infty.$$

When $p = \infty$, we shall interpret $\|u\|_{p,\vartheta}$ as ess-sup$|\vartheta(\xi)\hat{u}(\xi)|$.

We want to list some results about weight functions ϑ and spaces $B_{p,\vartheta}$. For details see [Hör69].

Lemma 6.5.2. *If $\vartheta \in \mathcal{K}$, then ϑ is continuous. If $\vartheta \in \mathcal{K}$ with constants C, m, then*

$$\vartheta(0)(1 + C|\xi|)^{-m} \le \vartheta(\xi) \le \vartheta(0)(1 + C|\xi|)^m.$$

It holds $\langle \xi \rangle := (1 + |\xi|^2)^{1/2} \in \mathcal{K}$ with $C = m = 1$.

Proposition 6.5.3. *If $\vartheta_1, \vartheta_2 \in \mathcal{K}$, then $\vartheta_1 + \vartheta_2 \in \mathcal{K}$, $\vartheta_1\vartheta_2 \in \mathcal{K}$, $\sup(\vartheta_1, \vartheta_2) \in \mathcal{K}$, $\inf(\vartheta_1, \vartheta_2) \in \mathcal{K}$. If $\vartheta \in \mathcal{K}$, then $\vartheta^s \in \mathcal{K}$ for every real s.*

Proposition 6.5.4. *$B_{p,\vartheta}$ is a Banach space. It holds $\mathcal{S} \subset B_{p,\vartheta} \subset \mathcal{S}'$, also in the topological sense. C_0^∞ is dense in $B_{p,\vartheta}$, if $p < \infty$.*

Proposition 6.5.5. *If $\vartheta_1, \vartheta_2 \in \mathcal{K}$ and $\vartheta_2(\xi) \leq C\vartheta_1(\xi)$ for all $\xi \in \mathbb{R}^n$, then $B_{p,\vartheta_1} \subset B_{p,\vartheta_2}$.*

Proposition 6.5.6. *If $1 < p < \infty$, then the dual space of $B_{p,\vartheta}$ is $B_{p',1/\vartheta}$, $1/p + 1/p' = 1$.*

Proposition 6.5.7. *Let $\varphi \in C_0^\infty$, $\int \varphi(x)\,dx = 1$, $\psi \in C_0^\infty$, $\psi(0) = 1$. We set $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ and $\psi_\varepsilon(x) = \psi(\varepsilon x)$. If $u \in B_{p,\vartheta}$ and $p < \infty$, then $u * \varphi_\varepsilon$ and $u\psi_\varepsilon$ converge to u in $B_{p,\vartheta}$, as ε tends to 0.*

Now we will utilise (some of) these cited properties to consider the spaces $B_{L_1L_2MK_1K_2}$.

Proposition 6.5.8. *For each fixed $t > 0$, $\vartheta_{L_1L_2MKK}(.,t)$ is a temperate weight in the sense of the Definition 6.5.1. The constants C and m are independent of t.*

Proof. We can write

$$\vartheta_{L_1L_2MKK}(\xi,t) = \max\left(\left(\frac{\varrho(\xi,t_\xi)}{\varrho(\xi,t)}\right)^{L_1}, 1\right) \max\left(\lambda(t_\xi), \lambda(t)\right)^{L_2} \times \min(J(t_\xi,t_0), J(t,t_0))\langle\xi\rangle^M t_\xi^K.$$

If we are able to show that

$$\varrho(\xi,t_\xi), \varrho(\xi,t), \lambda(t_\xi), J(t_\xi,t_0), t_\xi \in \mathcal{K},$$

then the proposition is proved. Let us start with $\lambda(t_\xi)$. From (6.3.2) it can be deduced that

$$\begin{aligned}
\frac{\lambda(t_{\xi+\eta})}{\lambda(t_\eta)} &\leq \left(\frac{\Lambda(t_{\xi+\eta})}{\Lambda(t_\eta)}\right)^{d_0} = \left(\frac{\langle\eta\rangle}{\langle\xi+\eta\rangle}\right)^{d_0} \leq (1+|\xi|)^{d_0}, \quad t_{\xi+\eta} \leq t_\eta, \\
\frac{\lambda(t_{\xi+\eta})}{\lambda(t_\eta)} &\leq \left(\frac{\Lambda(t_{\xi+\eta})}{\Lambda(t_\eta)}\right)^{d_1} = \left(\frac{\langle\eta\rangle}{\langle\xi+\eta\rangle}\right)^{d_1} \leq (1+|\xi|)^{d_1}, \quad t_\eta \leq t_{\xi+\eta}.
\end{aligned} \tag{6.5.1}$$

Hence we conclude that $\lambda(t_\xi) \in \mathcal{K}$ with $C = 1$, $m = d_1$. We know that $\langle\xi\rangle \in \mathcal{K}$ with $C = m = 1$. Then it follows that

$$\varrho(\xi,t_\xi) = \left(1 + \frac{1}{N}\lambda(t_\xi)^2\langle\xi\rangle^2\right)^{1/2} \in \mathcal{K},$$

with constants C and m independent of t. We also know that $1, \lambda(t)^2/\Lambda(t) \in \mathcal{K}$ with $C = m = 0$. Hence $\varrho(.,t) \in \mathcal{K}$ and again the constants C and m do not depend on t. By (6.3.14), (6.3.15), (6.3.16) and (6.5.1) we have

$$\frac{J(t_{\xi+\eta}, t_0)}{J(t_\eta, t_0)} = J(t_{\xi+\eta}, t_\eta) \leq \left(\frac{\lambda(t_\eta)}{\lambda(t_{|\xi|+|\eta|})}\right)^{K_0} \leq (1+|\xi|)^{d_1K_0}.$$

This gives $J(t_\xi, t_0) \in \mathcal{K}$. It remains to verify that $t_\xi \in \mathcal{K}$. In order to prove this, we show that $t_\xi\langle\xi\rangle \in \mathcal{K}$. From (6.3.6) and the mean value theorem we deduce that

$$\begin{aligned} t_{\xi+\eta}\langle\xi+\eta\rangle &\leq t_{|\xi|+|\eta|}\langle|\xi|+|\eta|\rangle \\ &= t_{|\eta|}\langle\eta\rangle + (t_{|\zeta|}\langle|\zeta|\rangle)'|\xi| \quad (|\eta| < |\zeta| < |\eta|+|\xi|) \\ &= t_{|\eta|}\langle\eta\rangle + \left(\frac{-\Lambda(t_{|\zeta|})}{\lambda(t_{|\zeta|})\langle|\zeta|\rangle} + t_{|\zeta|}\right)\frac{|\zeta|}{\langle\zeta\rangle}|\xi| \\ &\leq t_{|\eta|}\langle\eta\rangle + t_{|\zeta|}|\xi| \leq t_{|\eta|}\langle\eta\rangle + t_{|\eta|}|\xi| \leq (t_\eta\langle\eta\rangle)(1+|\xi|). \end{aligned}$$

Hence we obtain $t_\xi\langle\xi\rangle \in \mathcal{K}$ with $C = m = 1$. □

Thus, we can conclude that the space with temperate weight $\vartheta_{L_1L_2MKK} = \vartheta_{L_1L_2MKK}(.,t)$ is a Banach space for each frozen $t \geq 0$. It is easy to see that then $B_{L_1L_2MKK}$ is a Banach space, too.

In order to derive embedding results of the $B_{L_1L_2MKK}$–spaces into the usual spaces $C\left([0,T], H^s\right)$ (and vice versa), we estimate the weight. From $J(t,t_0) \geq 1$, (6.3.5), (6.3.6), (6.3.7), (6.3.10) and (6.3.13) we deduce that

$$\begin{aligned} &\vartheta_{L_1L_2MKK}(\xi,t) \geq \lambda(t_\xi)^{L_2}\langle\xi\rangle^M t_\xi^K \\ &\quad = (\lambda(t_\xi)\langle\xi\rangle^{d_1})^{L_2}\langle\xi\rangle^{M-K-d_1L_2}(t_\xi\langle\xi\rangle)^K \geq C\langle\xi\rangle^{M-K-d_1L_2}, \\ &\vartheta_{L_1L_2MKK}(\xi,t) \leq \varrho(\xi,t_\xi)^{L_1}\lambda(t_0)^{L_2}J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^K \\ &\quad \leq C\lambda(t_\xi)^{L_1}\left(\frac{\lambda(t_0)}{\lambda(t_\xi)}\right)^{K_0}\langle\xi\rangle^{M+L_1+K(d_0-1)} \\ &\quad = C\lambda(t_\xi)^{L_1-K_0}\langle\xi\rangle^{M+L_1+K(d_0-1)} \\ &\quad \leq C\langle\xi\rangle^{M-K(1-d_0)+L_1+d_1|L_1-K_0|}. \end{aligned}$$

Then it follows that

$$\begin{aligned} C\left([0,T], H^{M-K(1-d_0)+L_1+d_1|L_1-K_0|}\right) &\subset B_{L_1L_2MKK} \\ &\subset C\left([0,T], H^{M-K-d_1L_2}\right). \end{aligned}$$

Finally, we study embeddings from $C_{L_1L_2MK}$ into Sobolev spaces (and vice versa). In a similar way as for the B–spaces we get

$$\begin{aligned}\vartheta_{L_1L_2MK_1K_2}(\xi,0) &= \varrho(\xi,t_\xi)^{L_1}\lambda(t_\xi)^{L_2}J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^K\\ &\geq C\lambda(t_\xi)^{L_1+L_2}\langle\xi\rangle^{M+L_1}t_\xi^K\\ &= C(\lambda(t_\xi)\langle\xi\rangle^{d_1})^{L_1+L_2}\langle\xi\rangle^{M+L_1-d_1(L_1+L_2)-K}(t_\xi\langle\xi\rangle)^K\\ &\geq C\langle\xi\rangle^{M-K+L_1-d_1(L_1+L_2)},\\ \vartheta_{L_1L_2MK_1K_2}(\xi,0) &\leq C\lambda(t_\xi)^{L_1+L_2}\left(\frac{\lambda(t_0)}{\lambda(t_\xi)}\right)^{K_0}\langle\xi\rangle^{M+L_1+K(d_0-1)}\\ &= C\lambda(t_\xi)^{L_1+L_2-K_0}\langle\xi\rangle^{M+L_1+K(d_0-1)}\\ &\leq C\langle\xi\rangle^{M-K(1-d_0)+L_1+d_1|L_1+L_2-K_0|}.\end{aligned}$$

This implies

$$H^{M-K(1-d_0)+L_1+d_1|L_1+L_2-K_0|}\subset C_{L_1L_2MK}\subset H^{M-K+L_1-d_1(L_1+L_2)}.$$

6.5.2 The Algebra Property

The aim of this subsection is to show that $B_{L_1L_2MKK}$ is an algebra, if M is sufficiently large. We split the proof into three lemmata.

Lemma 6.5.9. *Let $B_{2,\vartheta(t)}$ be a space with temperate weight $\vartheta(\xi,t)$. If*

$$\sup_{[0,T]\times\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_\eta}\frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta =: C_\vartheta^2<\infty,$$

then $B_{2,\vartheta(t)}$ is an algebra and it holds

$$\|uv\|_{B_{2,\vartheta(t)}}\leq C_\vartheta\|u\|_{B_{2,\vartheta(t)}}\|v\|_{B_{2,\vartheta(t)}}.$$

Proof. The idea of this proof is taken from [BR84]. Let $w(\xi)\in L^2(\mathbb{R}^n)$ be an arbitrary function. Then, by the Inequality of Cauchy–Schwarz, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n_\xi}\vartheta(\xi,t)(uv)\hat{\ }(\xi,t)w(\xi)\,d\xi\right|\\ &=\left|\int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_\eta}\vartheta(\xi,t)\hat{u}(\eta,t)\hat{v}(\xi-\eta,t)w(\xi)\,d\eta\,d\xi\right|\end{aligned}$$

$$= \left| \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_\eta} \frac{\vartheta(\xi,t)}{\vartheta(\eta,t)\vartheta(\xi-\eta,t)} \left(\hat{u}(\eta,t)\vartheta(\eta,t)\right) \times \left(\hat{v}(\xi-\eta,t)\vartheta(\xi-\eta,t)\right) w(\xi)\, d\eta\, d\xi \right|$$

$$\leq \left(\int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_\eta} \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2} |w(\xi)|^2\, d\eta\, d\xi \right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^n_\eta} \int_{\mathbb{R}^n_\xi} \left|\hat{u}(\eta,t)\vartheta(\eta,t)\right|^2 \left|\hat{v}(\xi-\eta,t)\vartheta(\xi-\eta,t)\right|^2\, d\xi\, d\eta \right)^{1/2}$$

$$\leq C_\vartheta \|w\|_{L^2} \|u\|_{B_{2,\vartheta(t)}} \|v\|_{B_{2,\vartheta(t)}}.$$

This gives the assertion. □

Lemma 6.5.10. *Let the temperate weight $\vartheta(\xi,t)$ fulfil the conditions*

$$\sup_{[0,T]} \int_{\mathbb{R}^n_\eta} \vartheta(\eta,t)^{-2}\, d\eta =: C_1 < \infty,$$

$$\vartheta(\xi,t) \leq C_2 \vartheta(\xi/2,t) \quad \forall (t,\xi) \in [0,T] \times \mathbb{R}^n,$$

$\vartheta(\xi,t) = \vartheta(|\xi|,t)$ *is monotonically increasing in* $|\xi|$ *for each fixed* t.

Then a constant C_ϑ exists with

$$\sup_{[0,T]\times\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_\eta} \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\, d\eta =: C_\vartheta^2 < \infty.$$

Proof. Let $\xi \in \mathbb{R}^n$ be fixed. We split $\mathbb{R}^n_\eta$ into three parts:

$$A = \{\eta \in \mathbb{R}^n : |\eta| \geq 2|\xi|\},$$
$$B = \{\eta \in \mathbb{R}^n : |\eta| \leq 2|\xi|, |\xi-\eta| \leq |\eta|\},$$
$$C = \{\eta \in \mathbb{R}^n : |\eta| \leq 2|\xi|, |\xi-\eta| \geq |\eta|\}.$$

In A we have $|\xi| \leq |\eta|/2 \leq |\xi-\eta| \leq 3|\eta|/2$. This gives

$$\int_A \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\, d\eta \leq \int_A \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\eta/2,t)^2}\, d\eta \leq \int_A \frac{d\eta}{\vartheta(\eta,t)^2} \leq C_1.$$

In B it holds $|\eta| \geq |\xi|/2$, hence

$$\int_B \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\, d\eta \leq \int_B \frac{\vartheta(\xi,t)^2}{\vartheta(\xi/2,t)^2\vartheta(\xi-\eta,t)^2}\, d\eta \leq C_2^2 \int_B \frac{d\eta}{\vartheta(\xi-\eta,t)^2} \leq C_1 C_2^2.$$

And in C we have $|\xi - \eta| \geq |\xi|/2$, which similarly gives

$$\int_C \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta \leq C_1C_2^2.$$

The lemma is proved. □

Lemma 6.5.11. *If M is sufficiently large, then $\vartheta_{L_1L_2MKK}$ fulfils the conditions mentioned in the previous lemma.*

Proof. The estimate $\vartheta_{L_1L_2MKK}(\xi,t) \geq C\langle\xi\rangle^{M-K-d_1L_2}$ has been proved in the above subsection. If $M > K + d_1L_2 + n/2$, then

$$\sup_{[0,T]}\int_{\mathbb{R}^n_\eta} \vartheta_{L_1L_2MKK}(\eta,t)^{-2}\,d\eta < \infty.$$

To consider the second assertion, we distinguish three cases. If $(\xi,t) \in Z_{hyp}(N)$ and $(\xi/2,t) \in Z_{hyp}(N)$, then it is clear that

$$\lambda(t)^{L_2}J(t,t_0)\langle\xi\rangle^M t_\xi^M \leq C\lambda(t)^{L_2}J(t,t_0)\langle\xi/2\rangle^M t_{\xi/2}^M.$$

Now let $(\xi,t) \in Z_{pd}(N)$ and $(\xi/2,t) \in Z_{pd}(N)$. Then it is to show that

$$\begin{aligned}&\left(\frac{\varrho(\xi,t_\xi)}{\varrho(\xi,t)}\right)^{L_1}\lambda(t_\xi)^{L_2}J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^M \\ &\quad\leq C\left(\frac{\varrho(\xi/2,t_{\xi/2})}{\varrho(\xi/2,t)}\right)^{L_1}\lambda(t_{\xi/2})^{L_2}J(t_{\xi/2},t_0)\langle\xi/2\rangle^M t_{\xi/2}^M.\end{aligned} \tag{6.5.2}$$

We have $\varrho(\xi/2,t) \leq \varrho(\xi,t) \leq \sqrt{2}\varrho(\xi/2,t)$. From (6.3.2) we get

$$\left(\frac{\langle\xi/2\rangle}{\langle\xi\rangle}\right)^{d_1} \leq \frac{\lambda(t_\xi)}{\lambda(t_{\xi/2})} \leq \left(\frac{\langle\xi/2\rangle}{\langle\xi\rangle}\right)^{d_0},$$

hence $C_1\lambda(t_{\xi/2}) \leq \lambda(t_\xi) \leq C_2\lambda(t_{\xi/2})$. From this result and (6.3.10) follows

$$C_1'\varrho(\xi/2,t_{\xi/2}) \leq \varrho(\xi,t_\xi) \leq C_2'\varrho(\xi/2,t_{\xi/2}).$$

Furthermore, due to (6.3.2) it holds

$$\begin{aligned}\frac{J(t_\xi,t_0)}{J(t_{\xi/2},t_0)} &= J(t_\xi,t_{\xi/2}) \leq \left(\frac{\lambda(t_{\xi/2})}{\lambda(t_\xi)}\right)^{K_0} \\ &\leq \left(\frac{\Lambda(t_{\xi/2})}{\Lambda(t_\xi)}\right)^{d_1K_0} = \left(\frac{\langle\xi\rangle}{\langle\xi/2\rangle}\right)^{d_1K_0} \leq C.\end{aligned}$$

Finally, $t_\xi \le t_{\xi/2}$. Thus, (6.5.2) is proved. In the last case we have $(\xi/2, t) \in Z_{pd}(N)$ and $(\xi, t) \in Z_{hyp}(N)$. Then $t_\xi \le t \le t_{\xi/2}$ and, consequently,

$$\lambda(t)^{L_2} \le \lambda(t_{\xi/2})^{L_2} \le \lambda(t_{\xi/2})^{L_2} \left(\frac{\varrho(\xi/2, t_{\xi/2})}{\varrho(\xi/2, t)}\right)^{L_1}.$$

With $\langle\xi\rangle^M \le C\langle\xi/2\rangle^M$, $J(t_\xi, t_0) \le CJ(t_{\xi/2}, t_0)$ and $t_\xi \le t_{\xi/2}$ we get $\vartheta_{L_1L_2MKK}(\xi, t) \le \vartheta_{L_1L_2MKK}(\xi/2, t)$ in this case, too.
Finally, we prove that $\vartheta_{L_1L_2MKK}(\xi, t)$ is monotonically increasing in $|\xi|$. In the hyperbolic zone, the weight can be written as

$$\lambda(t)^{L_2}\langle\xi\rangle^{M-K}(\langle\xi\rangle t_\xi)^K.$$

Because $\langle\xi\rangle t_\xi$ is increasing in $\langle\xi\rangle$ (see (6.3.6)), we have the assertion, if $M \ge K$. Now let us consider the pseudodifferential zone. We can write the weight in the form

$$\begin{aligned}&\left(\varrho(\xi, t_\xi)\langle\xi\rangle^{d_1}\right)^{L_1} (\lambda(t_\xi)\langle\xi\rangle^{d_1})^{L_2} J(t_\xi, t_0)\left(\langle\xi\rangle t_\xi\right)^K \\ &\quad \times \langle\xi\rangle^{M-K-d_1L_1-d_1L_2}\varrho(\xi, t)^{-L_1}.\end{aligned}$$

From (6.3.13) we gain the monotonicity of the first two factors. The term $J(t_\xi, t_0)$ is obviously increasing in $\langle\xi\rangle$. Due to (6.3.6) we know that $(\langle\xi\rangle t_\xi)^K$ is increasing, too. It remains to show that the last factor $r(\langle\xi\rangle, t) := \langle\xi\rangle^{M'}\varrho(\xi, t)^{-L_1}$ increases in $\langle\xi\rangle$, $M' := M - K - d_1L_1 - d_1L_2$. We compute the derivative:

$$\begin{aligned} r_{\langle\xi\rangle}(\langle\xi\rangle, t) &= M' r(\langle\xi\rangle, t)\langle\xi\rangle^{-1} - r(\langle\xi\rangle, t)\frac{L_1\frac{\lambda(t)^2}{\Lambda(t)}}{2\varrho(\xi, t)^2} \\ &\ge M' r(\langle\xi\rangle, t)\langle\xi\rangle^{-1} - \frac{L_1}{2} r(\langle\xi\rangle, t)\langle\xi\rangle^{-1} > 0, \end{aligned}$$

if $M > K + (d_1 + 1/2)L_1 + d_1L_2$. □

From these lemmata we immediately get:

Theorem 6.5.12 (Algebra). *Let $M > \max(K + d_1L_2 + n/2, K + (d_1 + 1/2)L_1 + d_1L_2)$, then $B_{L_1L_2MKK}$ is an algebra and it holds*

$$\|uv\|_{B_{L_1L_2MKK}} \le C_{alg}\|u\|_{B_{L_1L_2MKK}}\|v\|_{B_{L_1L_2MKK}}$$

for all functions u, v from $B_{L_1L_2MKK}$.

Corollary 6.5.13 (Compositions). *Let the assumptions of the previous theorem be satisfied and let $f(u) = \sum_{j=1}^\infty f_j u^j$ be an entire analytic function with $f(0) = 0$. Then f maps bounded sets from $B_{L_1L_2MKK}$ into bounded sets from $B_{L_1L_2MKK}$ and it holds*

$$\|f(u)\|_{B_{L_1L_2MKK}} \le C(\|u\|_{B_{L_1L_2MKK}})\|u\|_{B_{L_1L_2MKK}}.$$

6.6 Existence of Solutions and Regularity

In Section 6.4 we have proved:

Theorem 6.6.1 (Linear case). *Let v be a solution of*

$$\begin{aligned} &Lv = f(x,t), \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \\ &L = \partial_{tt} + \sum_{j=1}^{n} c_j(t)\lambda(t)\partial_{x_j t} - \sum_{i,j=1}^{n} a_{ij}(t)\lambda(t)^2\partial_{x_i x_j} \\ &\quad + \sum_{j=1}^{n} b_j(t)\lambda'(t)\partial_{x_j} + c_0(t)\partial_t. \end{aligned}$$

Let $\varphi \in C_{MK}$, $\psi \in C_{M(K+1)}$ and $f \in B_{MKK}$, see the Definitions 6.4.1 and 6.4.2.
Then $Hv \in B_{MKK}$, $v_t \in B_{M(K+1)K}$, $v \in B_{01(M+1)KK}$ and

$$\begin{aligned} &\|Hv\|_{B_{MKK}} + \|v_t\|_{B_{M(K+1)K}} + \|v\|_{B_{01(M+1)KK}} \\ &\quad \le C_{apr}(T\|f\|_{B_{MKK}} + \|\varphi\|_{C_{MK}} + \|\psi\|_{C_{M(K+1)}}). \end{aligned}$$

For the definition of the first order operator H see Proposition 6.3.3. The spaces are sharp in the cases of the Examples 6.2.2 and 6.2.3.

The following theorem is devoted to the semilinear case.

Theorem 6.6.2 (Semilinear case). *Let $f(u)$ be an entire analytic function with $f(0) = 0$ and let L, φ, ψ, H be as in the previous theorem. If $T > 0$ is small enough and M is sufficiently large, then a local solution u of*

$$Lu = f(u), \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

exists. This solution and its first derivatives lie in the same spaces as the solution v of the linear problem

$$\begin{aligned} &Lv = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x) : \\ &Hu \in B_{MKK}, \quad u_t \in B_{M(K+1)K}, \quad u \in B_{01(M+1)KK}. \end{aligned}$$

Remark 6.6.3. *It is possible to prove the same result, if the right–hand side $f(u)$ is replaced by $f(u, Hu)$.*

Proof. In the Banach space $B := B_{MKK} \times B_{M(K+1)K}$ we choose the closed set

$$\begin{aligned} \mathcal{M}_D = &\{(u_1, u_2) : u_1(x,0) = \varphi(x), \quad u_2(x,0) = \psi(x), \\ &\|u_1\|_{B_{MKK}} + \|u_2\|_{B_{M(K+1)K}} \le D\}. \end{aligned}$$

If the constant D is large enough, then $\mathcal{M}_D$ is not empty. Then we consider the mapping $\mathcal{T}: \mathcal{M}_D \to B$, $\mathcal{T}: (v_1, v_2) \mapsto (u_1, u_2) = (Hu, u_t)$ with

$$Lu = f(v), \quad v = H(D_x, t)^{-1} v_1, \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

From $\varrho \geq 1$ and $\lambda(t)\langle\xi\rangle \geq \lambda(t_\xi)\langle\xi\rangle \geq \sqrt{N}\varrho(\xi, t_\xi)/C \geq C'$ we deduce that $0 < H(\xi,t)^{-1} \leq C''$, which results in $v \in B_{MKK}$, hence $f(v) \in B_{MKK}$ and $\|f(v)\|_{B_{MKK}} \leq C(D)\|v\|_{B_{MKK}}$. The estimate from Theorem 6.6.1 implies $(u_1, u_2) \in \mathcal{M}_D$ if T is small enough. Hence, $\mathcal{T}$ maps $\mathcal{M}_D$ into itself. If $V, V' \in \mathcal{M}_D$, $V = (v_1, v_2)$, $V' = (v_1', v_2')$ and $v = H^{-1}v_1$, $v' = H^{-1}v_1'$, then

$$\|f(v) - f(v')\|_{B_{MKK}} \leq C(D)' \|v - v'\|_{B_{MKK}} \leq C(D)'' \|V - V'\|_B,$$

since B_{MKK} is an algebra and f is an entire analytic function. If $\mathcal{T}V = (u_1, u_2)$ and $\mathcal{T}V' = (u_1', u_2')$, then Theorem 6.6.1 implies

$$\|u_1 - u_1'\|_{B_{MKK}} + \|u_2 - u_2'\|_{B_{M(K+1)K}} \leq C_{apr} T C(D)'' \|V - V'\|_B.$$

If T is sufficiently small, then the mapping $\mathcal{T}$ is contractive. The fixed point theorem of Banach gives the assertion. □

Finally, let us study the difference $u - v$. It satisfies

$$L(u - v) = f(u), \quad (u - v)(x, 0) = 0, \quad (u - v)_t(x, 0) = 0.$$

Theorem 6.6.4. *Under the assumptions of Theorem 6.6.2 it holds*

$$H(u - v) \in B_{M(K-1)(K-1)}, \quad (u - v)_t \in B_{M(K-1)(K-1)},$$
$$u - v \in B_{01(M+1)(K-1)(K-1)}.$$

Proof. Corollary 6.4.5 gives $u \in B_{01(M+1)KK}$. Since this space is an algebra, we have $f(u) \in B_{01(M+1)KK}$. Similar to the proof of Theorem 6.4.3, we estimate $\|H(u-v)\|_{B_{M(K-1)(K-1)}}$ and $\|(u-v)_t\|_{B_{M(K-1)(K-1)}}$. In the pseudodifferential zone it holds

$$\begin{aligned} |\varrho(\xi,t)(\hat{u} - \hat{v})(\xi,t)| &\leq C \int_0^t \varrho(\xi,t)(t-s)|(f \circ u)\hat{\ }(\xi,s)|\, ds \\ &\leq C\varrho(\xi,t) t_\xi \sqrt{t} \left(\int_0^t |(f\circ u)\hat{\ }(\xi,s)|^2\, ds \right)^{1/2}. \end{aligned}$$

From (6.3.10) it can be concluded that

$$\begin{aligned} &\left\| \varrho(\xi,t)(\hat{u} - \hat{v})(\xi,t)\vartheta_{M(K-1)(K-1)} \right\|^2_{L^2(|\xi| \leq R_0(t))} \\ &\leq Ct \int_0^t \int_{|\xi|\leq R_0(t)} |(f\circ u)\hat{\ }(\xi,s)|^2 J(t_\xi, t_0)^2 \lambda(t_\xi)^2 \langle\xi\rangle^2 \langle\xi\rangle^{2M} t_\xi^{2K}\, d\xi\, ds \\ &\leq Ct^2 \|f(u)\|^2_{B_{01(M+1)KK}}. \end{aligned}$$

The derivative $D_t(u-v)$ fulfils the estimate

$$\begin{aligned}
|D_t(\hat{u}-\hat{v})(\xi,t)| &\le C\int_0^t (1+(\lambda(t)-\lambda(s))(t-s)\langle\xi\rangle)|(f\circ u)\hat{}(\xi,s)|\,ds\\
&\le C\sqrt{t}\left(\int_0^t |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2}\\
&\quad + C\varrho(\xi,t)t\sqrt{t}\left(\int_0^t |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2}
\end{aligned}$$

in the pseudodifferential zone. From (6.3.10) we obtain

$$\begin{aligned}
&\left\|D_t(\hat{u}-\hat{v})\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\le R_0(t))} \le C\|f\|^2_{B_{MKK}}\\
&\quad + Ct\int_0^t\int_{|\xi|\le R_0(t)} |(f\circ u)\hat{}(\xi,s)|^2 t^2\lambda(t_\xi)^2\langle\xi\rangle^2 J(t_\xi,t_0)^2\langle\xi\rangle^{2M} t_\xi^{2K-2}\,d\xi\,ds\\
&\le C\|f\|^2_{B_{MKK}} + Ct^2\|f\|_{B_{01(M+1)KK}}.
\end{aligned}$$

In the hyperbolic zone we have

$$\begin{aligned}
&|\lambda(t)\xi(\hat{u}-\hat{v})(\xi,t)| + |(\hat{u}-\hat{v})_t(\xi,t)|\\
&\le C\int_{t_\xi}^t J(s,t)|(f\circ u)\hat{}(\xi,s)|\,ds\\
&\quad + CJ(t_\xi,t)\int_0^{t_\xi}(1+\varrho(\xi,t_\xi)(t_\xi-s))|(f\circ u)\hat{}(\xi,s)|\,ds\\
&\le C\sqrt{t}\left(\int_{t_\xi}^t |(f\circ u)\hat{}(\xi,s)|^2 J(s,t)^2\,ds\right)^{1/2}\\
&\quad + CJ(t_\xi,t)\sqrt{t_\xi}\left(\int_0^{t_\xi} |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2}\\
&\quad + CJ(t_\xi,t)\varrho(\xi,t_\xi)t_\xi\sqrt{t_\xi}\left(\int_0^{t_\xi} |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2}.
\end{aligned}$$

Making use of

$$1\le N=\Lambda(t_\xi)\langle\xi\rangle\le\lambda(t_\xi)t_\xi\langle\xi\rangle\le\sqrt{N}\varrho(\xi,t_\xi)t_\xi$$

we can drop the second term on the right. Then it follows that

$$\begin{aligned}
&\left\|\lambda(t)\xi(\hat{u}-\hat{v})\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\ge R_0(t))}\\
&\quad + \left\|(\hat{u}-\hat{v})_t\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\ge R_0(t))}
\end{aligned}$$

$$\begin{aligned}
&\leq Ct \int_{|\xi|\geq R_0(t)} \int_{t_\xi}^{t} |(f\circ u)\hat{}(\xi,s)|^2 \frac{\lambda(s)^2 J(s,t)^2 \langle\xi\rangle^{2(M+1)} t_\xi^{2K}}{\lambda(s)^2\langle\xi\rangle^2 t_\xi^2}\, ds\, d\xi \\
&\quad + Ct \int_{|\xi|\geq R_0(t)} \int_0^{t_\xi} |(f\circ u)\hat{}(\xi,s)|^2 \lambda(t_\xi)^2 J(t_\xi,t)^2 \langle\xi\rangle^{2(M+1)} t_\xi^{2K}\, ds\, d\xi \\
&\leq Ct \int_0^t \int_{|\xi|\geq R_0(t)} |(f\circ u)\hat{}(\xi,s)|^2 \vartheta_{01(M+1)KK}(\xi,s)^2\, d\xi\, ds \\
&\leq Ct^2 \|f\|^2_{B_{01(M+1)KK}},
\end{aligned}$$

since $\lambda(s)\langle\xi\rangle t_\xi \geq \Lambda(s)\langle\xi\rangle t_\xi/s \geq N$ in $Z_{hyp}(N)$. Using the ideas from the proof of Corollary 6.4.5 we deduce that $u - v \in B_{01(M+1)(K-1)(K-1)}$. □

6.7 An Example

Let us illustrate the results of this chapter by an example. In [DR98b], Section 6, Example 3, the Example of Qi Min-You has been extended to a Cauchy problem of the type

$$\begin{aligned}
&Lv = v_{tt} + ct^l v_{xt} - at^{2l} v_{xx} - blt^{l-1} v_x = 0, \quad l \in \mathbb{N}, \quad l \geq 2, \\
&v(x,0) = \varphi(x), \quad v_t(x,0) = 0.
\end{aligned}$$

The ansatz $v(x,t) = \sum_{k=0}^m C_k t^{(l+1)k} \partial_x^k \varphi(x + \mu t^{l+1})$ leads to

$$m = \frac{-l(l+1)\mu + bl}{2(l+1)^2\mu + (l+1)c}, \quad \mu_{1,2} = -\frac{1}{2(l+1)}\left(c \mp \sqrt{c^2+4a}\right).$$

This gives

$$m_{1,2} = \frac{l}{2(l+1)}\left(-1 \pm \frac{2b+c}{\sqrt{c^2+4a}}\right).$$

Let us assume that the constants l, a, b and c are chosen in such a way that either m_1 or m_2 is a positive integer. It is not possible that both numbers m_1 and m_2 are positive integers, because $m_1 + m_2 = -l/(l+1)$ and $-1 < -l/(l+1) < 0$. Under this assumption, singularities of the datum φ propagate along one characteristic only and the loss of Sobolev regularity is given by $\max(m_1, m_2)$; that is, $\varphi \in H^s(\mathbb{R})$ implies $v \in C(\mathbb{R}, H^{s-\max(m_1,m_2)}(\mathbb{R}))$ and these are the sharp spaces.

Let us now apply the general theory developed in this chapter. We have

$$
\begin{aligned}
&b(\xi,t) = b\frac{\xi}{|\xi|}, \quad c(\xi,t) = \frac{1}{2}c\frac{\xi}{|\xi|}, \quad a(\xi,t) = a,\\
&J(s,t) = \exp\left(\int_s^t \sup_\zeta \frac{\lambda'(\tau)}{2\lambda(\tau)}\left|1 \pm \frac{b(\zeta,\tau)+c(\zeta,\tau)}{\sqrt{c(\zeta,\tau)^2+a(\zeta,\tau)}}\right| d\tau\right)\\
&\qquad = \exp\left(\int_s^t \frac{\lambda'(\tau)}{2\lambda(\tau)}\left(1+\frac{|2b+c|}{\sqrt{c^2+4a}}\right)d\tau\right) = \left(\frac{\lambda(t)}{\lambda(s)}\right)^{\frac{1}{2}+\frac{|2b+c|}{2\sqrt{c^2+4a}}},\\
&t_\xi = O\left(\langle\xi\rangle^{-\frac{1}{l+1}}\right),\\
&\lambda(t_\xi) = O\left(\langle\xi\rangle^{-\frac{l}{l+1}}\right).
\end{aligned}
$$

This implies for the weight $\vartheta_{MKK}(\xi,t)$:

$$
\begin{aligned}
\vartheta_{MKK}(\xi,0) &= \varrho(\xi,t_\xi)J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^K\\
&= O(\lambda(t_\xi)\langle\xi\rangle)O(\lambda(t_\xi))^{-\frac{1}{2}-\frac{|2b+c|}{2\sqrt{c^2+4a}}}\langle\xi\rangle^M O(\langle\xi\rangle^{-\frac{K}{l+1}})\\
&= O\left(\langle\xi\rangle^{M+1+\frac{l}{2(l+1)}\left(-1+\frac{|2b+c|}{\sqrt{c^2+4a}}\right)-\frac{K}{l+1}}\right),\\
\vartheta_{MKK}(\xi,t) &= J(t,t_0)\langle\xi\rangle^M t_\xi^K = O(\langle\xi\rangle^{M-\frac{K}{l+1}})
\end{aligned}
$$

if $t > 0$ is fixed and $\langle\xi\rangle$ is large. We know that $\|Hv\|_{B_{MKK}} \le C\|\varphi\|_{C_{MK}}$, where H is a pseudodifferential operator that behaves like $\lambda(t)\partial_x$ if $t > 0$ is fixed. Then the theory presented in this chapter says that the loss of Sobolev regularity is

$$
\frac{l}{2(l+1)}\left(-1+\frac{|2b+c|}{\sqrt{c^2+4a}}\right).
$$

But this value is exactly $\max(m_1, m_2)$. In other words, the results of this chapter are sharp in the case of this linear model problem.
However, our theory says more, namely that the solution u of the semilinear problem

$$
\begin{aligned}
&Lu = f(u) = \sum_{j=1}^{\infty} f_j u^j,\\
&u(x,0) = \varphi(x), \quad u_t(x,0) = 0
\end{aligned}
$$

has the *same regularity* as v, and that the difference $u - v$ has *higher regularity* than u and v. The difference in the regularities is described by

$t_\xi^{-1} = O(\langle\xi\rangle^{1/(l+1)})$, cf. Theorems 6.6.2 and 6.6.4. Summarising, we have

$$\begin{aligned} &u, v \in C([0,T], H^{s-\max(m_1,m_2)}(\mathbb{R})), \\ &u - v \in C([0,T], H^{s-\max(m_1,m_2)+1/(l+1)}(\mathbb{R})) \end{aligned}$$

if $T > 0$ is sufficiently small and s is sufficiently large.
This allows to draw some conclusions about the propagation of singularities. Let us assume $\varphi \in H^s(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R}\setminus\{x_0\})$. There is no loss of generality if we assume $m_1 > m_2$. From the explicit representation of $v(x,t)$ we know that the singularity of φ at the point x_0 propagates along the characteristic

$$\mathcal{C} = \{(x,t) : x + \mu_1 t^{l+1} = x_0\}.$$

The function v is smooth in the complement set of this characteristic. From the above statements we get that

$$\emptyset \neq \text{sing–supp}_{H^{s-m_1+\varepsilon}}(v(.,t)) = \text{sing–supp}_{H^{s-m_1+\varepsilon}}(u(.,t)),$$

if $0 \le t \le T$ and $0 < \varepsilon \le 1/(l+1)$. In other words, u has H^{s-m_1} singularities on $\mathcal{C}$. The function u may have singularities away from $\mathcal{C}$, but these are weaker, at least of order $1/(l+1)$. The strongest singularities of u coincide with the singularities of v.

Remark 6.7.1. *The results of this chapter tell us that the strongest singularities of solutions to* semilinear *equations propagate in the same way as the singularities of solutions to* linear *equations. The linear case has been studied, e.g., in [TT80] and [Ale84].*

Appendix A

The Spherical Harmonics

For the study of the properties of classical pseudo–differential operators with symbols of limited smoothness it is very useful to expand the homogeneous components of the symbols into a series

$$a_j(x,\xi) = \langle\xi\rangle^j \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} a_{jlm}(x) Y_{lm}(\xi),$$

where $Y_{lm}(\xi)$ are the spherical harmonics and $(a_{jlm}(x))_l$, in some sense, is a rapidly decreasing sequence in l. In this section we will give a precise description of the spherical harmonic functions and such expansions. We restrict us to the case of at least 3 space dimensions. The spherical harmonics on the unit circle are the well known functions $\sin(l\xi)$, $\cos(l\xi)$, $l = 0, 1, \ldots$. The unit sphere in the space $\mathbb{R}^1$ consists of two points. These two cases are trivial.

The following results, definitions and notations are taken from [EMO53], Volume II.

Let $x = (x_1, \ldots, x_{p+2})$ be a point in the Euclidean space $\mathbb{R}^{p+2}$, $p \geq 1$. A polynomial $H_n(x)$ of degree n in $x_1, \ldots, x_{p+2}$ is called a *harmonic polynomial of degree* n, if it satisfies the Laplace equation in $\mathbb{R}^{p+2}$,

$$\triangle H_n(x) = 0 \quad \forall x \in \mathbb{R}^{p+2},$$

and is homogeneous of degree n,

$$H_n(\lambda x) = \lambda^n H_n(x) \quad \forall \lambda \in \mathbb{R}, \quad \forall x \in \mathbb{R}^{p+2}.$$

There are

$$h(n,p) = (2n+p)\frac{(n+p-1)!}{p!n!} = O(\langle n\rangle^p)$$

linearly independent homogeneous polynomials of degree n, see Section 11.2 in [EMO53], Volume II. They have the following form (Theorem 1 in Section 11.2 of [EMO53]):
Let $m_0, \ldots, m_p$ be integers with

$$n = m_0 \geq m_1 \geq \cdots \geq m_{p-1} \geq |m_p|,$$

and let r_k be defined by

$$r_k = \left(x_{k+1}^2 + \cdots + x_{p+2}^2\right)^{1/2}, \quad k = 0, \ldots, p, \quad r_0 = r.$$

Then

$$\begin{aligned} H(m_k, x) &:= h(n, m_1, \ldots, m_p, x) = \\ &= \left(\frac{x_{p+1}}{r_p} + i\frac{x_{p+2}}{r_p}\right)^{m_p} r_p^{m_p} \prod_{k=0}^{p-1} r_k^{m_k - m_{k+1}} C_{m_k - m_{k+1}}^{m_{k+1}+(p-k)/2} \left(\frac{x_{k+1}}{r_k}\right) \end{aligned}$$

form a complete set of $h(n,p)$ linearly independent harmonic polynomials of degree n. The functions $C_{m_k - m_{k+1}}^{m_{k+1}+(p-k)/2}$ are the Gegenbauer polynomials (cf. Section 3.5 of [EMO53], Volume I) and can be defined using hypergeometric functions ${}_2\mathrm{F}_1(a, b; c; z)$:

$$\begin{aligned} n! C_n^{\lambda}(x) &= (2\lambda)_n \, {}_2\mathrm{F}_1\left(-n, n + 2\lambda; \lambda + \frac{1}{2}; \frac{1-x}{2}\right), \\ {}_2\mathrm{F}_1(a, b; c; z) &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \\ (a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)}. \end{aligned}$$

Their restrictions on the unit sphere $\{|x| = 1\}$ form a *complete* set of *orthogonal* functions. In other words, they are an orthogonal basis of $L^2(\{|x| = 1\})$. These functions are called the spherical harmonics $Y(m_k, \theta, \varphi)$,

$$Y(m_k, \theta, \varphi) := r^{-n} H(m_k, x),$$

where $(r, \theta, \varphi) = (r, \theta_1, \ldots, \theta_p, \varphi)$ are the polar coordinates in $\mathbb{R}^{p+2}$:

$$\begin{aligned} x_1 &= r \cos\theta_1, \\ x_2 &= r \sin\theta_1 \cos\theta_2, \\ x_3 &= r \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ &\ldots, \end{aligned}$$

$$
\begin{aligned}
x_p &= r \sin\theta_1 \sin\theta_2 \dots \sin\theta_{p-1} \cos\theta_p, \\
x_{p+1} &= r \sin\theta_1 \sin\theta_2 \dots \sin\theta_{p-1} \sin\theta_p \cos\varphi, \\
x_{p+2} &= r \sin\theta_1 \sin\theta_2 \dots \sin\theta_{p-1} \sin\theta_p \sin\varphi, \\
&\quad 0 \le r < \infty, \quad 0 \le \theta_1, \dots, \theta_p \le \pi, \quad 0 \le \varphi \le 2\pi.
\end{aligned}
$$

The Laplace operator becomes (see [EMO53], Section 11.1)

$$
\begin{aligned}
\triangle &= \triangle_r + \frac{1}{r^2} \triangle_S \\
&= r^{-p-1} \frac{\partial}{\partial r} \left(r^{p+1} \frac{\partial}{\partial r} \right) \\
&\quad + \frac{1}{r^2} \frac{1}{(\sin\theta_1)^p} \frac{\partial}{\partial\theta_1} \left((\sin\theta_1)^p \frac{\partial}{\partial\theta_1} \right) \\
&\quad + \frac{1}{r^2} \frac{1}{(\sin\theta_1)^2} \frac{1}{(\sin\theta_2)^{p-1}} \frac{\partial}{\partial\theta_2} \left((\sin\theta_2)^{p-1} \frac{\partial}{\partial\theta_2} \right) \\
&\quad + \frac{1}{r^2} \frac{1}{(\sin\theta_1 \sin\theta_2)^2} \frac{1}{(\sin\theta_3)^{p-2}} \frac{\partial}{\partial\theta_3} \left((\sin\theta_3)^{p-2} \frac{\partial}{\partial\theta_3} \right) \\
&\quad \dots \\
&\quad + \frac{1}{r^2} \frac{1}{(\sin\theta_1 \sin\theta_2 \dots \sin\theta_{p-1})^2} \frac{1}{(\sin\theta_p)^1} \frac{\partial}{\partial\theta_p} \left((\sin\theta_p)^1 \frac{\partial}{\partial\theta_p} \right) \\
&\quad + \frac{1}{r^2} \frac{1}{(\sin\theta_1 \sin\theta_2 \dots \sin\theta_p)^2} \frac{\partial^2}{\partial\varphi^2}.
\end{aligned}
$$

The operator $\triangle_r$ differentiates with respect to r only and the operator $\triangle_S$ operates on the sphere.
The following proposition states that the spherical harmonics $Y(m_k, \theta, \varphi)$ are not only an L^2–basis on the sphere, they are even eigenfunctions of the spherical Laplacian $\triangle_S$.

Proposition A.0.1. *Let $Y(m_k, \theta, \varphi)$ be a spherical harmonic function of degree n. Then,*

$$
\triangle_S Y(m_k, \theta, \varphi) = -n(n+p) Y(m_k, \theta, \varphi).
$$

Proof. The function $H(m_k, x) = r^n Y(m_k, \theta, \varphi)$ is a harmonic polynomial of degree n, $\triangle H(m_k, x) = 0$. Hence we obtain

$$
\begin{aligned}
0 = \triangle H = \triangle(r^n Y) &= (\triangle_r r^n) Y + r^{n-2} \triangle_S Y \\
&= r^{n-2} (n(n+p) + \triangle_S) Y.
\end{aligned}
$$

□

We come back to our style of notation:

$$\begin{aligned}
&p+2 \longmapsto n,\\
&(n, m_1, \ldots, m_p) \longmapsto (l, m), \quad 1 \le m \le h(l, n-2) = O(\langle l\rangle^{n-2}),\\
&(x_1, \ldots, x_{p+2}) \longmapsto \xi \in \mathbb{R}^n,\\
&(\theta_1, \ldots, \theta_p, \varphi) \longmapsto \frac{\xi}{|\xi|} \in S^{n-1},\\
&Y(m_k, \theta, \varphi) \longmapsto Y_{lm}(\xi) \text{ or } Y_{lm}(\theta, \varphi).
\end{aligned}$$

Remark A.0.2. *We should write $Y_{lm}(\xi/|\xi|)$, but we want to keep the notation simple. The reader should keep in mind that the argument of Y_{lm} is normalised without further notice.*

We rewrite the results in the new notation:

$$\begin{aligned}
&\{Y_{lm}(\xi)\}_{l=0,\ldots,\infty,\ m=1,\ldots,h(l,n-2)} \text{ form an orthogonal basis of } L^2(S^{n-1}),\\
&h(l, n-2) = O(\langle l\rangle^{n-2}),\\
&Y_{lm}(\xi) \in C^\infty(S^{n-1}) \quad \forall l, m,\\
&-\triangle_S Y_{lm}(\xi) = l(l+n-2) Y_{lm}(\xi).
\end{aligned}$$

There is no loss of generality in assuming that $\|Y_{lm}\|_{L^2(S^{n-1})} = 1$ for all l, m. From [Mic78], Section 14.6, we take the estimate of the L^∞ norm:

$$\|Y_{lm}(\xi)\|_{L^\infty(S^{n-1})} \le C_n \langle l\rangle^{\frac{n}{2}-1}. \tag{A.0.1}$$

Now we are in a position to formulate the main result of this section.

Theorem A.0.3. *Let $a(x, D) \in OPX^s S^j_{cl}(M)$ with $X^s = C^s_b$ or $X^s = H^s$ be a positive homogeneous symbol of order j for $|\xi| \ge 1/2$. Then there is a constant C_0 with*

$$a(x, \xi) = \langle\xi\rangle^j \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} a_{lm}(x) Y_{lm}(\xi)$$

for all $|\xi| \ge C_0$ and

$$\|a_{lm}(x)\|_{X^s} \le C(n, k) \langle l\rangle^{-2k} \sup \left\{ \left\| D^\beta_\xi a(., \xi) \right\|_{X^s} : |\beta| \le 2k, |\xi| = 1 \right\}.$$

for all k, l, m. Additionally, it holds

$$\|Y_{lm}(D)\|_{H^t \to H^t} \le C \langle l\rangle^{n/2-1} \quad \forall t \in \mathbb{R}.$$

Proof. The notation $a(x,\xi) \in X^s S^j_{cl}(M)$ for homogeneous $a(x,\xi)$ means

$$
\begin{aligned}
&a(x,\lambda\xi) = \lambda^j a(x,\xi), \quad |\xi| \geq \frac{1}{2}, \quad \lambda \geq 1,\\
&\left\| D_\xi^\alpha a(.,\xi) \right\|_{X^s} \leq C_\alpha \langle\xi\rangle^{j-|\alpha|}.
\end{aligned}
$$

If $|\xi| \geq C_0$ and C_0 is sufficiently large, then

$$
a(x,\xi) = |\xi|^j a\left(x, \frac{\xi}{|\xi|}\right) = \langle\xi\rangle^j a\left(x, \frac{\xi}{|\xi|}\right).
$$

For each fixed x from M, $a\left(x, \frac{\xi}{|\xi|}\right)$ can be regarded as a function defined on the unit sphere S^{n-1}. Hence it can be written as

$$
\begin{aligned}
a\left(x, \frac{\xi}{|\xi|}\right) &= \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,n-2)} a_{lm}(x) Y_{lm}(\xi),\\
a_{lm}(x) &= \int_{S^{n-1}} a(x,\xi) Y_{lm}(\xi)\, dS_\xi.
\end{aligned}
$$

In the following step we derive the decay properties of the sequence $(a_{lm}(x))_{l,m}$. Let us consider the case of $X^s = C_b^s$ first. For simplicity we assume $s \in \mathbb{N}_0$. It holds (for $|\gamma| \leq s$)

$$
\begin{aligned}
D_x^\gamma a_{lm}(x) &= \int_{S^{n-1}} D_x^\gamma a(x,\xi) Y_{lm}(\xi)\, d\xi\\
&= \int_{S^{n-1}} D_x^\gamma a(x,\xi) \left(\frac{-\triangle_{S,\xi} + 1}{l(l+n-2)+1} \right)^k Y_{lm}(\xi)\, d\xi\\
&= (l(l+n-2)+1)^{-k} \int_{S^{n-1}} \left(D_x^\gamma \left(-\triangle_{S,\xi} + 1\right)^k a(x,\xi) \right) Y_{lm}(\xi)\, d\xi.
\end{aligned}
$$

Exploiting the Cauchy–Schwarz Inequality, we conclude that

$$
\begin{aligned}
&\|D_x^\gamma a_{lm}(x)\|_{L^\infty}\\
&\quad \leq C(n,k)\langle l\rangle^{-2k} \sup\{|D_x^\gamma(-\triangle_{S,\xi}+1)^k a(x,\xi)| : x \in M, |\xi| = 1\}.
\end{aligned}
$$

Second, we consider the case $X^s = H^s$. We have

$$
\begin{aligned}
\langle D_x\rangle^s a_{lm}(x) &= \int_{S^{n-1}} \langle D_x\rangle^s a(x,\xi) Y_{lm}(\xi)\, d\xi\\
&= \int_{S^{n-1}} \langle D_x\rangle^s a(x,\xi) \left(\frac{-\triangle_{S,\xi} + 1}{l(l+n-2)+1} \right)^k Y_{lm}(\xi)\, d\xi\\
&= (l(l+n-2)+1)^{-k} \int_{S^{n-1}} \left(\langle D_x\rangle^s \left(-\triangle_{S,\xi} + 1\right)^k a(x,\xi) \right) Y_{lm}(\xi)\, d\xi.
\end{aligned}
$$

Then it follows that

$$\begin{aligned}
&\|\langle D_x\rangle^s a_{lm}(x)\|^2_{L^2(M)} \\
&\quad \le C\langle l\rangle^{-4k} \int_M \left(\int_{S^{n-1}} \left(\langle D_x\rangle^s \left(-\triangle_{S,\xi} + 1\right)^k a(x,\xi) \right) Y_{lm}(\xi)\, d\xi \right)^2 dx \\
&\quad \le C\langle l\rangle^{-4k} \int_M \int_{S^{n-1}} \left(\langle D_x\rangle^s \left(-\triangle_{S,\xi} + 1\right)^k a(x,\xi) \right)^2 d\xi \, \|Y_{lm}\|^2_{L^2}\, dx \\
&\quad \le C\langle l\rangle^{-4k} \sup \left\{ \left\| \langle D_x\rangle^s (-\triangle_{S,\xi} + 1)^k a(x,\xi) \right\|_{L^2(M)} : |\xi| = 1 \right\}^2 .
\end{aligned}$$

It remains to estimate the suprema. If we express the spherical Laplacian $\triangle_S$ by means of the Laplacian in $\mathbb{R}^n_\xi$ and the radial Laplacian $\triangle_r$,

$$\triangle_S = r^2 \triangle - r^2 \frac{\partial^2}{\partial r^2} - (n-1) r \frac{\partial}{\partial r}, \quad |\xi| = r,$$

then we see that $(-\triangle_{S,\xi} + 1)^k a(x,\xi) \in X^s S^j_{cl}(M)$. It can be concluded that

$$\begin{aligned}
&\sup\{ |D^\gamma_x (-\triangle_{S,\xi} + 1)^k a(x,\xi)| : x \in M, |\xi| = 1 \} \\
&\quad \le C(n,k) \sup\{ |D^\gamma_x D^\beta_\xi a(x,\xi)| : x \in M, |\xi| = 1, \beta \le 2k \}, \\
&\sup \left\{ \left\| \langle D_x\rangle^s (-\triangle_{S,\xi} + 1)^k a(x,\xi) \right\|_{L^2(M)} : |\xi| = 1 \right\} \\
&\quad \le C(n,k) \sup \left\{ \left\| D^\beta_\xi a(x,\xi) \right\|_{H^s(M)} : |\xi| = 1, \beta \le 2k \right\} .
\end{aligned}$$

Taking into account all the above estimates we get

$$\|a_{lm}(x)\|_{X^s} \le C(n,k) \langle l\rangle^{-2k} \sup \left\{ \left\| D^\beta_\xi a(.,\xi) \right\|_{X^s} : |\xi| = 1, \beta \le 2k \right\}$$

for all $k \ge 0$. Finally, we have to study the mapping properties

$$Y_{lm}(D) : H^t(\mathbb{R}^n) \to H^t(\mathbb{R}^n), \quad t \in \mathbb{R}.$$

By the estimate (A.0.1) it follows that

$$\begin{aligned}
\|Y_{lm}(D)u\|_{H^t} &= \left\| \langle\xi\rangle^t (Y_{lm}(D)u)\hat{}(\xi) \right\|_{L^2} = \left\| \langle\xi\rangle^t Y_{lm}(\xi) \hat{u}(\xi) \right\|_{L^2} \\
&\le C_n \langle l\rangle^{n/2-1} \left\| \langle\xi\rangle^t \hat{u}(\xi) \right\|_{L^2} = C_n \langle l\rangle^{n/2-1} \|u\|_{H^t} .
\end{aligned}$$

The theorem is proved for $a(x,D) \in OPC^s S^j_{cl}(M)$ with integer $s \ge 0$ and $a(x,D) \in OPH^s S^j_{cl}(M)$ with arbitrary $s > n/2$. ∎

Appendix B

Miscellaneous

Here we provide some auxiliary results.

Proposition B.0.1. *Let $\Omega \subset \mathbb{R}^n_\xi$ be a bounded domain with smooth boundary. Then there is a basis $\{\beta_l(\xi)\}_{l=0}^{\infty}$ of $L^2(\Omega)$ with the property that every $a(x,\xi) \in C^m(M \times \Omega)$ can be decomposed in a series*

$$a(x,\xi) = \sum_{l=0}^{\infty} a_l(x)\beta_l(\xi).$$

Furthermore, the following statements hold:

$$a_l(x) = \int_\Omega a(x,\xi)\beta_l(\xi)\,d\xi,$$
$$\|\beta_l\|_{L^2} = 1,$$
$$\|\beta_l\|_{L^\infty} \le C\langle l\rangle,$$
$$\|a_l\|_{C^m} \le C_k \langle l\rangle^{1-\frac{2}{n}k} \sup\{\left\|D_\xi^\alpha a(.,\xi)\right\|_{C^m} : |\alpha| = 2k, \xi \in \Omega\} \quad \forall k.$$

Proof. We consider the operator $(-\triangle)|_\Omega$ with homogeneous Dirichlet conditions. This operator has eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots$ and eigenfunctions $\beta_1, \beta_2, \dots$ which form an orthonormal basis for $L^2(\Omega)$. It is well–known (see e.g. [ES91], Volume II, Part I, formula 8.23) that

$$\lambda_l \sim (2\pi)^2 \left(\frac{l}{\omega_n \operatorname{meas}\Omega}\right)^{2/n} (1 + O(l^{-1/n})), \quad l \to \infty.$$

First we estimate $\|\beta_l\|_{L^\infty}$. From $(-\triangle)^n\beta_l = \lambda_l^n\beta_l$ and the Sobolev Embedding Theorem it can be deduced that

$$c\,\|\beta_l\|_{L^\infty}^2 \le \|\beta_l\|_{H^n}^2 \le C\lambda_l^n \le C\langle l\rangle^2.$$

To estimate a_l, we take some $|\gamma| \le m$. Then

$$\begin{aligned}
|D_x^\gamma a_l(x)| &= \left| \int_\Omega D_x^\gamma a(x,\xi)\beta_l(\xi)\,d\xi \right| \\
&= \frac{1}{\lambda_l^k} \left| \int_\Omega D_x^\gamma a(x,\xi)(-\triangle_\xi)^k \beta_l(\xi)\,d\xi \right| \\
&\le \frac{1}{\lambda_l^k} \int_\Omega |D_x^\gamma (\triangle_\xi)^k a(x,\xi)||\beta_l(\xi)|\,d\xi \\
&\le C_k \langle l\rangle^{-\frac{2}{n}k} \sup\{ \left\| D_\xi^\alpha a(x,\xi) \right\|_{C^m} : |\alpha| = 2k, \xi \in \Omega\} \cdot \langle l\rangle.
\end{aligned}$$

The proof is complete. □

Proposition B.0.2. *Let the homogeneous differential operator $P(x,D)$ of order m be strictly hyperbolic at the point x_0 in the direction N, $|N| = 1$. By λ_{max} we denote the largest absolute value of the characteristic roots, i.e.,*

$$\lambda_{max} = \sup\{|\tau| : P(x_0, \tau N + \xi) = 0, |\xi| = 1, \xi \perp N\}.$$

Then P is strictly hyperbolic at x_0 in any direction $N + e$ with $N \perp e$, $|e|^{-1} > \lambda_{max}$.

Proof. We have to check two conditions:

$$\begin{aligned}
&P(x_0, N + e) \neq 0, \\
&P(x_0, \tau(N + e) + \xi) = 0, \quad |\xi| = 1, \quad \xi \perp N + e, \\
&\qquad\qquad \text{has } m \text{ real and distinct roots } \tau_1(\xi), \ldots, \tau_m(\xi).
\end{aligned}$$

The first condition holds obviously:

$$P(x_0, N + e) = |e|^m P\left(x_0, \frac{1}{|e|}N + \frac{e}{|e|}\right) \neq 0,$$

since $|e|^{-1} > \lambda_{max}$.
It remains to show the second condition: We consider the set of zeroes of $P(x_0, .)$:

$$Z = \{\tau N + \xi : P(x_0, \tau N + \xi) = 0, \xi \perp N\}.$$

It consists of m parts,

$$\begin{aligned}
&Z = Z_1 \cup \cdots \cup Z_m, \quad Z_i \cap Z_j = \{0\} \quad (i \neq j), \qquad \text{(B.0.1)} \\
&Z_j = \{\tau_j(\xi)N + \xi : P(x_0, \tau_j(\xi)N + \xi) = 0, \xi \perp N\}, \quad \tau_j \text{ continuous.}
\end{aligned}$$

The functions $\tau_j(\xi)$ are homogeneous functions of order 1,

$$\tau_j(\xi) = |\xi|\tau_j\left(\frac{\xi}{|\xi|}\right), \quad \xi \perp N, \quad \xi \neq 0.$$

The choice of λ_{max} gives $|\tau_j(\xi)| \leq \lambda_{max}|\xi|$. We fix a vector ξ, $|\xi| = 1$, $\xi \perp N+e$. It will be shown that the line

$$G = \{\varrho(N+e) + \xi : \varrho \in \mathbb{R}\}$$

intersects each set Z_j. The vector ξ can be decomposed as $\xi = \xi_\perp + \langle \xi, N\rangle N$ with $\langle \xi_\perp, N\rangle = 0$. Then we can write

$$\varrho(N+e) + \xi = (\varrho + \langle \xi, N\rangle)N + (\varrho e + \xi_\perp).$$

For large $|\varrho|$ we have

$$\frac{\varrho^2 + 2\varrho\langle \xi, N\rangle + \langle \xi, N\rangle^2}{\varrho^2|e|^2 + 2\varrho\langle e, \xi_\perp\rangle + |\xi_\perp|^2} > \lambda_{max}^2,$$

since $|e|^{-2} > \lambda_{max}^2$. This gives

$$\frac{|\varrho + \langle \xi, N\rangle|}{|\varrho e + \xi_\perp|} > \lambda_{max}$$

for large $|\varrho|$. It follows that the point

$$\varrho(N+e) + \xi = (\varrho + \langle \xi, N\rangle)N + (\varrho e + \xi_\perp)$$

lies "above" the set

$$\{\tau N + \eta : \tau = \lambda_{max}|\eta|, \eta \perp N\}$$

for $\varrho \gg 1$ and "below" the set

$$\{\tau N + \eta : \tau = -\lambda_{max}|\eta|, \eta \perp N\}$$

for $\varrho \ll -1$. Hence, the line G intersects each Z_j. We get m different intersection points, since (B.0.1) and $0 \notin G$. $\square$

The next lemma allows to extend a function $a(x,p)$ which is defined in $Q \times G$ (Q is a rectangular parallelepipedon in $\mathbb{R}^n$, $G \subset \mathbb{R}^r$ is a domain for the parameters $p = (p_1, \ldots, p_n)$) to a function $a_\varepsilon(x,p)$ which can be regarded as being periodic with respect to x. The functions $a(x,p)$ and $a_\varepsilon(x,p)$ coincide in the interior of Q if the point x is not near the boundary. The function a_ε has the same smoothness with respect to x and p as the function a. We will apply this lemma to the coefficients $a_{j,\alpha}(x,t)$ and $a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta} D_t^k u\})$ of weakly hyperbolic equations. The parameters p in these applications are t and the weighted derivatives of u.

Lemma B.0.3. *Let* $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ *be a rectangular parallelepipedon (RP for short) and* $a(x, p)$ *be a function defined in* $Q \times G$ *with* $G \subset \mathbb{R}^r$ *being a compact domain for the parameters* $p = (p_1, \ldots, p_r)$*;* a *is supposed to be continuous with respect to* (x, p)*. Let* Q' *be the RP* $Q' = [a_1, 2b_1 - a_1] \times \cdots \times [a_n, 2b_n - a_n]$*. Then for every* $\varepsilon > 0$ *a positive* δ *and a function* $a_\varepsilon(x, p)$ *(defined in* $Q' \times G$*) exist with*

$$a_\varepsilon(x,p) = a(x,p) \text{ if } x \in Q \text{ and } \operatorname{dist}(x, \partial Q) \geq \delta, \tag{B.0.2}$$

$$|a_\varepsilon(x,p) - a(x,p)| \leq \varepsilon \text{ if } x \in Q \text{ and } \operatorname{dist}(x, \partial Q) \leq \delta, \tag{B.0.3}$$

$$\frac{\partial}{\partial n} a_\varepsilon(x,p) = 0, \quad \operatorname{dist}(x, \partial Q') < \delta/2, \tag{B.0.4}$$

(n is the direction of the shortest connection from x to $\partial Q'$)

$$a_\varepsilon(.,p) \in C^\infty(Q' \cap \{\operatorname{dist}(x, \partial Q') < \delta/2\}) \text{ for each fixed } p, \tag{B.0.5}$$

$$\begin{aligned} &a_\varepsilon(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \\ &\quad = a_\varepsilon(x_1, \ldots, x_{i-1}, 2b_i - a_i, x_{i+1}, \ldots, x_n) \\ &\quad \forall i = 1, \ldots, n, \quad x_j \in (a_j, 2b_j - a_j). \end{aligned} \tag{B.0.6}$$

We write $Q' = \cup_{i=1}^{2^n} Q_i$ *with congruent RPs* Q_i *intersecting each other only at their boundaries. Then for each* i*,* $i = 1, \ldots, 2^n$*, an isometric bijective mapping* $\varphi_i : Q \to Q_i$ *exists with*

$$a_\varepsilon(\varphi_i x, p) = a_\varepsilon(x, p) \quad \forall (x,p) \in Q \times G.$$

Proof. At first we construct a function $a_\varepsilon(x, p)$ which is defined in $Q \times G$ and satisfies (B.0.2), (B.0.3) and (B.0.4), (B.0.5) with Q' replaced by Q. Then we will extend this function by means of reflections to $Q' \times G$, which are the above mentioned bijective mappings.
For $0 < \delta < (b_i - a_i)/2$ we define functions $\chi_i^+(s)$, $\chi_i^-(s) \in C^\infty(\mathbb{R})$ with

$$\chi_i^+(s) = \begin{cases} 0 & : s \leq b_i - \delta, \\ 1 & : s \geq b_i - \delta/2, \end{cases}$$

$$\chi_i^-(s) = \begin{cases} 0 & : s \geq a_i + \delta, \\ 1 & : s \leq a_i + \delta/2, \end{cases}$$

$$0 \leq \chi_i^+(s), \chi_i^-(s) \leq 1.$$

We recursively define functions a_0, a_0^-, a_0^+, a_1, $\ldots$, a_{n-1}^+, a_{n-1}^-, a_n by the following procedure:

- $a_0(x,p) := a(x,p)$,
- $i := 1$,
- choose $a^+_{i-1}(.,p) \in C^\infty(Q \cap \{x_i = b_i\})$ and (B.0.7)
  $a^-_{i-1}(.,p) \in C^\infty(Q \cap \{x_i = a_i\})$ (p is fixed) with
  $$|a^-_{i-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,p) - a_{i-1}(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n,p)| < \frac{\varepsilon}{2n},$$
  $$|a^+_{i-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,p) - a_{i-1}(x_1,\dots,x_{i-1},b_i,x_{i+1},\dots,x_n,p)| < \frac{\varepsilon}{2n},$$
- $a_i(x,p) := a^+_{i-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,p)\chi^+_i(x_i) + (a_{i-1}(x,p)(1-\chi^-_i(x_i)) + a^-_{i-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,p)\chi^-_i(x_i)) \times (1-\chi^+_i(x_i))$,
- replace i by $i+1$,
- if $i \le n$ then goto (B.0.7).

The functions $a^\pm_i(.,p)$ can be defined, e.g., by means of convolution with Friedrich's mollifiers. Then it is clear that the $a^\pm_i(.,p)$ have the same smoothness with respect to p as $a(.,p)$.
Then we can define

$$a_\varepsilon(x,p) := a_n(x,p) \quad (x,p) \in Q \times G.$$

This function has the desired properties in $Q \times G$. Now we are in a position to define a_ε in $(Q' \setminus Q) \times G$ by means of reflections. This is done recursively:

- $j := 1$,
- $\forall x \in \prod_{k=1}^{j-1}[a_k, 2b_k - a_k] \times [b_j, 2b_j - a_j] \times \prod_{k=j+1}^{n}[a_k, b_k]$ (B.0.8)
  we define:
  $a_\varepsilon(x_1,\dots,x_n,p) := a_\varepsilon(x_1,\dots,x_{j-1},2b_j - x_j, x_{j+1},\dots,x_n,p)$,
- replace j by $j+1$,
- if $j \le n$ then goto (B.0.8).

The bijective mappings mentioned in the lemma are suitable compositions of these reflections. □

The following lemma is a generalisation of Gronwall's Lemma to differential inequalities with a singular coefficient, see [Ner66].

Lemma B.0.4 (Nersesyan). *Let $y(t) \in C([0,T]) \cap C^1(0,T)$ be a solution of the differential inequality*

$$y'(t) \leq K(t)y(t) + f(t), \quad 0 < t < T,$$

where the functions $K(t)$ and $f(t)$ belong to $C(0,T)$. We assume for every $t \in (0,T)$ and every $\delta \in (0,t)$ that

$$\begin{aligned}
&\int_0^\delta K(\tau)\,d\tau = \infty, \quad \int_\delta^T K(\tau)\,d\tau < \infty, \\
&\lim_{\delta \to +0} \int_\delta^t \exp\left(\int_s^t K(\tau)\,d\tau\right) f(s)\,ds \quad \textit{exists}, \\
&\lim_{\delta \to +0} y(\delta) \exp\left(\int_s^t K(\tau)\,d\tau\right) = 0. \qquad \text{(B.0.9)}
\end{aligned}$$

Then it holds

$$y(t) \leq \int_0^t \exp\left(\int_s^t K(\tau)\,d\tau\right) f(s)\,ds.$$

For the estimate of solutions of ODEs we need the following two lemmata.

Lemma B.0.5. *Let $g, h \in C^2([s,T])$ be the solutions of*

$$\begin{aligned}
&h''(t) = B(t)h(t), \quad h(s) = H_0 \geq 0, \quad h'(s) = H_1 \geq 0, \\
&g''(t) = A(t)g(t), \quad g(s) = G_0 \geq 0, \quad g'(s) = G_1 \geq 0
\end{aligned}$$

with $|A(t)| \leq B(t)$, $G_0 < H_0$, $G_1 < H_1$. Then it holds

$$|g(t)| < h(t) \quad \forall s \leq t \leq T.$$

Proof. (indirect)
Let $|g(t_0)| \geq h(t_0)$ for some $s < t_0 \leq T$. We set

$$t_1 = \inf\{t \in [s,T] : |g(t)| \geq h(t)\}$$

and it follows that

$$|g(t_1)| = h(t_1), \quad |g(t)| < h(t) \quad s \leq t < t_1.$$

In the following we show $|g(t_1)| < h(t_1)$ which gives a contradiction. For $s \leq t \leq t_1$ it holds

$$g''(t) = A(t)g(t) \leq |A(t)| \cdot |g(t)| \leq B(t)h(t) = h''(t).$$

We integrate twice from s to t and get

$$g'(t) - G_1 \leq h'(t) - H_1, \quad s \leq t \leq t_1,$$
$$g(t) - G_1(t-s) - G_0 \leq h(t) - H_1(t-s) - H_0, \quad s \leq t \leq t_1,$$
$$g(t) < h(t), \quad s \leq t \leq t_1.$$

On the other hand, we have

$$g''(t) = A(t)g(t) \geq -|A(t)| \cdot |g(t)| \geq -B(t)h(t) = -h''(t).$$

We integrate twice and obtain

$$g'(t) - G_1 \geq H_1 - h'(t), \quad s \leq t \leq t_1,$$
$$g(t) - G_1(t-s) - G_0 \geq H_1(t-s) + H_0 - h(t), \quad s \leq t \leq t_1,$$
$$g(t) > -h(t), \quad s \leq t \leq t_1.$$

Especially it is true that $|g(t_1)| < h(t_1)$. □

We use this result to prove the following lemma.

Lemma B.0.6. *Let $g, h \in C^2([s,T])$ be the solutions of*

$$h''(t) = B(t)h(t), \quad h(s) = H_0 \geq 0, \quad h'(s) = H_1 \geq 0,$$
$$g''(t) = A(t)g(t), \quad g(s) = G_0 \geq 0, \quad g'(s) = G_1 \geq 0$$

with $|A(t)| \leq B(t)$, $G_0 \leq H_0$, $G_1 \leq H_1$. Then it holds

$$|g(t)| \leq h(t) \quad \forall s \leq t \leq T.$$

Proof. Let h^ε be the solution of

$$h^{\varepsilon\prime\prime}(t) = B(t)h^\varepsilon(t), \quad h^\varepsilon(s) = H_0 + \varepsilon, \quad h^{\varepsilon\prime}(s) = H_1 + \varepsilon$$

with $\varepsilon > 0$. Then the previous lemma reveals

$$|g(t)| < h^\varepsilon(t), \quad s \leq t \leq T.$$

Further, let $h_{1,1}(t)$ be the solution of

$$h''_{1,1}(t) = B(t)h_{1,1}(t), \quad h_{1,1}(s) = 1, \quad h'_{1,1}(s) = 1.$$

Then we have $h_{1,1}(t) > 0$ in $[s,T]$ and it holds

$$h^\varepsilon(t) = h(t) + \varepsilon h_{1,1}(t).$$

If ε tends to zero, we deduce that $|g(t)| \leq h(t)$. □

Appendix C

Propagation of Singularities — Auxiliary Results

Let us give the proof of Proposition 6.3.1.

Proof. The function λ is strictly monotonically increasing, hence

$$\Lambda(t) = \int_0^t \lambda(\tau)\, d\tau \leq t\lambda(t).$$

Integrating (6.1.6) from t to T_0 we get (6.3.2). It follows that

$$\Lambda(t) \leq t\lambda(t) \leq t\frac{\lambda(T_0)}{\Lambda(T_0)^{d_0}}\Lambda(t)^{d_0},$$
$$\text{const} = \frac{\Lambda(T_0)^{d_0}}{\lambda(T_0)} \leq t\Lambda(t)^{d_0-1} \quad \forall 0 < t \leq T_0.$$

This implies (6.3.3). If we differentiate the equation $\Lambda(t_\xi)\langle\xi\rangle = N$ with respect to $\langle\xi\rangle$, (6.3.4) follows. If we set $T_0 := T$ and $t = t_\xi$ in (6.3.2), we gain (6.3.5). The derivative of the function $p(\langle\xi\rangle)$ with respect to $\langle\xi\rangle$ satisfies

$$p'(\langle\xi\rangle) = t'_\xi\langle\xi\rangle + t_\xi = -\frac{\Lambda(t_\xi)}{\lambda(t_\xi)} + t_\xi \geq -\frac{t_\xi\lambda(t_\xi)}{\lambda(t_\xi)} + t_\xi = 0.$$

From $p(\langle\xi\rangle) \geq p(\langle 0\rangle)$ we conclude that $t_\xi \geq C_4\langle\xi\rangle^{-1}$. From (6.3.4) and (6.3.5) we see that

$$t'_\xi \leq -C\langle\xi\rangle^{d_0-2}.$$

Using $t_\xi = \int_{\langle\infty\rangle}^{\langle\xi\rangle} t'_\eta d\langle\eta\rangle = \int_{\infty}^{\langle\xi\rangle} t'_\eta d\langle\eta\rangle$ and (6.3.3) it can be deduced that

$$t_\xi \leq C_3\langle\xi\rangle^{d_0-1}.$$

By the inequality $\sqrt{1+a} \leq 1 + \sqrt{a}$, it follows that

$$\int_0^{t_\xi} \varrho(\xi, t)\, dt \leq t_\xi + \sqrt{\langle\xi\rangle} \int_0^{t_\xi} \frac{\lambda(t)}{\sqrt{\Lambda(t)}} dt$$
$$= t_\xi + 2\sqrt{\langle\xi\rangle\Lambda(t_\xi)} \leq t_0 + 2\sqrt{N} = C.$$

It holds $\Lambda(t)\langle\xi\rangle \leq N$ in $Z_{pd}(N)$, hence

$$\varrho(\xi, t)^2 = 1 + \frac{\lambda(t)^2}{\Lambda(t)}\langle\xi\rangle \geq \frac{\lambda(t)^2\langle\xi\rangle^2}{N}.$$

Then it immediately follows that

$$\varrho(\xi, t_\xi) \geq \frac{1}{\sqrt{N}}\lambda(t_\xi)\langle\xi\rangle.$$

We have $N^2 = \langle\xi\rangle^2\Lambda(t_\xi)^2 \leq \langle\xi\rangle^2 t_\xi^2 \lambda(t_\xi)^2$, thus,

$$1 \leq \frac{\langle\xi\rangle^2 t_\xi^2 \lambda(t_\xi)^2}{N^2}.$$

Then it can be seen that

$$\varrho(\xi, t_\xi)^2 \leq \frac{\langle\xi\rangle^2\lambda(t_\xi)^2}{N}\left(\frac{t_0^2}{N} + 1\right) \leq \frac{C}{N}\lambda(t_\xi)^2\langle\xi\rangle^2.$$

This proves (6.3.10). Employing partial integration twice and (6.1.6) we get

$$\int_0^t (t-s)^2 \varrho(\xi, s)^2\, ds = \int_0^t (t-s)^2\, ds + \langle\xi\rangle \int_0^t (t-s)^2 \frac{\lambda(s)^2}{\Lambda(s)}\, ds$$
$$\leq t^3 + C\langle\xi\rangle \int_0^t (t-s)^2 \lambda'(s)\, ds = t^3 + C\langle\xi\rangle \int_0^t \Lambda(s)\, ds$$
$$\leq t^3 + C\langle\xi\rangle\Lambda(t)t \leq Ct.$$

Since $d_0 > 1/2$, the derivative $\partial_t \varrho(t, \xi)$ fulfils

$$\partial_t \varrho(\xi, t) = \frac{\langle\xi\rangle}{2\varrho(\xi, t)} \frac{\lambda(t)}{\Lambda(t)} \left(2\lambda'(t) - \frac{\lambda(t)^2}{\Lambda(t)}\right)$$
$$\geq \frac{\langle\xi\rangle}{2\varrho(\xi, t)} \frac{\lambda(t)}{\Lambda(t)} (2d_0 - 1) \frac{\lambda(t)^2}{\Lambda(t)} > 0.$$

This proves (6.3.12). Finally, the derivative of $q(\langle\xi\rangle)$ satisfies

$$q'(\langle\xi\rangle) = \lambda'(t_\xi) t'_\xi \langle\xi\rangle^{d_1} + d_1 \lambda(t_\xi)\langle\xi\rangle^{d_1 - 1}$$
$$= \left(-\frac{\Lambda(t_\xi)}{\lambda(t_\xi)}\lambda'(t_\xi) + d_1\lambda(t_\xi)\right)\langle\xi\rangle^{d_1-1} \geq 0.$$

□

Bibliography

[Ale84] Aleksandrian, G. Parametrix and propagation of the wave front of a solution to a Cauchy problem for a model hyperbolic equation (in Russian). *Izv. Akad. Nauk Arm. SSR*, 19(3):219–232, 1984.

[AM84] Alinhac, S. and Metivier, G. Propagation de l'analyticité des solutions de systèmes hyperboliques non–linéaires. *Inv. math.*, 75:189–204, 1984.

[AS84] Abramowitz, M. and Stegun, I. *Handbook of Mathematical Functions*. Harri Deutsch, Frankfurt, 1984.

[Bon81] Bony, J.-M. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14:209–246, 1981.

[BR84] Beals, M. and Reed, M. Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems. *Trans. Amer. Math. Soc.*, 285(1):159–184, 1984.

[CJS83] Colombini, F., Jannelli, E., and Spagnolo, S. Well–posedness in the Gevrey classes of the Cauchy problem for a non–strictly hyperbolic equation with coefficients depending on time. *Ann. Scuola Norm. Sup. Pisa IV*, 10:291–312, 1983.

[CM78] Coifman, R. and Meyer, Y. Commutateurs d'intégrales singulières et opérateurs multilinéaires. *Ann. Inst. Fourier (Grenoble)*, 28(3):177–202, 1978.

[CS82] Colombini, F. and Spagnolo, S. An example of a weakly hyperbolic Cauchy problem not well posed in C^∞. *Acta Math.*, 148:243–253, 1982.

[CZ95] Cicognani, M. and Zanghirati, L. Quasi–Linear weakly hyperbolic equations with Gevrey–Levi conditions. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 41:5–31, 1995.

[CZ97] Cicognani, M. and Zanghirati, L. Quasilinear weakly hyperbolic equations with Levi's conditions. *Rend. Sem. Mat. Univ. Polit. Torino*, 55(2), 1997.

[D'A94a] D'Ancona, P. On a semilinear weakly hyperbolic equation with logarithmic nonlinearity. *Diff. Int. Eq.*, 7(1):121–132, 1994.

[D'A94b] D'Ancona, P. Well posedness in C^∞ for a weakly hyperbolic second order equation. *Rend. Sem. Mat. Univ. Padova*, 91:65–83, 1994.

[D'A95] D'Ancona, P. A note on a theorem of Jörgens. *Math. Zeitschr.*, 218:239–252, 1995.

[Dio62] Dionne, P. A. Sur les problèmes hyperboliques bien posés. *J. Analyse Math.*, 10:1–90, 1962.

[DR97] Dreher, M. and Reissig, M. About the C^∞–well posedness of fully nonlinear weakly hyperbolic equations of second order with spatial degeneracy. *Adv. Diff. Eq.*, 2(6):1029–1058, 1997.

[DR98a] D'Ancona, P. and Racke, R. Weakly hyperbolic equations in domains with boundaries. *Nonl. Anal.*, 33(5):455–472, 1998.

[DR98b] Dreher, M. and Reissig, M. Local solutions of fully nonlinear weakly hyperbolic differential equations in Sobolev spaces. *Hokk. Math. J.*, 27(2):337–381, 1998.

[DS91] D'Ancona, P. and Spagnolo, S. On the life span of the analytic solutions to quasilinear weakly hyperbolic equations. *Indiana Univ. Math. J.*, 40(1):71–99, 1991.

[EMO53] Erdelyi, A., Magnus, and Oberhettinger. *Higher Transcendental Functions. Bateman Manuscript Project*. McGraw–Hill, New York, 1953.

[ES91] Egorov, Yu. and Shubin, M.A. (editors). *Partial Differential Equations*. Springer, 1991.

[Gar57] Garding, L. *Cauchy's Problem for Hyperbolic Equations*. University Chicago, 1957.

[Gev13] Gevrey, M. Sur les équations aux derivées du type parabolique. *J. Math. Pures Appl.*, 9:305–471, 1913.

[GR] Gramchev, T. and Rodino, L. Gevrey solvability for semilinear partial differential equations with multiple characteristics. to appear in: *Bollettino U.M.I.*

[Heu92] Heuser, H. *Funktionalanalysis: Theorie und Anwendung*. Teubner, Stuttgart, 3rd edition, 1992.

[Hör69] Hörmander, L. *Linear Partial Differential Operators*. Springer, 1969.

[Hör85] Hörmander, L. *The Analysis of Linear Partial Differential Operators*. Springer, 1985.

[IP74] Ivrii, V.Ya. and Petkov, V.M. Necessary conditions for the Cauchy problem for non–strictly hyperbolic equations to be well–posed (in Russian). *Uspekhi Mat. Nauk*, 29(5):3–70, 1974. English translation: *Russian Math. Surveys* 29(5):1-70, 1974.

[Kaj83] Kajitani, K. Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes. *Hokk. Math. J.*, 12:434–460, 1983.

[Kg76] Kumano-go, H. A calculus of Fourier integral operators on $\mathbb{R}^n$ and the fundamental solution for an operator of hyperbolic type. *Comm. PDE*, 1(1):1–44, 1976.

[Kic96] Kichenassamy, S. *Nonlinear Wave Equations*. Monographs and textbooks in pure and applied mathematics. Marcel Dekker, Inc., New York, Basel, Hong Kong, 1996.

[Kla85] Klainerman, S. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *CPAM*, 38:321–332, 1985.

[KP88] Kato, T. and Ponce, G. Commutator estimates and the Euler and Navier–Stokes equations. *CPAM*, 41:891–907, 1988.

[KY98] Kajitani, K. and Yagdjian, K. Quasilinear hyperbolic operators with the characteristics of variable multiplicity. *Tsukuba J. Math.*, 22(1):49–85, 1998.

[Lax57] Lax, P. D. Asymptotic solutions of oscillatory initial value problems. *Duke Math. J.*, 24(4):627–646, 1957.

[LC88] Li, T.-T. and Chen, Y.-M. Initial value problems for nonlinear wave equations. *Comm. PDE*, 13(4):383–422, 1988.

[Ler54] Leray, J. *Hyperbolic Differential Equations*. Inst. Adv. Study, Princeton, 1954.

[Man94] Manfrin, R. A remark on global smooth solutions for quasilinear wave equations. *Rend. Sem. Mat. Univ. Padova*, 91:1–17, 1994.

[Man96] Manfrin, R. Analytic regularity for a class of semi–linear weakly hyperbolic equations of second order. *NoDEA*, 3(3):371–394, 1996.

[Mic78] Michlin, S. G. *Partielle Differentialgleichungen der Mathematischen Physik*. Akademie Verlag, Berlin, 1978.

[Miz61] Mizohata, S. Some remarks on the Cauchy problem. *J. Math. Kyoto Univ.*, 1(1):109–127, 1961.

[Miz73] Mizohata, S. *The Theory of Partial Differential Equations*. Cambridge University Press, 1973.

[MT96] Manfrin, R. and Tonin, F. On the Gevrey regularity for weakly hyperbolic equations with space–time degeneration of Oleinik type. *Rend. Mat. VII, (Roma)*, 16:203–231, 1996.

[Ner66] Nersesyan, A. On a Cauchy problem for degenerate hyperbolic equations of second order (in Russian). *Dokl. Akad. Nauk SSSR*, 166(6):1288–1291, 1966.

[Ole70] Oleinik, O. On the Cauchy problem for weakly hyperbolic equations. *CPAM*, 23:569–586, 1970.

[Pet38] Petrovskij, I. G. On the Cauchy problem for systems of linear partial differential equations. *Bull. Univ. Mosk. Ser. Int. Mat. Mekh.*, 1(7):1–74, 1938.

[Qi58] Qi, M.-Y. On the Cauchy problem for a class of hyperbolic equations with initial data on the parabolic degenerating line. *Acta Math. Sinica*, 8:521–529, 1958.

[Rac92] Racke, R. *Lectures on Nonlinear Evolution Equations. Initial Value Problems*. Vieweg Verlag, Braunschweig et. al., 1992.

[Rau79] Rauch, J. Singularities of solutions to semilinear wave equations. *J. Math. Pures Appl.*, 58:299–308, 1979.

[Rei97] Reissig, M. Weakly hyperbolic equations with time degeneracy in Sobolev spaces. *Abstract Appl. Anal.*, 2(3,4):239–256, 1997.

[RY] Reissig, M. and Yagdjian, K. $L_p - L_q$ estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients. to appear in: *Math. Nachr.*

[RY93] Reissig, M. and Yagdjian, K. On the Cauchy problem for quasilinear weakly hyperbolic equations with time degeneration. *J. Contemporary Math. Anal.*, 28(2):31–50, 1993.

[RY97] Reissig, M. and Yagdjian, K. Stability of global Gevrey solutions to weakly hyperbolic equations. *Chinese Ann. Math. Ser. B*, 18(1):1–14, 1997.

[RY99] Reissig, M. and Yagdjian, K. Weakly hyperbolic equations with fast oscillating coefficients. *Osaka J. Math.*, 36(2), 1999.

[Shi91] Shinkai, K. Stokes multipliers and a weakly hyperbolic operator. *Comm. PDE*, 16(4,5):667–682, 1991.

[Spa88] Spagnolo, S. Some results of analytic regularity for the semi–linear weakly hyperbolic equations of the second order. *Rend. Sem. Mat. Univ. Polit. Torino*, pages 203–229, 1988. Special edition.

[Tar95] Tarama, S. On the second order hyperbolic equations degenerating in the infinite order. — Example —. *Math. Japonica*, 42(3):523–533, 1995.

[Tay91] Taylor, M. E. *Pseudodifferential Operators and Nonlinear PDE*. Birkhäuser, Boston, 1991.

[Tri78] Triebel, H. *Interpolation Theory, Function Spaces, Differential Operators*. Deutscher Verlag der Wissenschaften, Berlin, 1978.

[TT80] Taniguchi, K. and Tozaki, Y. A hyperbolic equation with double characteristics which has a solution with branching singularities. *Math. Japonica*, 25(3):279–300, 1980.

[Yag78] Yagdjian, K. The Cauchy problem for weakly hyperbolic equations in the classes of Gevrey functions. *Izv. Akad. Nauk Arm. SSR*, 13(1):3–22, 1978.

[Yag96] Yagdjian, K. Gevrey asymptotic representation of the solutions of equations with one turning point. *Math. Nachr.*, 183:295–312, 1996.

[Yag97a] Yagdjian, K. *The Cauchy Problem for Hyperbolic Operators. Multiple Characteristics, Micro–Local Approach*. Akademie Verlag, Berlin, 1997.

[Yag97b] Yagdjian, K. Representation theorem for the solutions of equations with the turning point of infinite order. *Ann. mat. pura et appl.*, 173:13–30, 1997.

[Yos78] Yoshikawa, A. Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation I, II, III. *Hokk. Math. J.*, 6,7,7:313–344,1–26,127–141, 1977,1978,1978.

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