# Local Solutions to Quasilinear Weakly Hyperbolic Differential Equations

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## 1. Introduction

Let us consider the differential operator of order $m$

$$P(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j,$$

where we adopted the usual notation $D_t = -i\partial_t$, $D_x = -i\nabla_x$. This operator $P$ is called *hyperbolic in the direction* $t$ if the roots $\tau_j = \tau_j(x,t,\xi)$ of the equation

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\xi^\alpha \tau^j = 0$$

are real for all real $x, t, \xi$. The operator $P$ is said to be *strictly hyperbolic in the direction* $t$, if the roots $\tau_j$ are real and *distinct* for $\xi \in \mathbb{R}^n \setminus \{0\}$. If $P$ is hyperbolic, but not (necessarily) strictly hyperbolic, it is called *weakly hyperbolic*.

Hyperbolicity is a necessary condition for $C^\infty$ well–posedness of the Cauchy problem (see [19], [22]). Well–posedness (with respect to chosen topological spaces for the data, right–hand side and the solution) of a Cauchy problem means, as usual, the existence, uniqueness and continuous dependence (in the topologies of the given spaces) of the solution. However, hyperbolicity does not guarantee the well–posedness in, e.g., $C^\infty$ or Sobolev spaces. A sufficient condition for the well–posedness in $C^\infty$ and in Sobolev spaces is the strict hyperbolicity, see [26], [20] and [14].

Therefore it is a natural goal to find classes of weakly hyperbolic Cauchy problems which are $C^\infty$ well–posed.

In the weakly hyperbolic case, new phenomena occur which may prevent the Cauchy problem from being well–posed. These phenomena are the following:

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**Oscillations in the coefficients with respect to time**

- Colombini, Jannelli, and Spagnolo [6], [7], constructed a smooth function $a(t) \geq 0$ and smooth data $u_0(x)$, $u_1(x)$ with the property that the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

has no solution $u$ in the distribution space $\mathcal{D}'(\mathbb{R} \times [0,1])$. This coefficient $a(t)$ is positive for $t > 0$, oscillating for $t \to 0+0$ and vanishing for $t \leq 0$.
- Let $b(t)$ be a positive, periodic, smooth and non–constant function. Tarama [32] proved that the Cauchy problem

$$\begin{aligned} &u_{tt} - \exp(-2t^{-\alpha})b(t^{-1})^2 u_{xx} = 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \end{aligned}$$

is $C^\infty$ well–posed if and only if $\alpha \geq 1/2$.

**The influence of lower order terms**

- Ivrii and Petkov [16] showed that necessary conditions for the $C^\infty$ well–posedness of

$$v_{tt} - t^{2l}v_{xx} + t^k v_x = 0, \quad l, k \in \mathbb{N}_0, \tag{1.1}$$

$$u_{tt} - x^{2n}u_{xx} + x^m u_x = 0, \quad n, m \in \mathbb{N}_0, \tag{1.2}$$

are $k \geq l-1$ and $m \geq n$. The sufficiency of these conditions was proved by Oleinik [25].
- If one wants to study well–posedness in Sobolev spaces, one has to pay attention to another phenomenon, which occurs in the border case $k = l-1$ of the $C^\infty$ well–posedness: *the loss of Sobolev regularity*. Qi [27] showed by an explicit representation of the solution to the Cauchy problem

$$\begin{aligned} &u_{tt} - t^2 u_{xx} = bu_x, \\ &u(x,0) = \varphi(x), \quad u_t(x,0) = 0, \quad b = 4m+1, \quad m \in \mathbb{N} \end{aligned} \tag{1.3}$$

that $u(\cdot,t) \in H^{s-m}$ if $\varphi \in H^s$. Or, let us look from another point: choose an arbitrary data function $\varphi(x)$ with high Sobolev smoothness $s \gg 1$. Then a number $b$ exists such that there is no classical solution of (1.3). The solution only exists in distribution spaces.

The loss of regularity also occurs for equations of the form

$$\begin{aligned} &u_{tt} - t^{2l}u_{xx} - bt^{l-1}u_x = 0, \\ &u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \end{aligned} \tag{1.4}$$

as shown by Taniguchi and Tozaki [31]. Equations of the type (1.3) and (1.4) are interesting because singularities of their solutions may propagate in a non–standard way, see [12], [13], and [31].

There are different ways to exclude the phenomenon of oscillations and to restrict the influence of the lower order terms. We pick the equation

$$u_{tt} - a(x,t)u_{xx} + b(x,t)u_x + d(x,t)u_t + e(x,t)u = f(x,t)$$

as a model problem. First we consider the oscillations.

- If the degeneracy occurs for $t = 0$ only, we may assume ([25])
$$0 \le Ca(x,t) \pm \partial_t a(x,t), \quad t \ge 0, \quad C > 0.$$
- We can suppose that the coefficient $a(x,t)$ has the structure $a(x,t) = a_0(x,t)\sigma(x)^2\lambda(t)^2$ with some smooth $a_0(x,t) \ge \alpha > 0$, and $\lambda(0) = 0$, $\lambda'(t) > 0$ $(t > 0)$. The degeneration happens at the zeroes of the product $\sigma\lambda$. The functions $\sigma$ and $\lambda$ characterize the *spatial degeneracy* and *time degeneracy*, respectively. Assumptions of this type were made, e.g., in Nersesyan [24], Yagdjian [35], and [10], [11], [13]. *We will follow this idea and generalize it to quasilinear higher order equations in higher dimensions.*

Second, we consider the lower order terms. Conditions which restrict the influence of these terms are called *Levi conditions*. Our aim is to find conditions which do not exclude the interesting equations (1.3) and (1.4). The following Levi conditions have been used widely in the past:

- If the degeneracy occurs for $t = 0$ only, then we may take the condition
$$Btb(x,t)^2 \le Aa(x,t) + \partial_t a(x,t), \quad t \ge 0 \tag{1.5}$$
from [25]; $A$ and $B$ are some positive constants. This Levi condition is sharp in the case of finite degeneracy: if one fixes $a(x,t) = x^{2n}t^{2l}$ and $b(x,t) = x^m t^k$, (1.5) implies $m \ge n$, $k \ge l - 1$. These are exactly the necessary and sufficient conditions from Ivrii, Petkov and Oleinik. However, this condition is not sharp in the case of time degeneracy of infinite order. It exists an explicit representation of the solutions to
$$u_{tt} - e^{-\frac{2}{t}}\frac{1}{t^4}u_{xx} + be^{-\frac{1}{t}}\frac{1}{t^4}u_x = 0, \quad t \ge 0, \quad b = \text{const}, \tag{1.6}$$
see Aleksandrian [1], which implies that the Cauchy problem for this equation is $C^\infty$ well–posed. Yet, the coefficients from (1.6) do not satisfy (1.5). Similarly to (1.3) and (1.4), the solutions to (1.6) lose regularity, too; and their singularities can propagate in an astonishing way, see [1].
- If one wants to include more general degenerations, one may assume the rather general and crude conditions
$$b(x,t)^2 \le Ca(x,t),$$
$$a_t(x,t) \le Ca(x,t) \text{ or } a_t(x,t) \ge -Ca(x,t),$$
or, similarly,
$$Bb(x,t)^2 \le Aa(x,t) + a_t(x,t), \quad A, B > 0,$$
compare D'Ancona [8], Manfrin [21]. However, these conditions are not sharp; they exclude (1.3), (1.4) and (1.6).

- It can be presumed $a(x,t) = a_0(x,t)\sigma(x)^2\lambda(t)^2$ with $a_0(x,t) \geq \alpha > 0$ and $|b(x,t)| \leq C|\sigma(x)|\lambda'(t)$. Coefficients $a(x,t)$ and $b(x,t)$ satisfying such a Levi condition include the interesting cases (1.3), (1.4) and (1.6). *We will follow this way and generalize these conditions to higher order equations.*

Let us list the main results of this paper. We are concerned with the hyperbolic Cauchy problem

$$
\begin{aligned}
(1.7)\quad & D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta(c_{k,\beta}(x,t)D_t^k u)\})\lambda(t)^{|\alpha|} D_x^\alpha D_t^j(\sigma(x)^{|\alpha|}u) \\
&\qquad = f(x,t,\{D_x^\beta(c_{k,\beta}(x,t)D_t^k u)\}|_{k+|\beta|\leq m-1}), \quad m \geq 2, \\
& u(x,0) = \varphi_0(x), \dots, D_t^{m-1}u(x,0) = \varphi_{m-1}(x)
\end{aligned}
$$

for $(x,t) \in M \times [0,T]$; where $M$ is either $\mathbb{R}^n$ or a smooth closed $n$–dimensional manifold. The functions $\lambda = \lambda(t)$ and $\sigma = \sigma(x)$ describe the degeneration of the principal part of the differential operator, and the functions $c_{k,\beta} = c_{k,\beta}(x,t)$ are weight functions for the lower order terms and characterize the Levi conditions.

*Example.* The weight function $\lambda = \lambda(t)$ has to satisfy a certain condition (see Condition 4.1). Examples of such $\lambda$ are

$$
\begin{aligned}
&\lambda(t) = t^l, \quad l \in \mathbb{N}, \quad l \geq m-1, \\
&\lambda(t) = \Lambda'(t) \text{ with} \\
&\quad \Lambda(t) = \exp(-|t|^{-r}), \quad r > 0, \\
&\quad \Lambda(t) = \exp(-\exp(\exp(\exp(|t|^{-r})))), \quad r > 0.
\end{aligned}
$$

There are no restrictions on the choice of $\sigma$, any smooth real–valued function $\sigma = \sigma(x)$ is admissible.

The weight functions $c_{k,\beta}$ are connected with $\lambda$ and $\sigma$ via the relations (4.11) and (3.22)–(3.25), and special examples are

$$
c_{k,\beta}(x,t) = \begin{cases} \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}\sigma(x)^{|\beta|} & : |\beta| > 0, \\ 1 & : |\beta| = 0, \end{cases}
$$

where $\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau$.

The following results are proved for such Cauchy problems in this paper.

**Local existence in Sobolev spaces:** For given data in Sobolev spaces, a solution is found which suffers from the loss of Sobolev regularity, as motivated by Qi's example.

**Blow–up criterion:** We will prove that a blow–up of the solution in high order Sobolev spaces is only possible if the $C^1_*$ Zygmund norm of certain weighted derivatives (up to the order $m-1$) of the solution blows up. This is a generalization of a similar criterion from the strictly hyperbolic case, see Taylor [33].

**Local existence in $C^\infty$:** That blow–up criterion leads to the local existence of solutions in $C^\infty$ immediately.

**Domains of dependence:** A special feature of strictly hyperbolic equations is the finite propagation speed. In other words, the value of the solution at a given point at a given time only depends on the values of the initial data and right–hand side from a certain bounded domain, the so–called domain of dependence.

We will define and study domains of dependence for quasilinear weakly hyperbolic operators, and use them to prove the local in space and time existence and uniqueness of Sobolev solutions, and their $C^\infty$ regularity provided that the data are from $C^\infty$. Our concept of domains of dependence extends the concept of Alinhac and Metivier [2] from the strictly hyperbolic to the weakly hyperbolic case. Geometrically spoken, these domains can be described by the condition that the principal part of the operator be hyperbolic at each point of the boundary of the domain in the normal direction of the boundary. Since the coefficients of the principal part depend on the solution, the domain of dependence for the solution will be dependent on the solution itself.

Next we give some remarks concerning the used methods and tools.

A crucial step of the investigation of hyperbolic Cauchy problems is an *a priori* estimate of the solution in Sobolev spaces, which is usually proved by means of pseudodifferential operators. However, since the coefficients of the hyperbolic operator depend on the solution and its derivatives itself, and because the solution will be from some Sobolev space, the coefficients of this hyperbolic operator will not have $C^\infty$ smoothness. Thus, the theory of pseudodifferential operators with symbols of *infinite* smoothness seems not to be applicable; hence we present a theory of pseudodifferential operators with symbols of *finite* smoothness ($H^s$ or $C^1$ or merely $C^0$) in Section 2. We cite results of Taylor [33] concerning mapping properties, commutator estimates, adjoints and compositions.

Our methods for proving the local existence of a solution to (1.7) are a unification of ideas taken from [33] who studied quasilinear strictly hyperbolic equations; and Kajitani and Yagdjian [17] who studied quasilinear weakly hyperbolic equations with time degeneracy.

In Section 3, we study weakly hyperbolic Cauchy problems with pure spatial degeneracy, i.e., (1.7) without weight function $\lambda = \lambda(t)$. Our approach in this case is as follows. We construct some vector–valued function $U^*$ which contains weighted derivatives of $u$ up to the order $m - 1$ and solves a pseudodifferential hyperbolic system of first order

$$\partial_t U^* = K^*(x, t, U^*, D)(\sigma U^*) + F^*(x, t, U^*) \tag{1.8}$$

where $K^*$ is a *strictly* hyperbolic matrix pseudodifferential operator of order 1, and $F^*$ contains the right–hand side and some other terms. We insert some smoothing operators $J_\varepsilon$ into (1.8) such that its right–hand side becomes an operator of order 0, and the existence of an approximate solution $U_\varepsilon^*$ follows immediately from functional analytic arguments. Next we have to prove independent of $\varepsilon$ estimates of

$U_\varepsilon^*$ and its life–span. These estimates will allow us to show an interesting blow–up criterion:

*A blow–up of $U^*$ in the $H^s$ norm is impossible as long as the Zygmund space norm $\|U^*\|_{C_*^1}$ remains bounded.*

We are able to extend the results won by Dionne [9] and Taylor [33] from the strictly hyperbolic case to the weakly hyperbolic case.

The general weakly hyperbolic Cauchy problem with spatial and time degeneracy is treated in Section 4. Here we face new difficulties which are typical for the time degeneracy:

One such obstacle is a *singular coefficient in the energy inequality.* Consider, as an example, the weakly hyperbolic equation

$$u_{tt} - \lambda(t)^2 u_{xx} = f(x,t), \quad \lambda(0) = 0, \quad \lambda'(t) > 0 \quad (t > 0).$$

If we choose the energy in the usual way, $E(t) = \|u_t\|_{L^2}^2 + \|\lambda(t)u_x\|_{L^2}^2$, then we obtain, after some calculations,

$$E'(t) \le \|f(\cdot,t)\|_{L^2}^2 + \frac{\lambda'(t)}{\lambda(t)} E(t).$$

The lemma of Gronwall is not applicable, since the coefficient $\lambda'(t)/\lambda(t)$ becomes unbounded for $t \to 0$. But one can use Nersesyan's lemma (see Lemma 6.2) if the initial data vanish and $\|f(\cdot,t)\|_{L^2}$ has a zero of sufficiently high order at $t = 0$.

Another obstacle is the *loss of regularity.* The example of Qi [27] shows that the solution can lose Sobolev smoothness in comparison with the initial data. The number of lost derivatives depends (in the linear case) on the $L^\infty$–norm of the coefficients of some lower order terms. This makes the investigation of nonlinear Cauchy problems delicate, since the usual fixed point arguments can not be applied directly. The crucial tool for solving this difficulty is the reduction (Section 4.3) of the Cauchy problem (1.7) to another Cauchy problem which enjoys the so–called *strictly hyperbolic type property*: let $L$ be a weakly hyperbolic operator of order 2 (for simplicity). We say that a Cauchy problem

$$Lu(x,t) = f(x,t), \quad u(x,0) = u_t(x,0) = 0$$

has the *strictly hyperbolic type property* if there is a topological space $B$ and a weight function $\omega(x,t)$ such that $f \in B$ implies $\omega(x,t)\nabla_x u \in B$ and $u_t \in B$. Then the local existence in $B$ of a solution to a *quasilinear* version of the above Cauchy problem can be proved by standard arguments. In the strictly hyperbolic case, we choose $\omega \equiv 1$ and $B = C([0,T], H^s)$. In the weakly hyperbolic case, $\omega$ is chosen according to the degeneracy, and $B$ consists of functions which decay sufficiently fast for $t \to 0$, see Section 4.1. For other applications of such adapted Banach spaces to weakly hyperbolic equations, see [12], [13], and Reissig, Yagdjian [30].

Concerning the investigation of domains of dependence in Section 5, our technique is as follows: we exhaust the domain of dependence with hypersurfaces, and change the variables such that these hypersurfaces become planes of constant time. Outside some small domain, we then change the operator slightly, and transform the equation into a Cauchy problem on a torus which can be treated with the methods of Section 3.

Finally, we introduce some notations.

By $M$ we denote either $\mathbb{R}^n$ or a closed smooth $n$-dimensional manifold.

The Banach space of functions whose derivatives up to the order $k$ are bounded and continuous is denoted by $C_b^k(M)$, $k \in \mathbb{N}_0$. Similarly, we introduce the Hölder spaces $C_b^s(M)$, $s \in \mathbb{R}^+$, and write $Lip^1(M)$ for the space of Lipschitz continuous functions on $M$.

Let $C_*^s(M)$ denote the Hölder spaces for $s \notin \mathbb{N}$, and the *Zygmund spaces* for $s \in \mathbb{N}^+$. The Zygmund spaces $C_*^s$, $s \in \mathbb{N}^+$, consist of all functions $u$ with the property that $u \in C_b^{s-1}$ and (in local coordinates)

$$\sup_{x \neq y} \sum_{|\alpha|=s-1} \frac{|D^\alpha u(x) - 2(D^\alpha u)((x+y)/2) + D^\alpha u(y)|}{|x-y|} < \infty.$$

The spaces $C_b^k$ are continuously embedded in $C_*^k$, for $k \in \mathbb{N}^+$.

Let $\triangle$ be the Laplace–Beltrami operator on $M$ and set $\langle D \rangle = (1 - \triangle)^{1/2}$. In case of $M = \mathbb{R}^n$, $\langle D \rangle$ can be written as a pseudodifferential operator with symbol $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ (a thorough representation of the theory of pseudodifferential operators can be found in Hörmander [15]). Then we define the Sobolev spaces $H^s(M) = \langle D \rangle^{-s} L^2(M)$ for $s \in \mathbb{R}$, where $L^2(M)$ is the usual Lebesgue space of square integrable functions on $M$.

Assuming local coordinates $x = (x_1, \ldots, x_n)$ on $M$, we will employ the multi index notation:

$$D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad i^2 = -1.$$

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## 2. Pseudodifferential Operators with Finite Smoothness

### 2.1. Definition and Mapping Properties

In order to describe the smoothness of functions and pseudodifferential symbols, we introduce some scales $(X^s)_s$ of function spaces:

$$X^s = H^s(M), \quad \frac{n}{2} < s < \infty,$$
$$X^s = C_*^s(M), \quad 0 < s < \infty,$$
$$X^s = C_b^s(M), \quad 0 \leq s < \infty.$$

**Definition 2.1.1** (Space of symbols of finite smoothness)**.** The space $X^s S_{1,0}^m$ consists of all functions $p(x, \xi) : M \times \mathbb{R}^n \to \mathbb{C}$ with

$$\left\| D_\xi^\alpha p(\cdot, \xi) \right\|_{X^s} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|} \quad \alpha \geq 0.$$

In other words, for all $N \in \mathbb{N}_0$ it holds

$$\pi^m_{N,X^s}(p) = \sup\left\{ \left\| D^\alpha_\xi p(\cdot,\xi) \right\|_{X^s} \langle\xi\rangle^{-m+|\alpha|} : \xi \in \mathbb{R}^n, |\alpha| \le N \right\} < \infty.$$

**Definition 2.1.2** (Classical symbols of finite smoothness)**.** We say that $p(x,\xi) \in X^s S^m_{cl}$ if there is an asymptotic expansion $p(x,\xi) \sim \sum_{j\ge 0} \chi(\xi) p_j(x,\xi)$, where $p_j(x,\xi)$ are positive homogeneous of degree $m-j$ in $\xi$ and $p - \sum_{j=0}^{N-1} \chi p_j \in X^s S^{m-N}_{1,0}$. The function $\chi \in C^\infty(\mathbb{R}^n_\xi)$ vanishes in a neighborhood of 0 and equals 1 for $|\xi| \ge C > 0$.

**Definition 2.1.3** (Operators of finite smoothness)**.** Let $M = \mathbb{R}^n$. The operator spaces $OPX^sS^m_{1,0}$, $OPX^sS^m_{cl}$,respectively, consist of all operators $p(x,D)$ mapping $C^\infty_0(M)$ into the space of distributions $\mathcal{D}'(M)$ whose symbols $p(x,\xi)$ belong to $X^sS^m_{1,0}$, $X^sS^m_{cl}$, respectively, and satisfy

$$(p(x,D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{u}(\xi) d\xi \quad u \in C^\infty_0(\mathbb{R}^n).$$

If $M$ is a $C^\infty$ manifold, then the operator $p(x,D)$ is defined as follows. Let $(\Omega,\kappa)$ be a local chart of $M$, $\kappa\colon M \supset \Omega \to U \subset \mathbb{R}^n$. Define the pull–back $\kappa^*\colon C^\infty_0(U) \to C^\infty_0(\Omega)$ by $(\kappa^* u)(x) = u(\kappa(x))$, and the push–forward $\kappa_*\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(U)$ by $\langle \kappa_* F, u\rangle = \langle F, \kappa^* u\rangle$, $(F \in \mathcal{D}'(\Omega), u \in C^\infty_0(U))$. Then an operator $P\colon C^\infty_0(M) \to \mathcal{D}'(M)$ belongs to $OPX^sS^m_{1,0}$, $OPX^sS^m_{cl}$ if, for every local chart $(\Omega,\kappa)$, $\kappa_* \circ P \circ \kappa^*\colon C^\infty_0(U) \to \mathcal{D}'(U)$ belongs to $OPX^sS^m_{1,0}$, $OPX^sS^m_{cl}$, respectively.

The following two mapping properties are cited from [33] and [34], Chapter 13.9.

**Proposition 2.1.4.** *Let $p(x,D) \in OPC^s_* S^m_{1,0}$. Then $p(x,D)$ can be extended to an operator continuously mapping $C^{r+m}_*$ into $C^r_*$ $(-s < r \le s)$ and $H^{r+m}$ into $H^r$ $(-s < r < s)$, respectively.*

In the case of operators with coefficients from Sobolev spaces, we have less problems with the borderline case $r = s$:

**Proposition 2.1.5.** *If $p(x,D) \in OPH^sS^m_{1,0}$, then $p(x,D)$ can be extended to an operator which maps $H^{r+m}$ continuously into $H^r$ for $-s < r \le s$.*

### 2.2. Special Smoothing Operators

**Definition 2.2.1.** Denote the Laplace–Beltrami operator of $M$ by $\triangle$. For $0 < \varepsilon \le 1$, we define the smoothing operator $J_\varepsilon = (1 - \varepsilon\triangle)^{-1/2}$.

The proofs of the following lemmas are straightforward.

**Lemma 2.2.2.** *The operator $J_\varepsilon$ is invertible and commutes with $\langle D\rangle$.*

**Lemma 2.2.3.** *Let $X^s$ be either $H^s(M)$ with $s \in \mathbb{R}$, or $C^s_*(M)$ with $s > 0$. Then there is a constant $C > 0$ such that for every $0 < \varepsilon \le 1$,*

$$\|J_\varepsilon f\|_{X^{s+1}} \le C\varepsilon^{-1} \|f\|_{X^s},$$
$$\|f - J_\varepsilon f\|_{X^{s-t}} \le C\varepsilon^t \|f\|_{X^s}, \quad 0 \le t \le 1.$$

### 2.3. Commutator Estimates

We quote some estimates from Coifman, Meyer [5], Kato, Ponce [18] and Taylor [33]:

**Proposition 2.3.1.** *The following inequalities hold:*

$$\|[P,f]\|_{L^2\to L^2} \le C\,\|f\|_{Lip^1}\,, \quad P\in OPS^1_{1,0}, \quad f\in Lip^1,$$

$$\|[P,f]\|_{L^2\to H^1} \le C\,\|f\|_{Lip^1}\,, \quad P\in OPS^0_{1,0}, \quad f\in Lip^1,$$

$$\|[P,f]\|_{H^{-1}\to L^2} \le C\,\|f\|_{Lip^1} \quad P\in OPS^0_{1,0}, \quad f\in Lip^1,$$

$$\|P(fu)-fPu\|_{L^2} \le C\,\|f\|_{Lip^1}\,\|u\|_{H^{s-1}} + C\,\|f\|_{H^s}\,\|u\|_{L^\infty}\,, \tag{2.1}$$

$$P\in OPS^s_{1,0}, \quad s>0, \quad f\in Lip^1\cap H^s, \quad u\in L^\infty\cap H^{s-1}.$$

**Lemma 2.3.2.** *Let $J_\varepsilon$ be the smoothing operator from Definition 2.2.1. Then the assertions of the previous proposition hold for $P=J_\varepsilon$ with a constant $C$ independent of $\varepsilon$, $0<\varepsilon\le 1$.*

Exploiting the above estimates, we come to the central result of this section: commutator estimates for operators with *non-smooth, classical* symbols.

**Proposition 2.3.3.** *Let $a(x,D)\in OPC^1_bS^\alpha_{cl}$, $b(x,D)\in OPC^1_bS^\beta_{cl}$ with $\alpha,\beta\in\{0,1\}$. Then it holds (with some $N$)*

$$\|[a(x,D),b(x,D)]\|_{H^{\alpha+\beta-1}\to L^2}$$
$$\le C\left(\sum_{j=0}^{N-1}\pi^{\alpha-j}_{N,C^1_b}(a_j)+\pi^{\alpha-N}_{N,C^1_b}(r_{a,N})\right)\left(\sum_{j=0}^{N-1}\pi^{\beta-j}_{N,C^1_b}(b_j)+\pi^{\beta-N}_{N,C^1_b}(r_{b,N})\right),$$

*where $a_j$, $b_j$ are the homogeneous components of the expansions of $a$, $b$ with remainders $r_{a,N}$, $r_{b,N}$, respectively. If $\alpha=\beta=0$, then we additionally have*

$$\|[a(x,D),b(x,D)]\|_{L^2\to H^1}$$
$$\le C\left(\sum_{j=0}^{N-1}\pi^{\alpha-j}_{N,C^1_b}(a_j)+\pi^{\alpha-N}_{N,C^1_b}(r_{a,N})\right)\left(\sum_{j=0}^{N-1}\pi^{\beta-j}_{N,C^1_b}(b_j)+\pi^{\beta-N}_{N,C^1_b}(r_{b,N})\right).$$

The key idea of the proof is the following. Since the symbol $a(x,\xi)$ is *classical*, it allows the expansion (in local coordinates)

$$a(x,\xi)=\sum_{j=0}^{N-1}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,n)}a_{jlm}(x)Y_{lm}(\xi)\langle\xi\rangle^{\alpha-j}+r_{a,N}(x,\xi),$$

where $Y=Y_{lm}(\xi)$, $0\le l<\infty$, $1\le m\le h(l,n)$, are the spherical harmonics, i.e., the eigenfunctions of the Laplace–Beltrami operator on the unit sphere $S^{n-1}$, which form an orthogonal basis of $L^2(S^{n-1})$. The sequence $\{a_{jlm}\}_{l,m}$ is rapidly decreasing in the sense that

$$\sum_{m=1}^{h(l,n)}\|a_{jlm}\|_{C^1_b}\le C_k\pi^{\alpha-j}_{N,C^1_b}(a_j)(1+l)^{-k}, \quad k\ge 0,$$

for all $l \geq 0$, and $N$ sufficiently large. Moreover, $r_{a,N} \in C_b^1 S_{1,0}^{\alpha-N}$. Plugging this expansion and a similar one for $b(x,\xi)$ into the commutator $[a,b]$, and employing the above commutator estimates from Proposition 2.3.1, one can derive the desired estimates. For details, see [33].

Now we want to generalize (2.1), replacing $f \in Lip^1 \cap H^s$ by $A(x,D) \in OPC_b^1 S_{cl}^\alpha \cap OPH^{s_0} S_{cl}^\alpha$ with $s_0 > n/2$, $\alpha \in \mathbb{N}_0$. Here we run into a problem, since an operator $P$ from $OPS_{1,0}^s$ does generally not map $C_b^s(M)$ into $C_b^0(M)$. For this reason we introduce the space $C_{\sharp,K_0}^\alpha$ of all functions $u$ satisfying $\langle D\rangle^\alpha Y_{lm}(D)u \in C_b^0$ for all $l,m$ such that $\sup_{l,m}(1+l)^{-K_0} \left\|\langle D\rangle^\alpha Y_{lm}(D)u\right\|_{C_b^0} < \infty$. The constant $K_0$ is fixed in such a manner that $\left\|Y_{lm}(D)u\right\|_{C_b^0} \leq C(1+l)^{K_0} \left\|u\right\|_{C_b^0}$ for all $u \in C_0^\infty(M)$. The profit of this definition is that the mapping $B : C_{\sharp,K_0}^\alpha \to C_b^0$ is continuous for all $B \in OPS_{cl}^\alpha$. The embedding $C_b^{\alpha+\delta} \subset C_{\sharp,K_0}^\alpha$ is continuous for any positive $\delta$, see [33], p.126. We have the (set–theoretical) inclusions $C_{\sharp,K_0}^\alpha \subset C_b^\alpha \subset C_*^\alpha$.

**Proposition 2.3.4.** *Let $P \in OPS_{1,0}^s$, $A(x,D) \in OPC_b^1 S_{cl}^\alpha \cap OPH^{s_0} S_{cl}^\alpha$ with $s_0 > n/2$, $0 < s \leq s_0$, $\alpha \in \mathbb{N}_0$ and $K \geq K_0$. Then it holds*

$$\left\|[P, A(x,D)]\, u\right\|_{L^2} \leq C \left( \sum_{j=0}^{N-1} \pi_{N,C_b^1}^{\alpha-j}(a_j) + \pi_{N,C_b^1}^{\alpha-N}(r_N) \right) \left\|u\right\|_{H^{s+\alpha-1}} + C_K \left( \sum_{j=0}^{N-1} \pi_{N,H^s}^{-j}(a_j) + \pi_{N,H^s}^{\alpha-N}(r_N) \right) \left\|u\right\|_{C_{\sharp,K}^\alpha}$$

*with some constant $N$ and the terms $a_j$, $r_N$ from the asymptotic expansion of the classical operator $A$.*

A proof can be found in [33]. Now we list properties of the spaces $C_{\sharp,K}^\alpha$.

**Lemma 2.3.5.** *For every $\alpha \in \mathbb{N}_0$, a positive constant $C$ exists such that*

$$\left\|u\right\|_{C_b^\alpha} + \left\|\langle D\rangle^\alpha u\right\|_{C_b^0} \leq C \left\|u\right\|_{C_{\sharp,K_0}^\alpha}, \quad u \in C_{\sharp,K_0}^\alpha. \tag{2.2}$$

*Let $\sigma \in C_b^\infty$, $u \in C_{\sharp,K_0}^\alpha$ and $K_1 > K_0$ be sufficiently large. Then $\sigma u \in C_{\sharp,K_1}^\alpha$ and a constant $C = C(\sigma,\alpha)$ (independently of $u$) exists with*

$$\left\|\sigma u\right\|_{C_{\sharp,K_1}^\alpha} \leq C \left\|u\right\|_{C_{\sharp,K_0}^\alpha} \tag{2.3}$$

### 2.4. Adjoint Operators and Compositions

Pseudodifferential operators with symbols of finite smoothness form an algebra in the sense of the following propositions, whose proofs can be found in [33].

**Proposition 2.4.1.** *Let $K(x,D) \in OPC_b^1 S_{cl}^1$ be a matrix pseudodifferential operator. Then the adjoint operator $K^*(x,D)$ satisfies*

$$\operatorname{symb}(K^*(x,D) - R) = \overline{K(x,\xi)^T}, \quad \left\|RU\right\|_{L^2} \leq C\pi_{N,C_b^1}^1(K) \left\|U\right\|_{L^2}.$$

*with some operator $R$ and some $N > 0$.*

**Proposition 2.4.2.** *Let* $A(x,D) \in OPC_b^0 S_{cl}^j$, $B(x,D) \in OPC_b^1 S_{cl}^{1-j}$ *(*$j = 0$ *or* $j = 1$*) be pseudodifferential matrix operators. Then*

$$A(x,D)B(x,D) = C(x,D) + R,$$
$$C(x,\xi) = A(x,\xi)B(x,\xi) \in C_b^0 S_{cl}^1,$$
$$\|RU\|_{L^2} \le C\pi^j_{N,C_b^0}(A)\pi^{1-j}_{N,C_b^1}(B)\,\|U\|_{L^2}.$$

## 3. Weakly Hyperbolic Cauchy Problems with Spatial Degeneracy

### 3.1. The Linear Case

We are concerned with the linear Cauchy problem

$$D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j \left(\sigma(x)^\alpha u\right) = f(x,t), \tag{3.1}$$
$$u(x,t_0) = \varphi_0(x), \ldots, D_t^{m-1}u(x,t_0) = \varphi_{m-1}(x),$$

under the following condition of hyperbolicity:

**Condition 3.1.** *We assume that the roots* $\tau_j(x,t,\xi)$ *of the equation*

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\xi^\alpha \tau^j = 0$$

*are real and distinct,* $|\tau_j(x,t,\xi) - \tau_i(x,t,\xi)| \ge c|\xi|$, $c > 0$, *for* $i \ne j$, *and all* $(x,t,\xi)$.

*Remark* 3.1.1. The space variable $x$ lives on some manifold $M$, where either $M = \mathbb{R}^n$ or $M$ is a smooth closed $n$–dimensional manifold, and (3.1) is to be understood in local coordinates. Of special importance is the case of $M$ being a torus, $M = (\mathbb{R}/2\pi)^n$: during the investigation of domains of dependence (in Section 5), we will transfer a hyperbolic equation which is defined in some bounded domain of $\mathbb{R}^n \times [0,T]$ into a hyperbolic equation defined on $(\mathbb{R}/2\pi)^n \times [0,T]$, and bring into play the results to be proved now.

Marking the regularity of the data with subscript "d", we suppose that

$$\varphi_j \in H^{s_d+m-1-j}(M), \quad s_d \ge 0, \tag{3.2}$$
$$f \in C([t_0,T], H^{s_d}(M)). \tag{3.3}$$

The weight function $\sigma$ is presumed to be real–valued and smooth,

$$\sigma \in C_b^\infty(M,\mathbb{R}). \tag{3.4}$$

In case of $M$ being a closed manifold, we assume

$$a_{j,\alpha} \in C([t_0,T], H^{s_c}(M)) \cap C^1([t_0,T], H^{s_c-1}(M)), \quad s_c > \frac{n}{2} + 1, \tag{3.5}$$

where the subscript "c" means "coefficient". And if $M = \mathbb{R}^n$, we assume that there be constants $C_{j,\alpha}$ such that

$$a_{j,\alpha} - C_{j,\alpha} \in C([t_0,T], H^{s_c}(M)) \cap C^1([t_0,T], H^{s_c-1}(M)), \quad s_c > \frac{n}{2} + 1. \tag{3.6}$$

The reason for this distinction is that functions of $H^{s_c}(\mathbb{R}^n)$ have to decay at infinity, making Condition 3.1 impossible to hold. For unity of notation, we may define $C_{j,\alpha} = 0$ in the first case, and $A_{j,\alpha} = a_{j,\alpha} - C_{j,\alpha}$ for general $M$.

Our approach is as follows. We insert a smoothing operator (see Subsection 2.2) into (3.1), such that we obtain an ordinary differential equation for a function with values in a Banach space. Then this equation will be transformed into a first order pseudodifferential system. An *a priori* estimate and an existence result will be proved for this regularized system, see Proposition 3.1.3 (a), and an *a priori* estimate for the corresponding non-regularized system will be shown in Proposition 3.1.3 (b).

The question of *existence* of a solution to (3.1) will be answered in Section 3.2, *after* we have investigated a quasilinear version of (3.1).

**3.1.1.** CONSTRUCTION OF A FIRST ORDER SYSTEM For $0 < \varepsilon \le 1$, we consider a regularized version of (3.1):

$$\begin{aligned} D_t^m u_\varepsilon &= f(x,t) - J_\varepsilon \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_x^\alpha J_\varepsilon^{|\alpha|} D_t^j \left(\sigma(x)^\alpha u_\varepsilon\right), \\ u_\varepsilon(x,t_0) &= \varphi_0(x), \dots, D_t^{m-1} u_\varepsilon(x,t_0) = \varphi_{m-1}(x). \end{aligned} \tag{3.7}$$

The operator $J_\varepsilon$ maps $H^r$ into $H^{r+1}$ for any $r \in \mathbb{R}$ with norm $\mathcal{O}(\varepsilon^{-1})$. This allows us to regard (3.7) as a linear Banach space ODE which is globally solvable, and we acquire a unique solution

$$u_\varepsilon \in C^m([t_0,T], H^{\min(s_c,s_d)}(M)).$$

If we succeed in finding estimates of $u_\varepsilon$ which do not depend on $\varepsilon$, then there is hope that the limit $\lim_{\varepsilon\to 0} u_\varepsilon$ in the corresponding topologies exists and is a solution to (3.1). We recall that the $S^0_{1,0}$ seminorms of the pseudodifferential symbol of $J_\varepsilon$ can be estimated uniformly in $\varepsilon$, $0 \le \varepsilon \le 1$.

We define the vector of unknowns

$$U_\varepsilon = (U_{\varepsilon,1}, \dots, U_{\varepsilon,m})^T, \quad U_{\varepsilon,j} = (\langle D\rangle J_\varepsilon)^{m-j} \left(\sigma^{m-j} D_t^{j-1} u_\varepsilon\right), \tag{3.8}$$

and get the system

$$\begin{aligned} D_t U_{\varepsilon,j} =& \langle D\rangle J_\varepsilon (\sigma U_{\varepsilon,j+1}) + \langle D\rangle J_\varepsilon \left[ (\langle D\rangle J_\varepsilon)^{m-j-1}, \sigma\right] (\langle D\rangle J_\varepsilon)^{j+1-m} U_{\varepsilon,j+1}, \\ D_t U_{\varepsilon,m} =& f - J_\varepsilon \sum_{j+|\alpha|=m, j<m} a_{j,\alpha} P^\alpha \langle D\rangle (\sigma U_{\varepsilon,j+1}) \\ & - J_\varepsilon \sum_{j+|\alpha|=m, j<m} a_{j,\alpha} P^\alpha \langle D\rangle \left[ (\langle D\rangle J_\varepsilon)^{|\alpha|-1}, \sigma\right] (\langle D\rangle J_\varepsilon)^{1-|\alpha|} U_{\varepsilon,j+1} \end{aligned}$$

with $P_j = D_{x_j}\langle D\rangle^{-1}$, $P^\alpha = \prod_{j=1}^n P_j^{\alpha_j}$. This gives

(3.9) $$\partial_t U_\varepsilon = K_\varepsilon(\sigma U_\varepsilon) + B_\varepsilon U_\varepsilon + F, \quad U_\varepsilon(t_0) = \Phi_{\varepsilon,0},$$

(3.10) $$K_\varepsilon = J_\varepsilon K_0 \langle D\rangle = J_\varepsilon i \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ k_0 & k_1 & \dots & k_{m-1} \end{pmatrix} \langle D\rangle \in OPC_b^1 S_{cl}^1,$$

$$k_j = -\sum_{|\alpha|=m-j} a_{j,\alpha} P^\alpha,$$

(3.11) $$B_\varepsilon = i \begin{pmatrix} 0 & b_\varepsilon^{(2)} & 0 & \dots & 0 \\ 0 & 0 & b_\varepsilon^{(3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_\varepsilon^{(m)} \\ b_{\varepsilon,1} & b_{\varepsilon,2} & b_{\varepsilon,3} & \dots & 0 \end{pmatrix} \in OPC_b^1 S_{cl}^0,$$

$$b_{\varepsilon,k} = -J_\varepsilon \sum_{|\alpha|=m+1-k} a_{k-1,\alpha} P^\alpha \langle D\rangle \left[(\langle D\rangle J_\varepsilon)^{m-k}, \sigma\right] (\langle D\rangle J_\varepsilon)^{k-m},$$

$$b_\varepsilon^{(j)} = \langle D\rangle J_\varepsilon \left[(\langle D\rangle J_\varepsilon)^{m-j}, \sigma\right] (\langle D\rangle J_\varepsilon)^{j-m},$$

(3.12) $$F = (0, 0, \dots, 0, if)^T \in C\left([t_0, T], H^{s_d}\right),$$

(3.13) $$\Phi_{\varepsilon,0} = ((\langle D\rangle J_\varepsilon)^{m-1}(\sigma^{m-1}\varphi_0), \dots, \varphi_{m-1})^T \in H^{s_d}.$$

We remark that $K_0\langle D\rangle$ is a *strictly* hyperbolic operator, and does not depend on $\varepsilon$. Next we construct a symmetrizer for $K_0$, using ideas from Leray [20]. We introduce the notations $p_j = \xi_j\langle\xi\rangle^{-1}$, $p^\alpha = \prod_{j=1}^n p_j^{\alpha_j}$, and denote the eigenvalues of $K_0(x,t,p)$ by $i\tau_j(x,t,p)$. Obviously,

$$K_0\left(1, \tau_j, \dots, \tau_j^{m-1}\right)^T = i\tau_j\left(1, \tau_j, \dots, \tau_j^{m-1}\right)^T.$$

Let $S_0 = V(\tau_1(x,t,p), \dots, \tau_m(x,t,p))$ be the Vandermonde matrix of the numbers $(\tau_1, \dots, \tau_m)$, which satisfies

$$K_0 S_0 = i S_0 \operatorname{diag}(\tau_1, \dots, \tau_m) = i S_0 D.$$

We put $s_k(x,t,p) = \sum_{j=1}^m \tau_j(x,t,p)^k$, and see that the matrix

$$S = S_0 S_0^T = \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_{m-1} \\ s_1 & s_2 & s_3 & \dots & s_m \\ s_2 & s_3 & s_4 & \dots & s_{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_m & s_{m+1} & \dots & s_{2m-2} \end{pmatrix}$$

is symmetric and positive definite. Vieta's theorem reveals that the $s_k$ are some polynomials in $a_{j,\alpha}p^\alpha$. The symmetrizer is defined as $R = \det(S)S^{-1}$, and is obviously a symmetric positive definite matrix. It remains to check that $RK_0$ is symmetric: The matrix $K_0S$ is symmetric since

$$K_0S = K_0S_0S_0^T = iS_0DS_0^T = (iS_0DS_0^T)^T = (K_0S)^T.$$

Setting $c = \det(S)$, we see that $R$ is a symmetrizer for $K_0$:

$$\begin{aligned} RK_0 &= cS^{-1}K_0 = cS^{-1}(K_0S)S^{-1} = cS^{-1}(K_0S)(S^{-1})^T \\ &= (cS^{-1}(K_0S)(S^{-1})^T)^T = (RK_0)^T \end{aligned}$$

The components $r_{ij}$ of $R$ are some polynomials of the $a_{j,\alpha}p^\alpha$, that is,

$$r_{ij}(x,t,p) = \sum_{l\in B_{ij}} c_{ijl}\Bigg(\prod_{(j,\alpha)\in D_{ijl}} a_{j,\alpha}(x,t)\Bigg)\Bigg(\prod_{(j,\alpha)\in D_{ijl}} p^\alpha\Bigg), \tag{3.14}$$

with $c_{ijl} \in \mathbb{C}$ and some finite index sets $B_{ij}$ and $D_{ijl}$. Since the $\tau_k(x,t,p)$ depend on $p_j = \xi_j\langle\xi\rangle^{-1}$, we have $R(x,t,\xi) \in C_b^1 S_{cl}^0$. The property of $R$ being a symmetrizer implies (see [20])

$$C^{-1}\|V\|_{L^2}^2 \le (RV,V)_{L^2(M)} \le C\|V\|_{L^2}^2, \quad V \in L^2. \tag{3.15}$$

The product structure of the $k_{ij}$ and $r_{ij}$ gives us the estimates

$$\max\{\|K_0\|_{L^2\to L^2}, \|K_0\langle D\rangle\|_{C_\sharp^1\to C_b^0}\} \le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}), \tag{3.16}$$

$$\|R\|_{L^2\to L^2} \le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}), \tag{3.17}$$

where the term $C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})$ denotes a universal constant which depends on $\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0}$ in a nonlinear way.

Next we consider $(K_0\langle D\rangle)^*R + RK_0\langle D\rangle$. Proposition 2.4.1 gives us an expression of $(K_0\langle D\rangle)^*$, and Proposition 2.4.2 tells us how to compose $(K_0\langle D\rangle)^*$ and $R$, as well as $R$ and $K_0\langle D\rangle$. This way, we see that the principal symbol of $(K_0\langle D\rangle)^*R + RK_0\langle D\rangle$ falls out, and obtain

$$\|(K_0\langle D\rangle)^*R + RK_0\langle D\rangle\|_{L^2\to L^2} \le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1}+1). \tag{3.18}$$

Finally, mapping properties of the matrix operator $B_\varepsilon$ are studied. The assumption $\sigma \in C_b^\infty$ implies $b_\varepsilon^{(j)} \in OPS_{cl}^0$; hence $\left\|b_\varepsilon^{(j)}v\right\|_{H^s} \le C\|v\|_{H^s}$, uniformly in $\varepsilon$. Similarly, $\|b_{\varepsilon,k}v\|_{L^2} \le C\max_{j,\alpha}\|a_{j,\alpha}\|_{L^\infty}\|v\|_{L^2}$, which yields

$$\|B_\varepsilon U\|_{L^2} \le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\|U\|_{L^2}. \tag{3.19}$$

If $s > 0$, then we can make use of formula (3.1.59) from [33],

$$\|uv\|_{H^s} \le C\left(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}\right), \quad s > 0,$$

and the estimates

$$\begin{aligned} \left\|P^\alpha\langle D\rangle\left[(\langle D\rangle J_\varepsilon)^{m-k},\sigma\right](\langle D\rangle J_\varepsilon)^{k-m}v\right\|_{H^s} &\le C\|v\|_{H^s}, \\ \left\|P^\alpha\langle D\rangle\left[(\langle D\rangle J_\varepsilon)^{m-k},\sigma\right](\langle D\rangle J_\varepsilon)^{k-m}v\right\|_{L^\infty} &\le C\|v\|_{C^0_{\sharp,K_0}}, \end{aligned}$$

which give us the uniform in $\varepsilon$ estimates

$$\|B_\varepsilon U\|_{H^s} \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \|U\|_{H^s} + C \max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s} + 1) \|U\|_{C_\sharp^0} . \tag{3.20}$$

Let us summarize the results:

**Proposition 3.1.2.** *The regularized linear Cauchy problem* (3.1) *can be transformed into the equivalent system* (3.9) *with* $U_\varepsilon$, $K_\varepsilon$, $B_\varepsilon$, $F$, $\Phi_{\varepsilon,0}$ *from* (3.8), (3.10), (3.11), (3.12) *and* (3.13), *respectively.*

*The matrix operator* $K_0\langle D\rangle$ *is a* strictly *hyperbolic pseudodifferential operator with finite smoothness,* $K_0\langle D\rangle \in OPC_b^1 S_{cl}^1$. *In case* $M = \mathbb{R}^n$, *a matrix pseudodifferential operator* $\tilde{K}$ *exists, whose symbol does not depend on* $x$, *such that* $K_0\langle D\rangle - \tilde{K} \in OPH^{s_c} S_{cl}^1$. *In case of a compact manifold* $M$, *we have* $K_0\langle D\rangle \in OPH^{s_c} S_{cl}^1$.

*Furthermore, a symmetrizer* $R$ *assigned to* $K_0\langle D\rangle$ *exists, which is a zero order pseudodifferential operator with symbol of finite smoothness,* $R \in OPC_b^1 S_{cl}^0$, *and induces a norm in* $L^2$ *which is equivalent to the usual norm, see* (3.15).

*The operators* $K_0\langle D\rangle$, $R$, $(K_0\langle D\rangle)^* R + RK_0\langle D\rangle$ *and* $B_\varepsilon$ *have the mapping properties given in* (3.16)–(3.20), *respectively.*

**3.1.2.** A–Priori Estimates Now we have all tools to show an *a priori* estimate of strictly hyperbolic type:

**Proposition 3.1.3.** **(a)** *The linear system* (3.9) *has a unique global solution* $U_\varepsilon \in C^1([t_0,T], H^{\min(s_c,s_d)}(M))$ *which satisfies the following estimates for* $0 \le s \le \min(s_c, s_d)$*:*

$$\begin{aligned}
&\partial_t \left(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon\right) \\
&\quad \le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0}) \|U_\varepsilon\|_{H^s}^2 + 2\sqrt{(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon)}\sqrt{(R\langle D\rangle^s F, \langle D\rangle^s F)} \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U_\varepsilon\|_{H^s}^2 \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s} + 1) \|U_\varepsilon\|_{H^s} \|U_\varepsilon\|_{C_{\sharp,K_0}^1} .
\end{aligned}$$

**(b)** *Consider the system of type* (3.9) *which we obtain from* (3.7) *in the way of Subsection 3.1.1, replacing everywhere* $J_\varepsilon$ *by the identity operator. Let* $U \in C([t_0,T], H^{\min(s_c,s_d)}(M)) \cap C^1([t_0,T], H^{\min(s_c,s_d)-1}(M))$ *be a solution of such a system and* $0 \le s \le \min(s_c, s_d) - 1$. *Then*

$$\begin{aligned}
&\partial_t \left(R\langle D\rangle^s U, \langle D\rangle^s U\right) \\
&\quad \le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0}) \|U\|_{H^s}^2 + 2\sqrt{(R\langle D\rangle^s U, \langle D\rangle^s U)}\sqrt{(R\langle D\rangle^s F, \langle D\rangle^s F)} \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U\|_{H^s}^2 \\
&\qquad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s} + 1) \|U\|_{H^s} \|U\|_{C_{\sharp,K_0}^1} .
\end{aligned}$$

*Proof of (a)* We can write

$$\begin{aligned}&\partial_t\,(R\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)\\&=(R_t\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)+(R\langle D\rangle^s(K_\varepsilon\sigma U_\varepsilon+B_\varepsilon U_\varepsilon+F),\langle D\rangle^s U_\varepsilon)\\&\quad+(R\langle D\rangle^s U_\varepsilon,\langle D\rangle^s(K_\varepsilon\sigma U_\varepsilon+B_\varepsilon U_\varepsilon+F))\,.\end{aligned}$$

It is easy to estimate the first term on the right:

$$\|R_t\langle D\rangle^s U_\varepsilon\|_{L^2}\le C(\max_{j,\alpha}\|\partial_t a_{j,\alpha}\|_{C_b^0})\,\|U_\varepsilon\|_{H^s}\,.$$

Since $(R\cdot,\cdot)$ is a scalar product of $L^2$, the Cauchy–Schwarz inequality yields

$$\begin{aligned}&|\,(R\langle D\rangle^s F,\langle D\rangle^s U_\varepsilon)+(R\langle D\rangle^s U_\varepsilon,\langle D\rangle^s F)\,|\\&\qquad\le 2\sqrt{(R\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)}\sqrt{(R\langle D\rangle^s F,\langle D\rangle^s F)}.\end{aligned}$$

From the formulas (3.20) and (3.17) we see that

$$\begin{aligned}&|\,(R\langle D\rangle^s B_\varepsilon U_\varepsilon,\langle D\rangle^s U_\varepsilon)+(R\langle D\rangle^s U_\varepsilon,\langle D\rangle^s B_\varepsilon U_\varepsilon)\,|\\&\quad\le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\,\|U_\varepsilon\|_{H^s}^2+C\max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s}+1)\,\|U_\varepsilon\|_{C^0_{\sharp,K_0}}\,\|U_\varepsilon\|_{H^s}\,.\end{aligned}$$

It remains to consider the terms

$$I_1=(R\langle D\rangle^s K_\varepsilon\sigma U_\varepsilon,\langle D\rangle^s U_\varepsilon)\,,\quad I_2=(R\langle D\rangle^s U_\varepsilon,\langle D\rangle^s K_\varepsilon\sigma U_\varepsilon)\,.$$

The scalar product $I_1$ can be written in the form

$$\begin{aligned}I_1=&I_{11}+I_{12}+I_{13}+I_{14}+I_{15}+I_{16}+I_{17}+I_{18}+I_{19}\\=&(RJ_\varepsilon\,[\langle D\rangle^s,K_0]\,\langle D\rangle\sigma U_\varepsilon,\langle D\rangle^s U_\varepsilon)+\left(RJ_\varepsilon K_0\left[\langle D\rangle^{s+1},\sigma\right]U_\varepsilon,\langle D\rangle^s U_\varepsilon\right)\\&+\left(RJ_\varepsilon\,[K_0,\sigma]\,\langle D\rangle^{s+1}U_\varepsilon,\langle D\rangle^s U_\varepsilon\right)+(RJ_\varepsilon\sigma\,[K_0,\langle D\rangle]\,\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)\\&+(R\,[J_\varepsilon,\sigma]\,\langle D\rangle K_0\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)+([R,\sigma]\,\langle D\rangle J_\varepsilon K_0\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)\\&+(\sigma\,[R,J_\varepsilon]\,\langle D\rangle K_0\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)+(\sigma J_\varepsilon R\,[\langle D\rangle,K_0]\,\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)\\&+(\sigma J_\varepsilon RK_0\langle D\rangle\langle D\rangle^s U_\varepsilon,\langle D\rangle^s U_\varepsilon)\,.\end{aligned}$$

We estimate now $I_{11},\dots,I_{18}$. From (3.17), Proposition 2.3.4 and Lemma 2.3.5 it can be deduced that

$$\begin{aligned}|I_{11}|\le&\, C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1}+1)\,\|U_\varepsilon\|_{H^s}^2\\&+C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s}+1)\,\|\sigma U_\varepsilon\|_{C^1_{\sharp,K_1}}\,\|U_\varepsilon\|_{H^s}\\\le&\, C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1}+1)\,\|U_\varepsilon\|_{H^s}^2\\&+C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s}+1)\,\|U_\varepsilon\|_{C^1_{\sharp,K_0}}\,\|U_\varepsilon\|_{H^s}\,.\end{aligned}$$

From (3.16), (3.17), and $\left[\langle D\rangle^{s+1},\sigma\right]\in OPS^s_{1,0}$ we can conclude that

$$|I_{12}|\le C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\,\|U_\varepsilon\|_{H^s}^2\,.$$

From (3.17) and Proposition 2.3.3 ($\alpha = \beta = 0$) it follows that

$$|I_{13}| \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|\sigma\|_{C_b^1} \, \|U_\varepsilon\|_{H^s}^2 \, .$$

By (3.17) and Proposition 2.3.3 ($\alpha = 0$, $\beta = 1$) we have

$$|I_{14}| + |I_{18}| \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \, .$$

We use (3.17), Lemma 2.3.2, Proposition 2.3.3 ($\alpha = \beta = 0$), (3.16), and get

$$\begin{aligned} &|I_{15}| + |I_{16}| + |I_{17}| \\ &\quad \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|\langle D\rangle K_0 \langle D\rangle^s U_\varepsilon\|_{H^{-1}} \, \|U_\varepsilon\|_{H^s} \\ &\quad \le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \, . \end{aligned}$$

Summing up shows

$$\begin{aligned} |I_1 - I_{19}| &\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \\ &\quad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s} + 1) \, \|U_\varepsilon\|_{H^s} \, \|U_\varepsilon\|_{C^1_{\sharp,K_0}} \, . \end{aligned}$$

The scalar product $I_2$ can be decomposed into

$$\begin{aligned} &(R\langle D\rangle^s U_\varepsilon, \langle D\rangle^s J_\varepsilon K_0 \langle D\rangle \sigma U_\varepsilon) = I_{21} + I_{22} + I_{23} + I_{24} \\ &\quad = (R\langle D\rangle^s U_\varepsilon, J_\varepsilon \, [\langle D\rangle^s, K_0] \, \langle D\rangle \sigma U_\varepsilon) + (R\langle D\rangle^s U_\varepsilon, J_\varepsilon K_0 \langle D\rangle \, [\langle D\rangle^s, \sigma] \, U_\varepsilon) \\ &\quad + (R\langle D\rangle^s U_\varepsilon, [J_\varepsilon, K_0\langle D\rangle] \, \sigma \langle D\rangle^s U_\varepsilon) + (\sigma J_\varepsilon (K_0\langle D\rangle)^* R \langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \, , \end{aligned}$$

where we used the self–adjointness of $J_\varepsilon$ and the fact that $\sigma$ is real–valued. Similarly as above we obtain

$$\begin{aligned} |I_2 - I_{24}| &\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \\ &\quad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s} + 1) \, \|U_\varepsilon\|_{H^s} \, \|U_\varepsilon\|_{C^1_{\sharp,K_0}} \, . \end{aligned}$$

Finally, (3.18) yields

$$\begin{aligned} |I_{19} + I_{24}| &= |\, (\sigma J_\varepsilon (R K_0 \langle D\rangle + (K_0\langle D\rangle)^* R) \langle D\rangle^s U_\varepsilon, \langle D\rangle^s U_\varepsilon) \,| \\ &\le C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \, \|U_\varepsilon\|_{H^s}^2 \, . \end{aligned}$$

Summing up we obtain the estimate of **(a)**.

*Proof of (b).* We verify this estimate in a similar way as the previous one replacing the operators $J_\varepsilon$ by the identity operator.

The Proposition 3.1.3 is proved. □

*Remark* 3.1.4. The restriction $s \le \min(s_c, s_d) - 1$ in the part (b) (instead of $s \le \min(s_c, s_d)$ in the part (a)) can be explained as follows: The attempt to estimate $\partial_t \, (R\langle D\rangle^{s_0} U, \langle D\rangle^{s_0} U)$ ($s_0 = \min(s_c, s_d)$) leads to a term $(\langle D\rangle^{s_0} K \sigma U, \langle D\rangle^{s_0} U)$ which does in general not exist, if $U(\cdot, t) \in H^{s_0}$.

Or, seen from a different perspective: it is well–known [9] that the assumptions (3.2), (3.3), (3.5), (3.6) lead to a solution $U \in C([0,T], H^{s_0})$ in the strictly hyperbolic case $\sigma \equiv 1$. Then the energy $E_{s_0}(t) = (R(t)\langle D\rangle^{s_0}U(t), \langle D\rangle^{s_0}U(t))$ is a continuous function of $t$, but in general not $C^1$. Hence one can not expect the estimate from the part (b) to hold for $s = s_0$.

*Remark* 3.1.5. In case of $s = 0$, we can slightly improve the estimates of Proposition 3.1.3: we may replace (3.20) by (3.19); and in the estimates of $I_{11}$ and $I_{21}$, we substitute Proposition 2.3.4 with Proposition 2.3.3, leading to

$$\begin{aligned}\partial_t (RU, U) &\le C(\max_{j,\alpha} \|\partial_t a_{j,\alpha}\|_{C_b^0}) \|U\|_{L^2}^2 + 2\sqrt{(RU,U)}\sqrt{(RF,F)} \\ &\quad + C(\max_{j,\alpha} \|a_{j,\alpha}\|_{C_b^0}) \max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1} + 1) \|U\|_{L^2}^2\end{aligned}$$

for $s_d \ge 1$. The advantage is that no Sobolev norm of the coefficients $a_{j,\alpha}$ appears, and we can weaken the assumptions (3.5), (3.6) to $a_{j,\alpha} \in C_b^1([t_0,T]\times M)$. A similar estimate holds for operators with lower order terms $D_x^\alpha D_t^j(\sigma^{|\alpha|}u)$, $j + |\alpha| \le m-1$, and will be used to study domains of dependence.

### 3.2. The Quasilinear Case

Now we consider the quasilinear Cauchy problem with spatial degeneracy

$$\begin{aligned}&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta(c_{k,\beta}^0(x,t)D_t^k u)\}) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|}u\right) \\ &\quad = f(x,t,\{D_x^\beta(c_{k,\beta}^0(x,t)D_t^k u)\}|_{k+|\beta|\le m-1}), \quad m \ge 2, \\ &u(x,t_0) = \varphi_0(x), \dots, D_t^{m-1}u(x,t_0) = \varphi_{m-1}(x),\end{aligned} \tag{3.21}$$

where the real–valued function $\sigma \in C_b^\infty(M)$ describes the degeneracy, which occurs at the zeroes of $\sigma$, and the weight functions $c_{k,\beta}^0$ characterize the Levi conditions as follows:

$$c_{k,\beta}^0 \in C^1([t_0,T], H^{s_c+|\beta|}(M)), \tag{3.22}$$

$$\left\|(\partial_t c_{k,\beta}^0(\cdot,t))v(\cdot)\right\|_{H^{s+|\beta|}} \le C\left\|c_{k,\beta}^0(\cdot,t)v(\cdot)\right\|_{H^{s+|\beta|}}, \tag{3.23}$$

$$\left\|c_{k,\beta}^0(\cdot,t)v(\cdot)\right\|_{H^{s+|\beta|}} \le C\left\|c_{k+1,\beta}^0(\cdot,t)v(\cdot)\right\|_{H^{s+|\beta|}}, \tag{3.24}$$

$$c_{k,\beta}^0(x,t) = \sigma(x)^{|\beta|}, \quad k + |\beta| = m-1, \tag{3.25}$$

for $s_c > n/2 + 1$, $s_c \ge s \ge 0$.

*Example.* We give some examples of $c_{k,\beta}^0$, $0 \le k + |\beta| \le m-1$:

- $c_{k,\beta}^0(x,t) = \sigma(x)^{|\beta|}$
- $c_{k,\beta}^0 \in C^1([t_0,T], H^{s_c+|\beta|}(M))$ with the property that $(\partial_t c_{k,\beta}^0)/c_{k,\beta}^0$ and $c_{k,\beta}^0/c_{k+1,\beta}^0$ belong to $C^1([t_0,T], H^{s_c+|\beta|}(M)) + C^1([t_0,T], C_b^{s_c+|\beta|}(M))$

However, coefficients $c_{k,\beta}^0$ with non-continuous quotients $(\partial_t c_{k,\beta}^0)/c_{k,\beta}^0$ and $c_{k,\beta}^0/c_{k+1,\beta}^0$ are also possible: let $M = \mathbb{R}/2\pi$ be the unit circle, and set $\sigma(x) =$

$\exp(-(\sin(x))^{-2})$, which has zeroes of infinite order for $x \in \{0, \pi\}$. Then the following weight functions $c^0_{k,\beta} \in C_b^\infty([0,T] \times M)$ are admissible:

$$c^0_{k,\beta}(x,t) = \begin{cases} \sigma(x)^{|\beta|} & : k + |\beta| = m-1, \\ (t+k+1)\sigma(x)^{|\beta|} & : x \in [0,\pi), \quad k+|\beta| \leq m-2, \\ (2t+k+2)\sigma(x)^{|\beta|} & : x \in [\pi, 2\pi), \quad k+|\beta| \leq m-2. \end{cases}$$

We suppose that the coefficients and the right–hand side are defined in a suitable neighborhood $K_G$ of the initial data,

$$K_G = \{(x, \{v_{k,\beta}\}) \in M \times \mathbb{R}^{n_0} : |v_{k,\beta} - D_x^\beta(c^0_{k,\beta}(x,t_0)\varphi_k(x))| \leq G\}, \tag{3.26}$$

for some $G > 0$. Next, we introduce some set $X_G \subset H^{m-1}(M \times [t_0, T])$ containing those functions $v$ which can be reasonably inserted into $a_{j,\alpha}$ and $f$:

$$v \in X_G \Leftrightarrow (x, \{D_x^\beta(c^0_{k,\beta}(x,t)D_t^k v(x,t))\}) \in K_G \quad \forall (x,t) \in M \times [t_0, T].$$

Our regularity assumptions on the right–hand side $f$ and coefficients $a_{j,\alpha}$ are:

- The mapping

$$\begin{aligned} &\cap_{k=0}^{m-1} C^k([t_0,T], H^{s+m-1-k}(M)) \cap X_G \to C([t_0,T], H^s(M)), \\ &v(x,t) \mapsto f(x,t,\{D_x^\beta(c^0_{k,\beta}(t,x)D_t^k v(t,x))\}), \end{aligned} \tag{3.27}$$

  is bounded and continuous for every $s$ with $n/2 + 1 < s \leq s_c$.
- There are constants $C_{j,\alpha}$ such that the mappings

$$\begin{aligned} &\cap_{k=0}^{m-1} C^k([t_0,T], H^{s+m-1-k}) \cap X_G \to \cap_{k=0}^1 C^k([t_0,T], H^{s-k}), \\ &v(x,t) \mapsto a_{j,\alpha}(x,t,\{D_x^\beta(c^0_{k,\beta}(t,x)D_t^k v(t,x))\}) - C_{j,\alpha}, \end{aligned} \tag{3.28}$$

  are bounded and continuous for every $s$ with $n/2 + 1 < s \leq s_c$.

*Remark* 3.2.1. In case of a bounded manifold $M$, these conditions are satisfied if $f \in C([t_0,T], C^{s_c}(K_G))$ and $a_{j,\alpha} \in C^1([t_0,T], C^{s_c}(K_G))$. If $M = \mathbb{R}^n$, appropriate decays of $f(x,t,\{v_{k,\beta}\})$ and $a_{j,\alpha}(x,t,\{v_{k,\beta}\}) - C_{j,\alpha}$ for $|x| \to \infty$ are required.

The initial data are supposed to satisfy

$$\varphi_j \in H^{s_c+m-1-j}(M), \quad j \leq m-1. \tag{3.29}$$

Finally, we assume the hyperbolicity of the Cauchy problem (3.21):

**Condition 3.2.** *The roots* $\tau_j(x,t,v,\xi)$ *of*

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,v)\xi^\alpha \tau^j = 0$$

*are real and distinct,* $|\tau_j(x,t,v,\xi) - \tau_i(x,t,v,\xi)| \geq c|\xi|$, $c > 0$, $i \neq j$, *for all* $(t,x,v,\xi) \in [t_0,T] \times K_G \times \mathbb{R}^n$.

The main result of this subsection is the following theorem:

**Theorem 3.2.2.** *Under the above assumptions, there is a $T_0$, $t_0 < T_0 \le T$, such that the Cauchy problem* (3.21) *has a uniquely determined solution $u$ with*

$$\langle D\rangle^k(\sigma^k D_t^{m-1-k}u) \in C([t_0,T_0],H^{s_c}(M)) \cap C^1([t_0,T_0],H^{s_c-1}(M))$$

*for $0 \le k \le m-1$. This solution persists as long as the vector of weighted derivatives $(x,\{D_x^\beta(c^0_{k,\beta}(x,t)D_t^k u(x,t))\})$ stays in $K_G$ for all $x$ and the norms $\left\|\langle D\rangle^k(\sigma^k D_t^{m-k-1}u)\right\|_{C^1_*}$ remain finite, $0 \le k \le m-1$.*

The following well–posedness result in $C^\infty$ is an immediate consequence:

**Theorem 3.2.3.** *Consider the Cauchy problem* (3.21) *assuming* (3.4) *and $\varphi_j \in H^\infty(M)$. Furthermore, we assume that* (3.22)*–*(3.25)*,* (3.27)*,* (3.28) *hold for all $s \ge 0$, $s_c > n/2+1$ and that Condition 3.2 is satisfied. Then the Cauchy problem* (3.21) *has a unique solution*

$$u \in C^m([t_0,T_0],C^\infty(M)).$$

If the equation is linear, then we have *global* existence:

*Remark* 3.2.4. Proposition 3.1.3 and Theorem 3.2.2 give us a unique solution $u$ to the linear Cauchy problem (3.1) which can not blow up, i.e.,

$$\langle D\rangle^k(\sigma^k D_t^{m-1-k}u) \in C([t_0,T],H^{s_0}(M)) \cap C^1([t_0,T],H^{s_0-1}(M))$$

for $0 \le k \le m-1$ and $s_0 = \min(s_c,s_d)$.

Of course, we can include lower order terms $D_x^\beta(c^0_{k,\beta}(x,t)D_t^k u(x,t))$ as long as the equation remains linear.

The proof of Theorem 3.2.2 comprises the Propositions 3.2.5–3.2.8. At first, we construct a regularized hyperbolic first order system for a vector of weighted derivatives up to the order $m-1$. Employing the ideas from Subsection 3.1 we prove the existence and estimates of the solution $U_\varepsilon^*$ to this perturbed system. Then we show that the life–span of the $U_\varepsilon^*$ does not tend to zero as $\varepsilon$ approaches zero. For $\varepsilon \to 0$, the $U_\varepsilon$ converge to a solution $U$ of the asserted regularity. Last, the blow–up criterion will be proved.

**3.2.1.** Construction of a First Order System We start with a regularized version of (3.21),

$$\begin{aligned} &D_t^m u_\varepsilon + J_\varepsilon \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta(c^0_{k,\beta}J_\varepsilon^{|\beta|}D_t^k u_\varepsilon)\})D_x^\alpha J_\varepsilon^\alpha D_t^j\left(\sigma^{|\alpha|}u_\varepsilon\right)\\ &\quad = f(x,t,\{D_x^\beta(c^0_{k,\beta}J_\varepsilon^{|\beta|}D_t^k u_\varepsilon)\}), \qquad (3.30)\\ &u_\varepsilon(x,t_0) = \varphi_0(x),\dots,D_t^{m-1}u_\varepsilon(x,t_0) = \varphi_{m-1}(x). \end{aligned}$$

This is a quasilinear ODE for a function $u_\varepsilon$ with values in the Banach space $H^{s_c}(M)$; hence it has a solution

$$u_\varepsilon \in C^m([t_0,T_\varepsilon],H^{s_c}(M)), \quad T_\varepsilon > t_0.$$

We define $U_{\varepsilon,k} = (\langle D\rangle J_\varepsilon)^{m-k}(\sigma^{m-k} D_t^{k-1} u_\varepsilon)$ as in (3.8), and

$$U_{\varepsilon,l,k}(x,t) = (\langle D\rangle J_\varepsilon)^l (\sigma(x)^l D_t^k u_\varepsilon(x,t)), \quad k+l \le m-2,$$
$$U_{\varepsilon,k,\beta}(x,t) = \langle D\rangle^{|\beta|}(c^0_{k,\beta}(x,t) J_\varepsilon^{|\beta|} D_t^k u_\varepsilon(x,t)), \quad k+|\beta| \le m-2.$$

Obviously,

$$\partial_t U_{\varepsilon,k,\beta} = \langle D\rangle^{|\beta|}((\partial_t c^0_{k,\beta}) J_\varepsilon^{|\beta|} D_t^k u_\varepsilon) + i\langle D\rangle^{|\beta|}(c^0_{k,\beta} J_\varepsilon^{|\beta|} D_t^{k+1} u_\varepsilon).$$

Assuming $k+|\beta| \le m-3$, the $H^s$ norm of the right–hand side is bounded by

$$\begin{aligned}(3.31)\qquad & \left\|(\partial_t c^0_{k,\beta}) J_\varepsilon^{|\beta|} D_t^k u_\varepsilon\right\|_{H^{s+|\beta|}} + \left\|c^0_{k,\beta} J_\varepsilon^{|\beta|} D_t^{k+1} u_\varepsilon\right\|_{H^{s+|\beta|}} \\ & \le C\left\|c^0_{k,\beta} J_\varepsilon^{|\beta|} D_t^k u_\varepsilon\right\|_{H^{s+|\beta|}} + C\left\|c^0_{k+1,\beta} J_\varepsilon^{|\beta|} D_t^{k+1} u_\varepsilon\right\|_{H^{s+|\beta|}} \\ & = C\left\|U_{\varepsilon,k,\beta}\right\|_{H^s} + C\left\|U_{\varepsilon,k+1,\beta}\right\|_{H^s}.\end{aligned}$$

And for $k+|\beta| = m-2$, we obtain

$$\begin{aligned}(3.32)\qquad & \left\|(\partial_t c^0_{k,\beta}) J_\varepsilon^{|\beta|} D_t^k u_\varepsilon\right\|_{H^{s+|\beta|}} + \left\|c^0_{k,\beta} J_\varepsilon^{|\beta|} D_t^{k+1} u_\varepsilon\right\|_{H^{s+|\beta|}} \\ & \le C\left\|c^0_{k,\beta} J_\varepsilon^{|\beta|} D_t^k u_\varepsilon\right\|_{H^{s+|\beta|}} + C\left\|\sigma^{|\beta|} J_\varepsilon^{|\beta|} D_t^{k+1} u_\varepsilon\right\|_{H^{s+|\beta|}} \\ & \le C\left\|U_{\varepsilon,k,\beta}\right\|_{H^s} + C\sum_{l\le|\beta|}\left\|(\langle D\rangle J_\varepsilon)^l(\sigma^l D_t^{k+1} u_\varepsilon)\right\|_{H^s},\end{aligned}$$

see (3.22)–(3.25). We introduce the vector

$$U^*_\varepsilon = (\{U_{\varepsilon,k,\beta}\},\ \{U_{\varepsilon,l,k}\},\ U_\varepsilon^T)^T$$

and obtain

$$\partial_t U^*_\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & K_\varepsilon(x,t,U^*_\varepsilon,D)\end{pmatrix}(\sigma U^*_\varepsilon) + \begin{pmatrix} G_\varepsilon(x,t,U^*_\varepsilon) \\ F_\varepsilon(x,t,U^*_\varepsilon)\end{pmatrix}$$

with $\|G_\varepsilon\|_{H^s} \le C\,\|U^*_\varepsilon\|_{H^s}$, cf. (3.31), (3.32). This system can be written as

$$(3.33)\qquad \partial_t U^*_\varepsilon = K^*_\varepsilon(x,t,U^*_\varepsilon,D)(\sigma U^*_\varepsilon) + F^*_\varepsilon(x,t,U^*_\varepsilon), \quad U^*_\varepsilon(t_0) = \Phi^*_\varepsilon.$$

The matrix $R^*(x,t,U^*,D) = \operatorname{diag}(E, R(x,t,U^*,D))$ is a symmetrizer for $K^*$, where $E$ is the identity matrix and $R$ is the (independent of $\varepsilon$) symmetrizer from Subsection 3.1.

**3.2.2.** Estimates and Common Existence Interval According to Theorem 3.2.2, the solution $U^*_\varepsilon \in C^1([t_0,T_\varepsilon], H^{s_c})$ persists as long as it stays in $K_G$

and as long as $\|U_\varepsilon^*\|_{H^{s_c}} < \infty$. Applying Proposition 3.1.3 and $C^{1,\delta} \subset C^1_{\sharp,K_0}$ we get

$$\begin{aligned}
&\partial_t\left(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*\right)\\
&\quad\le C(\max_{j,\alpha}\|\partial_t a_{j,\alpha}\|_{C_b^0})\,\|U_\varepsilon^*\|_{H^s}^2\\
&\qquad+2\sqrt{(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*)}\sqrt{(R^*\langle D\rangle^s F_\varepsilon^*, \langle D\rangle^s F_\varepsilon^*)}\\
&\qquad+C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|a_{j,\alpha}\|_{C_b^1}+1)\,\|U_\varepsilon^*\|_{H^s}^2\\
&\qquad+C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\max_{j,\alpha}(\|A_{j,\alpha}\|_{H^s}+1)\,\|U_\varepsilon^*\|_{H^s}\,\|U_\varepsilon^*\|_{C_b^{1,\delta}}
\end{aligned}$$

for $s_c \ge s > n/2+1+\delta$, $0<\delta<1$. The Moser–type estimates

$$\begin{aligned}
\|a_{j,\alpha}\|_{C_b^1} &\le C(\|U_\varepsilon^*\|_{C_b^0})(\|U_\varepsilon^*\|_{C_b^1}+1),\\
\|A_{j,\alpha}\|_{H^s} &\le C(\|U_\varepsilon^*\|_{L^\infty})(\|U_\varepsilon^*\|_{H^s}+1),
\end{aligned}$$

and the embedding inequality $\|U_\varepsilon^*\|_{C_b^{1,\delta}} \le C\,\|U_\varepsilon^*\|_{H^s}$ can be applied on the right. Let us consider the term $C(\max_{j,\alpha}\|\partial_t a_{j,\alpha}\|_{C_b^0})$ which denotes some constant that depends in a nonlinear way on $\max_{j,\alpha}\|\partial_t a_{j,\alpha}\|_{C_b^0}$. The computations which lead to this term show that it has the form

$$C(\max_{j,\alpha}\|a_{j,\alpha}\|_{C_b^0})\cdot\max_{j,\alpha}(\|\partial_t a_{j,\alpha}\|_{C_b^0}+1).$$

Appealing to Lemma 2.3.5 and (3.16), we have

$$\begin{aligned}
\|\partial_t a_{j,\alpha}\|_{C_b^0} &\le C(1+\|\partial_t U_\varepsilon^*\|_{C_b^0}) \le C(1+\|K_\varepsilon^*\sigma U_\varepsilon^*\|_{C_b^0}+\|F_\varepsilon^*\|_{C_b^0})\\
&\le C(1+\|U_\varepsilon^*\|_{C_b^{1,\delta}}).
\end{aligned}$$

Taking into account all these inequalities we obtain

$$\begin{aligned}
\partial_t\left(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*\right) &\le C(\|U_\varepsilon^*\|_{L^\infty})(\|U_\varepsilon^*\|_{C_b^{1,\alpha}}+1)\,\|U_\varepsilon^*\|_{H^s}^2\\
&\quad+2\sqrt{(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*)}\sqrt{(R^*\langle D\rangle^s F_\varepsilon^*, \langle D\rangle^s F_\varepsilon^*)}.
\end{aligned} \tag{3.34}$$

In the next step we show indirectly that there is a common existence interval of the solutions $U_\varepsilon^*$. We define $t_\varepsilon \in (t_0, T_\varepsilon]$ by the inequality $\|U_\varepsilon^*(t)\|_{H^s} \le 2\sup_{0\le\varepsilon'\le\varepsilon_0}\|\Phi_{\varepsilon'}^*\|_{H^s}+1$, $t_0 \le t \le t_\varepsilon$; and by the condition that the components of the vector $U_\varepsilon^*(t)$ be in the interior of the domain of definition of the coefficients $a_{j,\alpha}$ and the right–hand side $f$, for $t_0 \le t \le t_\varepsilon$.

To obtain a contradiction, let us assume that for every small $\gamma>0$ an $\varepsilon=\varepsilon(\gamma)$ exists with $t_0 < t_\varepsilon \le t_0+\gamma$. Now we study estimates of $\|U_\varepsilon^*\|_{H^s}$ and $\|U_\varepsilon^*-\Phi_\varepsilon^*\|_{L^\infty}$.

The norms $\|V\|_{L^2}$ and $\sqrt{(R^*V,V)}$ are equivalent as long as $R^*$ is defined (i.e., for $t \le t_\varepsilon$). From (3.34) and $\|F_\varepsilon^*\|_{H^s} \le C(\|U_\varepsilon^*\|_{L^\infty})(\|U_\varepsilon^*\|_{H^s}+1)$ we see that

$$\partial_t\left(R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*\right) \le Q((R^*\langle D\rangle^s U_\varepsilon^*, \langle D\rangle^s U_\varepsilon^*)),$$

where $Q$ is a smooth nonlinear increasing function, independent of $\varepsilon$. Let $T_0$ be a number with the property that the nonnegative solutions of

$$\begin{aligned} &\partial_t y(t) \le Q(y(t)), \\ &y(t_0) = (R^*(x, t_0, \Phi^*, D)\langle D\rangle^s \Phi^*_\varepsilon, \langle D\rangle^s \Phi^*_\varepsilon), \quad 0 \le \varepsilon \le \varepsilon_0, \end{aligned} \tag{3.35}$$

satisfy $y(t) \le 2 \sup_{0 \le \varepsilon' \le \varepsilon_0} \|\Phi^*\|_{H^s} + 1$ for $t_0 \le t \le T_0$.

To estimate $\|U^*_\varepsilon - \Phi^*_\varepsilon\|_{L^\infty}$, we write $U^*_\varepsilon = \Phi^*_\varepsilon + V^*_\varepsilon$ and get, by Proposition 3.1.3,

$$\begin{aligned} &\partial_t \left(R^* V^*_\varepsilon, V^*_\varepsilon\right) \le C(\|U^*_\varepsilon\|_{L^\infty})((R^* V^*_\varepsilon, V^*_\varepsilon) + (R^* F^*_\varepsilon, F^*_\varepsilon)), \\ &\left(R^* V^*_\varepsilon, V^*_\varepsilon\right)(t_0) = 0. \end{aligned}$$

The norms $\|U^*_\varepsilon\|_{L^\infty}$ are uniformly bounded for $t \le t_\varepsilon$, due to the definition of $t_\varepsilon$. From Gronwall's lemma it can be concluded that

$$\left(R^* V^*_\varepsilon, V^*_\varepsilon\right)(t) \le g(t)^2, \quad t_0 \le t \le \min(T_0, t_\varepsilon),$$

$g(t_0) = 0$, $g$ continuous and increasing. We obtain $\|V^*_\varepsilon(t)\|^2_{L^2} \le C g(t)^2$; and an interpolation argument gives us a continuous function $g_1$, such that

$$\|V^*_\varepsilon(t)\|_{L^\infty} \le g_1(t), \quad g_1(t_0) = 0. \tag{3.36}$$

This demonstrates that $t_\varepsilon$ cannot come arbitrarily close to $t_0$, which is a contradiction. Hence there is a common existence interval.

We have proved:

**Proposition 3.2.5.** *There is a constant $T_0 > t_0$ with the property that the systems* (3.33) *have unique solutions $U^*_\varepsilon \in C^1([t_0, T_0], H^s)$ for $0 < \varepsilon \le \varepsilon_0$ and $s_c \ge s > n/2 + 1$. It holds for all $0 < \varepsilon \le \varepsilon_0$ and all $t_0 \le t \le T_0$*

$$\|U^*_\varepsilon(t)\|_{H^s} \le C, \quad \|U^*_\varepsilon(t) - \Phi^*\|_{L^\infty} \le g_1(t)$$

*with some continuous function $g_1(t)$, $g_1(t_0) = 0$.*

**3.2.3.** Convergence and Regularity of the Limit Our previous *a priori* estimates allow us to show

$$\|U^*_\varepsilon(\cdot, t) - U^*_{\varepsilon'}(\cdot, t)\|^2_{L^2} \le C(T_0 - t_0)(\varepsilon + \varepsilon'), \quad t_0 \le t \le T_0.$$

By the uniform bound $\|U^*_\varepsilon\|_{H^{s_c}} \le C$ and interpolation, the following result is easily established:

**Proposition 3.2.6.** *The above constructed sequence $(U^*_\varepsilon) \subset C^1([t_0, T_0], H^{s_c})$ converges in $C([t_0, T_0], H^s)$ and $C([t_0, T_0], C_b^{1,\delta})$ for any $s$ and $\delta$ with $n/2 + 1 + \delta < s < s_c$. The limit $U^*$ belongs to $C^1\left([t_0, T_0], H^{s-1}\right)$ and is a solution of* (3.33) *with $\varepsilon = 0$.*

It remains to study the regularity of $U^*$. Here we make use of standard techniques, which can be found e.g. in [28]. The uniform estimate of $U^*_\varepsilon$ in $H^{s_c}$ gives $U^* \in L^\infty([t_0, T_0], H^{s_c})$ and $U^* \in Lip^1([t_0, T_0], H^{s_c - 1})$. We want to show that the $H^{s_c}$ norm of $U^*$ is not only bounded, but also continuous:

**Proposition 3.2.7.** *The above constructed solution $U^*$ to* (3.33) *with $\varepsilon = 0$ belongs to $C\left([t_0, T_0], H^{s_c}\right) \cap C^1\left([t_0, T_0], H^{s_c-1}\right)$.*

*Proof.* We fix $t_0 \le t_1 < T_0$ and will show the continuity of $\|U^*(\cdot, t)\|_{H^{s_c}}$ at the point $t_1$. To this end, we consider the forward Cauchy problem (recycling the variable $U_\varepsilon^*$ which we do not need anymore)

$$\partial_t U_\varepsilon^* = K_\varepsilon^*(x, t, U_\varepsilon^*, D)(\sigma U_\varepsilon^*) + F_\varepsilon^*(x, t, U_\varepsilon^*), \quad U_\varepsilon^*(t_1) = U^*(t_1),$$

for some small $\varepsilon > 0$. The only difference to (3.33) is the initial condition. Defining the equivalent norm for the Hilbert space $H^{s_c}$

$$\|V\|^2_{H^{s_c}, t_1} = (R^*(t_1)\langle D\rangle^{s_c} V, \langle D\rangle^{s_c} V) \tag{3.37}$$

we deduce from (3.34) that

$$\partial_t \|U_\varepsilon^*(t)\|^2_{H^{s_c}, t} \le C'(\|U_\varepsilon^*\|_{C_b^{1,\delta}}) \|U_\varepsilon^*(t)\|^2_{H^{s_c}, t} + C \|F_\varepsilon^*(t)\|^2_{H^{s_c}}.$$

Gronwall's lemma gives, with some $C' = C'(\|U_\varepsilon^*\|_{C_b^{1,\delta}})$,

$$\|U_\varepsilon^*(t)\|^2_{H^{s_c}, t} \le \|U^*(t_1)\|^2_{H^{s_c}, t_1} e^{C'(t-t_1)} + C \int_{t_1}^t e^{C'(t-\tau)} \|F_\varepsilon^*(\tau)\|^2_{H^{s_c}} d\tau.$$

The weak compactness of bounded subsets in Hilbert spaces implies ($\varepsilon \to 0$)

$$\|U^*(t)\|^2_{H^{s_c}, t} \le \|U^*(t_1)\|^2_{H^{s_c}, t_1} e^{C'(t-t_1)} + C|t - t_1|.$$

The function $\|V\|_{H^{s_c}, t}$ defines an equivalent norm in the Hilbert space $H^{s_c}$ if $t$ is fixed. For proving continuity in $t$ we need a norm which does not depend on $t$. Using Lipschitz continuity of $R^*$, we find

$$\|U^*(t)\|^2_{H^{s_c}, t_1} \le \|U^*(t_1)\|^2_{H^{s_c}, t_1} e^{C'(t-t_1)} + C''|t - t_1|,$$

resulting in

$$\limsup_{t \to t_1 + 0} \|U^*(t)\|^2_{H^{s_c}, t_1} \le \|U^*(t_1)\|^2_{H^{s_c}, t_1} \le \liminf_{t \to t_1 + 0} \|U^*(t)\|^2_{H^{s_c}, t_1},$$

which gives the $H^{s_c}$–continuity of $U^*$ at $t_1$ from the right. Inverting the time direction the reader can show the continuity from the left. Since $t_1$ can be chosen arbitrarily, the proof is complete. □

In the last step a criterion for the blow–up is given. The idea of the proof is taken from [33], Proposition 5.1.F.

**Proposition 3.2.8.** *Let $U^* \in C\left([t_0, T), H^{s_c}\right) \cap C^1\left([t_0, T), H^{s_c-1}\right)$ be a solution of* (3.33) ($\varepsilon = 0$) *with*

$$\sup_{t \in [t_0, T)} \|U^*(t)\|_{C^1_*} < \infty,$$

$$\inf_{t \in [t_0, T)} \operatorname{dist}((x, \{U_{k,\beta}(x, t)\}), \partial K_G) \ge \delta > 0.$$

*Then a constant $T_1 > T$ exists with $U^* \in C\left([t_0, T_1], H^{s_c}\right) \cap C^1\left([t_0, T_1], H^{s_c-1}\right)$.*

*Proof.* We consider the non–regularized version of (3.33), apply $J_\varepsilon$ to it, and estimate the terms on the right as in the proof of Proposition 3.1.3. This gives

$$\begin{aligned} (3.38)\qquad &\partial_t \left(R^* \langle D\rangle^{s_c} J_\varepsilon U^*, \langle D\rangle^{s_c} J_\varepsilon U^*\right) \\ &\qquad \le C(\|U^*\|_{L^\infty})(1 + \|U^*\|_{C_b^1} + \|U^*\|_{C^1_{\sharp,K_0}})\, \|U^*\|^2_{H^{s_c}} + \|F^*\|^2_{H^{s_c}}. \end{aligned}$$

We suppose that the $H^{s_c}$ norm is arranged in such a way that $\|V^*\|_{C^1_*} \le \|V^*\|_{H^{s_c}}$ holds for every function $V^* \in H^{s_c}$. Then the inequality

$$\|V^*\|_{C^1_{\sharp,K_0}} \le C \|V^*\|_{C^1_*} \left(1 + \ln\left(\frac{\|V^*\|_{H^{s_c}}}{\|V^*\|_{C^1_*}}\right)\right)$$

can be shown, see [33], (B.2.12). Consequently,

$$\|U^*\|_{C^1_{\sharp,K_0}} \le C \|U^*\|_{C^1_*} (1 + \ln^+ \|U^*\|_{H^{s_c}}) + C.$$

According to Lemma 2.3.5, we have $\|U^*\|_{C_b^1} \le C \|U^*\|_{C^1_{\sharp,K_0}}$, and from $\|F^*\|^2_{H^{s_c}} \le C(\|U^*\|_{L^\infty})(e + \|U^*\|^2_{H^{s_c}})$ it follows that

$$\begin{aligned} &\partial_t \left(R^* \langle D\rangle^{s_c} J_\varepsilon U^*, \langle D\rangle^{s_c} J_\varepsilon U^*\right) \\ &\qquad \le C(\|U^*\|_{L^\infty})(1 + \|U^*\|_{C^1_*})(1 + \ln^+ \|U^*\|^2_{H^{s_c}})(e + \|U^*\|^2_{H^{s_c}}). \end{aligned}$$

Using the equivalent norm $\|\cdot\|_{H^{s_c},t}$ from (3.37) and $\|U^*\|_{C^1_*} \le C$, we get

$$\partial_t \|J_\varepsilon U^*(t)\|_{H^{s_c},t} \le C_0(1 + \ln^+ \|U^*(t)\|^2_{H^{s_c},t})(e + \|U^*(t)\|_{H^{s_c},t}).$$

We integrate, let $\varepsilon$ tend to 0 and see that

$$\begin{aligned} \|U^*(t)\|^2_{H^{s_c},t} &\le \|U^*(t_0)\|^2_{H^{s_c},t_0} \\ &\qquad + C_0 \int_{t_0}^t (1 + \ln^+ \|U^*(\tau)\|^2_{H^{s_c},\tau})(e + \|U^*(\tau)\|^2_{H^{s_c},\tau})\, d\tau. \end{aligned}$$

Introducing $N(t) = e + \|U^*(t)\|^2_{H^{s_c},t}$ we deduce that

$$N(t) \le N(t_0) + 2C_0 \int_{t_0}^t \ln(N(\tau))N(\tau)\, d\tau = Q(t).$$

The continuity of $N(t)$ yields $Q(t) \in C^1[t_0, T)$. Obviously,

$$Q'(t) = 2C_0 \ln(N(t))N(t) \le 2C_0 \ln(Q(t))Q(t);$$

hence $Q(t) \le C$ for all $t_0 \le t < T$. Taking into account all these inequalities we find $\|U^*(t)\|_{H^{s_c}} \le C'$ for $t_0 \le t < T$.

Next we have to extend $U^*$ continuously to some longer time interval. Therefore, we review the regularized Cauchy problems (3.33) with data $U^*_\varepsilon(T - \gamma) = U^*(T - \gamma)$ for some small $\gamma > 0$. The functions $U^*_\varepsilon(t)$ persist as long as their $H^{s_c}$ norms remain bounded and as long as each component of these vectors stays in the domain of the functions $a_{j,\alpha}$ and $f$. An estimate of the life–span of $U^*_\varepsilon$ is provided by (3.35) and (3.36), which are autonomous (differential) inequalities (independent of $\varepsilon$), hence the length of the existence interval only depends on $\delta$ and $C'$. It follows that for small $\gamma$ the point $T$ is contained in the common existence interval of the $U_\varepsilon^*$, and consequently, of the limit $U$. □

This completes the proof of Theorem 3.2.2.

*Remark* 3.2.9. Crucial for the proof of Proposition 3.2.8 was the fact that the right–hand side of (3.38) grows at most linearly in $\|U^*\|_{C^1_*}$ and $\|U^*\|_{C^1_{\sharp,K_0}}$.

*Remark* 3.2.10. Using similar arguments as above, one can easily show that the solution of a quasilinear weakly hyperbolic Cauchy problem continuously depends on the data, weight functions, coefficients and right–hand side.

## 4. Weakly Hyperbolic Cauchy Problems with Spatial and Time Degeneracy

Now we are ready to study the general equation (1.7) which incorporates both types of degeneracy: spatial degeneracy and time degeneracy.

Our approach is divided into three steps:

- First, we study a *linear* Cauchy problem with vanishing initial data, whose right–hand side has a zero of sufficiently high order at $t = 0$. We establish an estimate of *strictly hyperbolic type* in Theorem 4.1.1.
- Secondly, we consider a *quasilinear* Cauchy problem. Its initial data also vanish, and the right–hand side has a zero of high order at $t = 0$. The estimates of strictly hyperbolic type for the linear problem and the usual iteration procedure imply local existence, see Theorem 4.2.6.
- Thirdly, we transform the general equation (1.7) into a special equation which can be treated with the methods of the second step, see Theorem 4.3.1.

The idea of transforming a weakly hyperbolic problem with *general* right–hand side into another weakly hyperbolic problem with *special* right–hand side has been widely used, for example in Kajitani, Yagdjian [17], Oleinik [25] and Reissig [29].

### 4.1. A Special Linear Case

We analyze the Cauchy problem

$$D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\lambda(t)^{|\alpha|} D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) = f(x,t), \tag{4.1}$$
$$u(x,0) = \cdots = D_t^{m-1} u(x,0) = 0$$

under the decay assumption

$$\|f(\cdot,t)\|_{H^{s_0}} \le C_f \lambda(t)^p \lambda'(t). \tag{4.2}$$

Later, the number $p$ will be presumed sufficiently large. Additionally, we suppose Condition 3.1, (3.3), (3.4), and (3.5) or (3.6). For the function $\lambda = \lambda(t)$ we assume:

**Condition 4.1.** *Fix* $\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau$. *The function* $\lambda = \lambda(t)$ *satisfies one of the following conditions:*

- $\lambda(t) = t^l$, $l \in \mathbb{N}$, $l \geq m-1$,
- $\lambda \in C^2([0,T])$, $\lambda(0) = 0$, $\lambda'(t) > 0$ $(t > 0)$, $\lambda''(t) \geq 0$ $(t \geq 0)$,

$$\frac{\lambda(t)}{\Lambda(t)} \leq C_\lambda \frac{\lambda'(t)}{\lambda(t)}, \quad \frac{\lambda'(t)}{\lambda(t)} \leq C'_\lambda \frac{\lambda(t)}{\Lambda(t)}, \quad C_\lambda < \frac{m}{m-1}.$$

Examples for the second case are $\Lambda(t) = \exp(-|t|^{-r})$, $r > 0$.

The central result is the following theorem:

**Theorem 4.1.1.** *If the constant p is sufficiently large, then the Cauchy problem* (4.1) *has a solution u with the property that*

$$U \in C\left([0,T], H^{s_c}\right) \cap C^1\left([0,T], H^{s_c-1}\right),$$
$$U = \{\langle D\rangle^{m-i}((\lambda\sigma)^{m-i} D_t^{i-1} u) : i = 1, \ldots, m\}.$$

*There are constants* $C_1$, $C_2$, $C_3$ *(independent of u) such that the estimate*

$$\|U(t)\|_{H^{s_c}} \leq C_3 \int_0^t e^{C_1(t-\tau)} \left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{C_2} \|f(\tau)\|_{H^{s_c}}\,d\tau$$

*holds for* $p > C_2 + 1$.

*Proof.* We obtain the system

$$\begin{aligned}
\partial_t U_j &= (m-j)\frac{\lambda'}{\lambda} U_j + \lambda i \langle D\rangle(\sigma U_{j+1}) \\
&\qquad + \lambda i \langle D\rangle \left[\langle D\rangle^{m-j-1}, \sigma\right] \langle D\rangle^{j+1-m} U_{j+1}, \quad 1 \leq j < m, \\
\partial_t U_m &= -i \sum_{j+|\alpha|=m, j<m} a_{j,\alpha} \lambda^{|\alpha|} D_x^\alpha D_t^j \left(\sigma^{|\alpha|} u\right) + if \\
&= -i\lambda \sum_{j+|\alpha|=m, j<m} a_{j,\alpha} P^\alpha \langle D\rangle(\sigma U_{j+1}) \\
&\quad - i\lambda \sum_{j+|\alpha|=m, j<m} a_{j,\alpha} P^\alpha \langle D\rangle \left[\langle D\rangle^{|\alpha|-1}, \sigma\right] \langle D\rangle^{1-|\alpha|} U_{j+1} + if.
\end{aligned}$$

This leads to

$$\partial_t U = \lambda K_0 \langle D\rangle(\sigma U) + \lambda B U + H\frac{\lambda'}{\lambda} U + F, \quad U(0) = 0 \tag{4.3}$$

with $K_0\langle D\rangle$, $B$, $F$ as in (3.10), (3.11), (3.12) and $H = \operatorname{diag}(m-1, m-2, \ldots, 1, 0)$. We regularize this system and eliminate the time degeneracy:

$$\begin{aligned}
&\partial_t U_{\delta\varepsilon} = J_\varepsilon(\lambda+\delta) K_0 \langle D\rangle(\sigma U_{\delta\varepsilon}) + \lambda B U_{\delta\varepsilon} + H\frac{\lambda'}{\lambda+\delta} U_{\delta\varepsilon} + F, \\
&U_{\delta\varepsilon}(0) = 0,
\end{aligned} \tag{4.4}$$

where $\delta > 0$ is small. This is a weakly hyperbolic system with pure spatial degeneracy; therefore we can take advantage of the methods of Subsection 3.1. We may choose the same symmetrizer $R$, since the function $\lambda+\delta$ has no influence on the operator $K_0\langle D\rangle$. This operator does not feel the time degeneracy.

We choose $s=0$ or $n/2+1<s\le s_c$, and define the norm

$$H_s(U_{\delta\varepsilon}(t))=\sqrt{(R(t)\langle D\rangle^s U_{\delta\varepsilon}(t),\langle D\rangle^s U_{\delta\varepsilon}(t))}.$$

Then Proposition 3.1.3 gives us the estimate

$$\partial_t H_s(U_{\delta\varepsilon})\le C_1(H_s(U_{\delta\varepsilon})+\|F\|_{H^s})+C_2\frac{\lambda'}{\lambda+\delta}H_s(U_{\delta\varepsilon}). \tag{4.5}$$

By Gronwall's lemma we see that

$$\begin{aligned}H_s(U_{\delta\varepsilon}(t))&\le\int_0^t e^{C_{2,\delta}(t-\tau)}C_1\|F(\tau)\|_{H^s}\,d\tau\\&\le\exp\left(\frac{CT}{\delta}\right)C\int_0^t\lambda'(\tau)\lambda(\tau)^p d\tau\le C_\delta\lambda(t)^{p+1},\\C_{2,\delta}&=\sup_{t\in[0,T]}\left\{C_1+C_2\tfrac{\lambda'(t)}{\lambda(t)+\delta}\right\}=O(\delta^{-1}).\end{aligned}$$

This allows us to apply Nersesyan's lemma (see Lemma 6.2) to (4.5) if we assume $p>C_2+1$. The result is

$$\begin{aligned}H_s(U_{\delta\varepsilon}(t))&\le\int_0^t e^{C_1(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{C_2}C_1\|F(\tau)\|_{H^s}\,d\tau\\&\le C\lambda(t)^{C_2}\int_0^t e^{C_1(t-\tau)}\lambda(\tau)^{p-C_2}\lambda'(\tau)d\tau\\&\le\frac{Ce^{C_1 t}}{p-C_2+1}\lambda(t)^{p+1}.\end{aligned}$$

We emphasize that this estimate is independent of $\delta$ and $\varepsilon$. It is known that $U_{\delta\varepsilon}$ belongs to the spaces $C([0,T],H^{s_c})$ and $C^1([0,T],H^{s_c-1})$. Employing the methods from Subsection 3.2 one can show that there is a limit $U_\delta=\lim_{\varepsilon\to0}U_{\delta\varepsilon}$ which belongs to the same spaces and solves

$$\partial_t U_\delta=(\lambda+\delta)K_0\langle D\rangle(\sigma U_\delta)+\lambda BU_\delta+H\frac{\lambda'}{\lambda+\delta}U_\delta+F,\quad U_\delta(0)=0, \tag{4.6}$$

and that the following *a priori* estimate holds for $n/2+1<s\le s_c$ and $s=0$:

$$H_s(U_\delta(t))\le C_1\int_0^t e^{C_1(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{C_2}\|F(\tau)\|_{H^s}\,d\tau. \tag{4.7}$$

In the next step we send $\delta$ to 0 and study the convergence properties of the sequence $(U_\delta)$. The difference $U_\delta - U_{\delta'}$ solves the equation

$$\begin{aligned}
&\partial_t(U_\delta - U_{\delta'}) \\
&\quad = (\lambda+\delta')K_0\langle D\rangle(\sigma(U_\delta - U_{\delta'})) + \lambda B(U_\delta - U_{\delta'}) + H\frac{\lambda'}{\lambda+\delta'}(U_\delta - U_{\delta'}) \\
&\qquad + (\delta-\delta')\left(K_0\langle D\rangle(\sigma U_\delta) - H\frac{\lambda'}{(\lambda+\delta)(\lambda+\delta')}U_\delta\right).
\end{aligned}$$

From (4.7) with $s=0$ it can be concluded that

$$H_0((U_\delta - U_{\delta'})(t)) \le C|\delta-\delta'|\lambda(t)^p.$$

It is standard to verify that the sequence $(U_\delta)$ converges to a limit $U$ which is a solution of (4.3) and satisfies

$$U \in C\left([\gamma,T], H^{s_c}\right) \cap C^1\left([\gamma,T], H^{s_c-1}\right) \quad \forall\gamma>0.$$

On the other hand, we have $U(0)=0$ and $\|U(t)\|_{H^{s_c}} \le C\lambda(t)^{p+1}$. This gives the continuity for $t=0$ and the theorem is proved. □

### 4.2. A Special Quasilinear Case

Now we assume that the initial data vanish and that the right–hand side decays sufficiently fast for $t\to 0$. More precisely, we consider the Cauchy problem

$$\begin{aligned}
&D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\})\lambda(t)^{|\alpha|}D_x^\alpha D_t^j(\sigma(x)^{|\alpha|}u) \\
&\qquad = f(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\}|_{k+|\beta|\le m-1}), \qquad (4.8)\\
&u(x,0) = \cdots = D_t^{m-1}u(x,0) = 0
\end{aligned}$$

with the following asymptotic behavior for $f$:

$$\|f(\cdot,t,0)\|_{H^{s_c}} \le C_{f0}\lambda(t)^p\lambda'(t), \qquad (4.9)$$

$$\sum_{k+|\beta|\le m-1}\left\|\frac{\partial f}{\partial g_{k,\beta}}(\cdot,t,\{g_{k,\beta}\})\right\|_{L^\infty} \le C_f \qquad (4.10)$$

for all $t\in[0,T]$ and all $\{g_{k,\beta}\}\in\mathbb{R}^{n_0}$ from a suitable chosen compact set near zero. Furthermore, we suppose (3.4), (3.27), (3.28), (replace $c^0_{k,\beta}$ by $c_{k,\beta}$, $t_0$ by 0 and $K_G$ by some compact set near zero) and Condition 3.2. Concerning the weight functions $c_{k,\beta}$, we assume the Levi conditions

$$c_{k,\beta}(x,t) = \begin{cases} \lambda(t)^{m-k}\Lambda(t)^{k+|\beta|-m}c^0_{k,\beta}(x,t) & : |\beta|>0, \\ c^0_{k,\beta}(x,t) & : |\beta|=0, \end{cases} \qquad (4.11)$$

where $\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau$; and (3.22)–(3.25).

Our intention is to show the local existence of a solution to (4.8), see Theorem 4.2.6. For this purpose we transform the equation into an equivalent system of first order and study a linearized version of this system. In other words, we introduce $U = (U_1, \dots, U_m)^T$, $V = (V_1, \dots, V_m)^T$, $V^* = (V_{k,\beta}^T, V^T)^T$,

$$U_j = \langle D\rangle^{m-j}((\lambda\sigma)^{m-j} D_t^{j-1} u), \qquad V_j = \langle D\rangle^{m-j}((\lambda\sigma)^{m-j} D_t^{j-1} v),$$

$$V_{k,\beta}(x,t) = D_x^\beta(c_{k,\beta}(x,t) D_t^k v(x,t)) \tag{4.12}$$

and get the system

$$\partial_t U = \lambda K_0(x,t,V^*,D)\langle D\rangle\sigma U + \lambda B(x,t,V^*,D)U + F(x,t,V^*) + H\frac{\lambda'}{\lambda}U,$$
$$U(0) = 0.$$

We obtain a mapping $U = \Phi(V)$, defined via $V \mapsto V^* \mapsto U$, and can construct a sequence $\{V^k\}$ by $V^k = \Phi(V^{k-1})$, $V^0 = 0$. This sequence will be shown to converge for large $p$ and small times, after some preparatory estimates.

**4.2.1.** Auxiliary Estimates We proceed with estimating $V^*$ in terms of $V$, and $K_0$, $B$, $F$ in terms of $V^*$.

**Lemma 4.2.1.** *Let $T < 1$ and $\|V(t)\|_{H^s} \le C_{Vp}\lambda(t)^{p+1}$. Then there is a constant $C_4$, independent of $V$ and $p$, such that*

$$\left\|\langle D\rangle^{|\beta|} c_{k,\beta}(\cdot,t) D_t^k v(\cdot,t)\right\|_{H^s} \le C_4 C_{Vp}\lambda(t)^p\lambda'(t). \tag{4.13}$$

*Proof.* The assertion is obvious for $|\beta| = 0$, $k = m-1$. Now let $|\beta| = 0$, $k \le m-2$. Then (3.23) and (3.24) imply

$$\begin{aligned}\partial_t \left\|c_{k,\beta} D_t^k v\right\|_{H^s} &\le \left\|(\partial_t c_{k,\beta}) D_t^k v\right\|_{H^s} + \left\|c_{k,\beta} D_t^{k+1} v\right\|_{H^s} \\ &\le C_c \left\|c_{k,\beta} D_t^k v\right\|_{H^s} + C_c \left\|c_{k+1,\beta} D_t^{k+1} v\right\|_{H^s}.\end{aligned}$$

By Gronwall's lemma and induction, we get (4.13) for $|\beta| = 0$ and $k \le m-2$. Now $|\beta| > 0$. We choose $k = m-1-|\beta|$ as base of induction, and deduce that

$$\begin{aligned}\left\|\langle D\rangle^{|\beta|} c_{k,\beta}(\cdot,t) D_t^k v\right\|_{H^s} &= \frac{\lambda(t)}{\Lambda(t)}\left\|\lambda(t)^{|\beta|}\langle D\rangle^{|\beta|}(\sigma^{|\beta|} D_t^k v)\right\|_{H^s} \\ &\le C_\lambda \frac{\lambda'(t)}{\lambda(t)}\|V(t)\|_{H^s} \le C_\lambda C_{Vp}\lambda(t)^p\lambda'(t).\end{aligned}$$

Let $k + |\beta| \le m-2$. Making use of $\lambda' \sim \lambda^2\Lambda$ (see Condition 4.1) and the induction hypothesis, we obtain

$$\begin{aligned}\left\|c_{k+1,\beta}^0 D_t^{k+1} v\right\|_{H^{s+|\beta|}} &\le CC_{Vp}\lambda(t)^{p+k+1-m}\lambda'(t)\Lambda(t)^{m-k-|\beta|-1} \\ &\le CC_{Vp}\lambda(t)^{p+k-1-m}(\lambda'(t))^2\Lambda(t)^{m-k-|\beta|}.\end{aligned}$$

By (3.23) and (3.24) we then conclude that

$$\begin{aligned}&\partial_t \left\|c_{k,\beta}^0 D_t^k v\right\|_{H^{s+|\beta|}} \\ &\quad\le CC_{Vp}\lambda(t)^{p+k-1-m}(\lambda'(t))^2\Lambda(t)^{m-k-|\beta|} + C\left\|c_{k,\beta}^0 D_t^k v\right\|_{H^{s+|\beta|}}.\end{aligned}$$

Bringing Gronwall's lemma into play, we then find

$$\begin{aligned}\left\|c^0_{k,\beta}D^k_t v\right\|_{H^{s+|\beta|}} &\leq CC_{Vp}\int_0^t \lambda(\tau)^{p+k-1-m}(\lambda'(\tau))^2\Lambda(\tau)^{m-k-|\beta|}\,d\tau \\ &\leq CC_{Vp}\lambda(t)^{p+k-m}\Lambda(t)^{m-k-|\beta|}\lambda'(t),\end{aligned}$$

which concludes the proof. □

*Remark* 4.2.2. The conclusion of this lemma can be sharpened in the following way. If $0 \leq s \leq s_c$, then the estimates

$$\|V_{k,\beta}\|_{H^s} \leq \left\|\langle D\rangle^{|\beta|}(c_{k,\beta}D^k_t v)\right\|_{H^s} \leq \lambda(t)^p\lambda'(t)C_4 \sup_{\tau\in[0,t]}\frac{\|V(\tau)\|_{H^s}}{\lambda(\tau)^{p+1}}$$

hold for $k+|\beta| = m-1$. And for $k+|\beta| \leq m-2$ we have

$$\|V_{k,\beta}\|_{H^s} \leq \lambda(t)^p\lambda'(t)C_4 \sup_{\tau\in[0,t]}\frac{\left\|\langle D\rangle^{|\beta|}(c_{k+1,\beta}D^{k+1}_t v)\right\|_{H^s}}{\lambda(\tau)^p\lambda'(\tau)}.$$

This lemma gives us estimates for $U^*$ and $V^*$ if bounds of $U$ and $V$ are known. An estimate of $U$ in the terms of the right–hand side is given by Theorem 4.1.1. The next lemma will be practical to find an estimate of $F$ in terms of $V^*$.

**Lemma 4.2.3.** *Let $K \subset \mathbb{R}^{n_0}$ be compact and $M$ be an $n$-dimensional smooth closed manifold. Let $f \in C^N(M\times K)$ and $v_i \in H^N(M)$ with $(x, v_1(x), \dots, v_{n_0}(x)) \in M\times K$ for $x \in M$. If $N$ is sufficiently large and $0 \leq m < n_0$, then a constant $N_1 < N$ (independent of $N$) exists with*

$$\begin{aligned}&\|f(\cdot, v_1(\cdot), \dots, v_{n_0}(\cdot))\|'_{H^N} \\ &\quad\leq \varphi_N(\|v_1\|_{C^{N_1}_b}, \dots, \|v_{n_0}\|_{C^{N_1}_b}) \\ &\qquad\times(\|v_1\|_{H^N} + \dots + \|v_m\|_{H^N} + \|v_{m+1}\|_{H^{N-1}} + \dots + \|v_{n_0}\|_{H^{N-1}}) \\ &\qquad+\sum_{j=m+1}^{n_0}\left\|\frac{\partial f}{\partial v_j}(\cdot, v_1(\cdot), \dots, v_{n_0}(\cdot))\right\|_{L^\infty}\sum_{|\alpha|=N}\|\partial^\alpha_x v_j\|_{L^2} \\ &\qquad+\sum_{|\alpha|\leq N}\left\|f^{(\alpha,0,\dots,0)}(\cdot, v_1(\cdot), \dots, v_{n_0}(\cdot))\right\|_{L^2}.\end{aligned}$$

*Here we used the equivalent norm $\|w\|'_{H^N} = \sum_{|\alpha|\leq N}\|\partial^\alpha_x w\|_{L^2}$,*

$$C_N^{-1}\|w\|'_{H^N} \leq \|w\|_{H^N} \leq C_N\|w\|'_{H^N}. \tag{4.14}$$

This lemma generalizes Remark A.1 in [11] and describes precisely the dependence of $\|f(\cdot, v_1, \dots, v_n)\|'_{H^N}$ on the highest orders of some $v_j$ (see the terms $\|\partial^\alpha_x v_j\|_{L^2}$). The proof is omitted. We will use this lemma to determine the loss of Sobolev regularity (it depends on $\left\|f_{v_j}\right\|_{L^\infty}$), or, in other words, to determine the space in which the solution exists.

From now on we assume $s_c = N \in \mathbb{N}$ and set $s_1 = N_1$.

**Lemma 4.2.4.** *Assuming* $\|\{V_{k,\beta}\}\|_{C_b^{s_1}} \le 1$, *there are constants* $C_5$ *and* $T^*$, *such that for* $0 \le t \le T^*$,

$$\|f(x,t,\{V_{k,\beta}\})\|_{H^{s_c}} \le C_5 \lambda(t)^p \lambda'(t) \left( C_f \sup_{\tau\in[0,t]} \frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}} + C_{f0} \right). \tag{4.15}$$

*Proof.* Lemma 4.2.3 and (4.14) allow us to estimate

$$\begin{aligned}
&\|f(x,t,\{V_{k,\beta}\})\|_{H^{s_c}} \le C_{s_c} \|f(x,t,\{V_{k,\beta}\})\|'_{H^{s_c}} \\
&\quad \le C_{s_c} \varphi_{s_c}(\|\{V_{k,\beta}\}\|_{C_b^{s_1}}) \|\{V_{k,\beta}\}\|_{H^{s_c-1}} + C_{s_c} \sum_{|\alpha|\le s_c} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\}) \right\|_{L^2} \\
&\qquad + C_{s_c} \sum_{k+|\beta|\le m-1} \left\| \frac{\partial f}{\partial V_{k,\beta}} \right\|_{L^\infty} \sum_{|\alpha|=s_c} \|\partial_x^\alpha V_{k,\beta}\|_{L^2}.
\end{aligned}$$

Repeated application of Remark 4.2.2 yields (for $|\beta|>0$)

$$\begin{aligned}
\|V_{k,\beta}\|_{H^{s_c-1}} &\le C\lambda(t)^p\lambda'(t) \sup_{\tau\in[0,t]} \frac{\left\| \langle D\rangle^{|\beta|} c_{m-1-|\beta|,\beta} D_t^{m-1-|\beta|} v \right\|_{H^{s_c-1}}}{\lambda(\tau)^p \lambda'(\tau)} \\
&\le C\lambda(t)^p\lambda'(t) \sup_{\tau\in[0,t]} \lambda(\tau)^{-p} \left\| \langle D\rangle^{|\beta|-1} (\lambda\sigma)^{|\beta|-1} D_t^{m-1-|\beta|} v \right\|_{H^{s_c}}.
\end{aligned}$$

Considering the time derivative of the last norm and exploiting Nersesyan's lemma, we find (restricting the time interval)

$$\|\{V_{k,\beta}\}\|_{H^{s_c-1}} \le \lambda(t)^{p+1} \sup_{\tau\in[0,t]} \frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}}.$$

Again by Remark 4.2.2,

$$\sum_{|\alpha|=s_c} \|\partial_x^\alpha V_{k,\beta}\|_{L^2} \le C\lambda(t)^p\lambda'(t) \sup_{\tau\in[0,t]} \frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}}.$$

Lastly, from Hadamard's formula and (4.9) it can be deduced that

$$\begin{aligned}
&\sum_{|\alpha|\le s_c} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\}) \right\|_{L^2} \\
&\quad \le \sum_{|\alpha|\le s_c} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,\{V_{k,\beta}\}) - \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,0) \right\|_{L^2} + \sum_{|\alpha|\le s_c} \left\| \frac{\partial^\alpha f}{\partial x^\alpha}(x,t,0) \right\|_{L^2} \\
&\quad \le C \|V^*(t)\|_{H^{s_c-1}} + CC_{f0}\lambda(t)^p\lambda'(t).
\end{aligned}$$

Summing up, we can estimate for small $t$:

$$\begin{aligned}\|f(x,t,\{V_{k,\beta}\})\|_{H^{s_c}} &\le C_{s_c}(2+\varphi_{s_c}(1))\lambda(t)^{p+1}\sup_{\tau\in[0,t]}\frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}}\\ &\quad + CC_f\lambda(t)^p\lambda'(t)\sup_{\tau\in[0,t]}\frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}} + CC_{f0}\lambda(t)^p\lambda'(t)\\ &\le C_5\lambda(t)^p\lambda'(t)\left(C_f\sup_{\tau\in[0,t]}\frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}}+C_{f0}\right).\end{aligned}$$

□

**4.2.2.** Iteration and Convergence Now we have all tools to find a bound for the mapping $V\mapsto V^*\mapsto U$.

**Lemma 4.2.5.** *We assume that $p$ is sufficiently large and that $T^*$ is sufficiently small. If $\|V(t)\|_{H^{s_c}}\le\lambda(t)^{p+1}$ for $0\le t\le T^*$, then it holds $\|U(t)\|_{H^{s_c}}\le\lambda(t)^{p+1}$ for $0\le t\le T^*$.*

*Proof.* Due to Theorem 4.1.1 and (4.15) we have

$$\begin{aligned}\|U(t)\|_{H^{s_c}} &\le C_3\int_0^t e^{C_1(t-s)}\left(\frac{\lambda(t)}{\lambda(s)}\right)^{C_2}\|f(s)\|_{H^{s_c}}\,ds\\ &\le C_3e^{C_1t}\int_0^t\left(\frac{\lambda(t)}{\lambda(s)}\right)^{C_2}C_5\lambda(s)^p\lambda'(s)\left(C_f\sup_{\tau\in[0,s]}\frac{\|V(\tau)\|_{H^{s_c}}}{\lambda(\tau)^{p+1}}+C_{f0}\right)ds\\ &\le C_3C_5(C_f+C_{f0})e^{C_1t}\frac{\lambda(t)^{p+1}}{p-C_2+1}\\ &\le\lambda(t)^{p+1}\end{aligned}$$

if $e^{C_1t}\le 2$ and $p$ is large. □

Now we restrict the constant $T^*$ in such a manner, that the assumption $\|V(t)\|_{H^{s_c}}\le\lambda(t)^{p+1}$ for all $t\in[0,T^*]$ implies $(x,(V_{k,\beta}(x,t)))\in K_G$ for all $(x,t)\in M\times[0,T^*]$. All these results enable us to define a sequence $(V^l)\subset C\left([0,T^*],H^{s_c}\right)\cap C^1\left([0,T^*],H^{s_c-1}\right)$ by $V^0(t)\equiv 0$ and

$$\begin{aligned}&V^l(0)=0,\quad l\ge 1,\\ &\partial_tV^l=\lambda K_0(x,t,V^{*,l-1},D)\langle D\rangle\sigma V^l+\lambda B(x,t,V^{*,l-1},D)V^l\\ &\qquad\quad+F(x,t,V^{*,l-1})+H\frac{\lambda'}{\lambda}V^l,\quad l\ge 1.\end{aligned}$$

Due to Lemma 4.2.5 and Remark 4.2.2 the functions $V^l$ fulfill

$$\left\|V^l(t)\right\|_{H^{s_c}}\le\lambda(t)^{p+1},\quad\left\|V^l_{k,\beta}(t)\right\|_{H^{s_c}}\le C\lambda(t)^p\lambda'(t).$$

Using the above technique once more and choosing $p$ larger if necessary, we are able to show the estimate

$$\sup_{t\in[0,T^*]} \frac{\left\|V^{l+1}(t)-V^l(t)\right\|_{L^2}}{\lambda(t)^{p+1}} \le \frac{1}{2} \sup_{t\in[0,T^*]} \frac{\left\|V^{l}(t)-V^{l-1}(t)\right\|_{L^2}}{\lambda(t)^{p+1}}.$$

This confirms that the sequence $(V^l)$ converges in $C\left([0,T^*],H^0\right)$. By interpolation we see that $(V^l)$ converges in $C\left([0,T^*],H^{s_c-1}\right)$, too; and we can prove in a standard way that the limit $U$ is a solution of (4.8). Exploiting the arguments which gave Proposition 3.2.7, we get $U\in C\left([0,T^*],H^{s_c}\right)$. Thus, we have proved:

**Theorem 4.2.6** (Existence)**.** *Let the conditions mentioned at the beginning of this section be fulfilled. Let $s_c$, $p\in\mathbb{N}$ be sufficiently large and $T^*>0$ be sufficiently small. Then the Cauchy problem* (4.8) *has a solution $u$ with $U\in C\left([0,T^*],H^{s_c}\right)\cap C^1\left([0,T^*],H^{s_c-1}\right)$.*

### 4.3. Reduction of a General Quasilinear Equation to a Quasilinear Equation with Special Right–Hand Side

In this subsection we reflect upon a general quasilinear weakly hyperbolic Cauchy problem and find a solution using the technique of the previous subsection. Namely, we will transform the Cauchy problem

$$\begin{aligned} &D_t^m u + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\})\lambda(t)^{|\alpha|}D_x^\alpha D_t^j(\sigma(x)^{|\alpha|}u) \\ &\quad = f(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k u\}), \qquad (4.16)\\ &u(x,0)=\varphi_0(x),\dots,D_t^{m-1}u(x,0)=\varphi_{m-1}(x) \end{aligned}$$

into another Cauchy problem

$$\begin{aligned} &D_t^m v + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha,p'}(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k v\})\lambda(t)^{|\alpha|}D_x^\alpha D_t^j(\sigma(x)^{|\alpha|}v) \\ &\quad = f_{p'}(x,t,\{D_x^\beta c_{k,\beta}(x,t)D_t^k v\}), \qquad (4.17)\\ &v(x,0)=\dots=D_t^{m-1}v(x,0)=0, \end{aligned}$$

whose right–hand side $f_{p'}$ fulfills the requirements (4.9) and (4.10) with $p=p(p')$. It will be shown that these two Cauchy problems are equivalent in the sense that functions $u_1$, $u_2$, …, $u_{p'}$ exist with

$$u=u_1+u_2+\dots+u_{p'}+v.$$

The functions $u_1$, $u_2$, …, $u_{p'}$ are solutions of ODEs in $t$ with parameter $x$. If $p$ is large enough, then Theorem 4.2.6 guarantees the existence of a solution to (4.17). This idea has been used in [17] and [29].

The example of Qi [27] shows that a loss of Sobolev regularity (in comparison to the data) must be expected. This corresponds to the fact that the smoothness of the $u_j$ decreases by $m$, as $j$ increases by 1.

We list the assumptions.

The functions $c_{k,\beta}$ are assumed to satisfy (4.11) together with (3.22)–(3.25). The functions $a_{j,\alpha}$ and the right–hand side $f$ are defined in the set $K_G$; see (3.26). Finally, we suppose (3.4), (3.27), (3.28), (3.29), Condition 3.2 and Condition 4.1.

**Theorem 4.3.1** (Existence)**.** *If $s_c \in \mathbb{N}$ is large enough, then some number $T^* \in (0, T]$ and some $\gamma > 0$ (independent of $s_c$) exist with the property that there is a solution $u$ of* (4.16) *with*

$$U \in C\left([0, T^*], H^{s_c-\gamma}\right) \cap C^1\left([0, T]^*, H^{s_c-\gamma-1}\right),$$
$$U = \{\langle D\rangle^{m-i}((\lambda\sigma)^{m-i} D_t^{i-1} u) : i = 1, \dots, m\}.$$

*Proof.* We define

$$\varepsilon_{l,i,\beta} = \begin{cases} 1 & : |\beta| = 0 \text{ or } l > i, \\ 0 & : |\beta| > 0 \text{ and } l = i \end{cases}$$

and consider the system of ODEs in $t$ with parameter $x$

$$\begin{aligned}
&D_t^m u_1(x,t) = f(x, t, \{D_x^\beta c_{k,\beta}(x,t) D_t^k u_1(x,t)\varepsilon_{1,1,\beta}\}), \\
&u_1(x,0) = \varphi_0(x), \dots, D_t^{m-1} u_1(x,0) = \varphi_{m-1}(x), \\
&D_t^m u_l(x,t) = g_l(x, t, u_l(x,t), \dots, D_t^{m-1} u_l(x,t)) \\
&\quad = f\Big(x, t, \Big\{D_x^\beta c_{k,\beta}(x,t) D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i(x,t)\Big\}\Big) \\
&\qquad - f\Big(x, t, \Big\{D_x^\beta c_{k,\beta}(x,t) D_t^k \sum_{i=1}^{l-1} \varepsilon_{l-1,i,\beta} u_i(x,t)\Big\}\Big) - \\
&\qquad - \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l}(x,t) \sum_{i=1}^{l-1} \lambda(t)^{|\alpha|} D_x^\alpha D_t^j (\sigma(x)^{|\alpha|} u_i(x,t)) \\
&\qquad + \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l-1}(x,t) \sum_{i=1}^{l-2} \lambda(t)^{|\alpha|} D_x^\alpha D_t^j (\sigma(x)^{|\alpha|} u_i(x,t)), \\
&u_l(x,0) = \cdots = D_t^{m-1} u_l(x,0) = 0, \quad l = 2, \dots, p.
\end{aligned}$$

The following abbreviations have been used here:

$$\tilde{a}_{j,\alpha,l}(x,t) = a_{j,\alpha}\Big(x, t, \Big\{D_x^\beta c_{k,\beta}(x,t) D_t^k \sum_{i=1}^{l} u_i(x,t)\varepsilon_{l,i,\beta}\Big\}\Big), \quad \sum_{i=1}^{0} = 0.$$

These equations can be solved step by step, and they possess solutions $u_l \in C^m\left([0, T_l], H^{s_c-m(l-1)}\right)$, $s_c - ml > n/2 + 1$. For a proof, see Theorem 3.2.2 with $\sigma \equiv 0$.

The functions $u_l$ have a special asymptotical behavior for $t \to 0$. If $\lambda$ satisfies the second set of assumptions of Condition 4.1, we fix some positive number $\varepsilon$ with

$2\varepsilon < m-(m-1)C_\lambda$. Otherwise, $\varepsilon = 0$. Inductively, we prove

$$\sum_{j=0}^{m-1} \left\| D_t^j u_l(\cdot,t) \right\|_{H^{s_c-m(l-1)}} \le C_6 \left( t^{1/2}\lambda(t)^\varepsilon \right)^{l-1}$$

with $s_c - ml > n/2+1$. This is obvious for $l=1$. To prove a corresponding estimate for $u_{l+1}$, we consider $\|g_{l+1}(t)\|_{H^{s_c-ml}}$. Hadamard's formula yields

$$\begin{aligned} &f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l+1} \varepsilon_{l+1,i,\beta} u_i\Big\}\Big) - f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big) \\ &\quad = \sum_k d_{1lk}(x,t) c_{k,0} D_t^k u_{l+1} + \sum_{|\beta|>0,k} d_{2lk\beta}(x,t) D_x^\beta c_{k,\beta} D_t^k u_l. \end{aligned}$$

The $H^{s_c-ml}$–norms of the functions $d_{1lk}$, $d_{2lk\beta}$ are bounded, by the induction assumption and $u_{l+1} \in C^m\left([0,T_{l+1}], H^{s_c-ml}\right)$. The function $\lambda(t)^{m-2\varepsilon}\Lambda(t)^{1-m}$ is monotonically increasing due to the choice of $\varepsilon$, see Condition 4.1. This implies

$$\|c_{k,\beta}(\cdot,t)\|_{H^s} \le C_s \lambda(t)^{2\varepsilon}, \quad s \in \mathbb{R}^+, \quad |\beta|>0.$$

The other contributions to $g_{l+1}$ can be discussed similarly. We conclude that $u_{l+1}$ is a solution of

$$D_t^m u_{l+1} + \sum_{k<m} d_{3lk}(x,t) D_t^k u_{l+1} = h_{l+1}(x,t)$$

with

$$\|h_{l+1}(\cdot,t)\|_{H^{s_c-ml}} \le C_l \left(t^{1/2}\lambda(t)^\varepsilon\right)^{l-1} \lambda(t)^{2\varepsilon}, \quad \|d_{3lk}(\cdot,t)\|_{H^{s_c-ml}} \le C_l. \tag{4.18}$$

Utilizing a standard technique one shows (restricting the time interval)

$$\sum_{j=0}^{m-1} \left\| D_t^j u_{l+1}(\cdot,t) \right\|_{H^{s_c-ml}} \le C_l \int_0^t \|h_{l+1}(\cdot,\tau)\|_{H^{s_c-ml}}\, d\tau \le \left(t^{1/2}\lambda(t)^\varepsilon\right)^l.$$

This is the desired estimate. Summing up the differential equations for $u_1, \ldots, u_l$ we deduce that

$$\begin{aligned} D_t^m(u_1+\cdots+u_l) &= f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big) \\ &\quad - \sum_{j+|\alpha|=m, j<m} \tilde{a}_{j,\alpha,l} \sum_{i=1}^{l-1} \lambda^{|\alpha|} D_x^\alpha D_t^j (\sigma^{|\alpha|} u_i), \\ D_t^j(u_1+\cdots+u_l)(x,0) &= \varphi_j(x), \quad j=0,\ldots,m-1. \end{aligned}$$

We define

$$a_{j,\alpha,l}(x,t,\{D_x^\beta c_{k,\beta} D_t^k v\}) = a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big),$$

$$\begin{aligned} &f_l(x,t,\{D_x^\beta c_{k,\beta} D_t^k v\}) \\ &= f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big) - f\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big) \\ &- \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \Big(\sum_{i=1}^{l} u_i + v\Big)\Big\}\Big)\lambda^{|\alpha|} D_x^\alpha D_t^j \Big(\sigma^{|\alpha|} \sum_{i=1}^{l} u_i\Big) \\ &+ \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}\Big(x,t,\Big\{D_x^\beta c_{k,\beta} D_t^k \sum_{i=1}^{l} \varepsilon_{l,i,\beta} u_i\Big\}\Big)\lambda^{|\alpha|} D_x^\alpha D_t^j \Big(\sigma^{|\alpha|} \sum_{i=1}^{l-1} u_i\Big). \end{aligned}$$

If a function $v$ has homogeneous initial data and satisfies

$$\begin{aligned} D_t^m v + &\sum_{j+|\alpha|=m, j<m} a_{j,\alpha,l}(x,t,\{D_x^\beta c_{k,\beta} D_t^k v\})\lambda^{|\alpha|} D_x^\alpha D_t^j (\sigma^{|\alpha|} v) \\ &= f_l(x,t,\{D_x^\beta c_{k,\beta} D_t^k v\}) \end{aligned}$$

(which is (4.17) with $p' = l$), then the function $u = \sum_{i=1}^{l} u_i + v$ solves (4.16).

It remains to verify the conditions (4.9) and (4.10) for this function $f_l$: We fix $\delta > 0$ and restrict all the intervals $[0, T_l]$ in such a way that

$$\Big\| D_t^j \sum_{i=1}^{l} u_i(\cdot,t) - \varphi_j(\cdot) \Big\|_{H^{s_c - ml}} \le \delta, \quad 0 \le t \le T_l, \quad 0 \le j \le m-1.$$

The condition (4.10) is obviously satisfied for $t \le T_l$ and the constant $C_f$ only depends on $\delta$, but not on $l$. In a similar way as in the proof of (4.18) one shows

$$\|f_l(\cdot,t,0)\|_{H^{s_c - ml}} \le C\left(t^{1/2}\lambda(t)^\varepsilon\right)^l,$$

which proves (4.9).

Theorem 4.2.6 gives the existence of a solution $v$ to (4.17) which satisfies $\langle D\rangle^k((\lambda\sigma)^k v) \in C\left([0,T^*], H^{s_c - ml}\right)$, if $p$ and $s_c - ml$ are large enough. □

*Remark* 4.3.2. Since the time degeneracy occurs only for $t = 0$, the blow–up criterion of Proposition 3.2.8 is still valid; we only have to take into account the number of lost derivatives.

## 5. Domains of Dependence

We will construct so–called *domains of dependence*. It turns out that our definition generalizes the definition of [2] from the strictly hyperbolic case to the weakly hyperbolic case. These domains can be exhausted with hypersurfaces, and the

Cauchy problem is weakly hyperbolic in the normal direction at each point of each hypersurface, see Definition 5.1.2. One example of such domains is a cone, whose slope does not exceed some critical value. This concept will be applied to prove some results of uniqueness, finite propagation speed and regularity:

**Global uniqueness for linear equations:** A solution to a linear Cauchy problem is unique in any domain of dependence, see Theorem 5.2.1.

**Local uniqueness for quasilinear equations:** For every ball in the initial plane one can find a cone (with suitably small slope) over this ball with the property that a solution is unique in this cone, see Theorem 5.3.2.

**Local existence for quasilinear equations:** For every rectangle in the initial plane one can find a rectangular parallelepipedon over the rectangle with the property that a Sobolev solution of the quasilinear equation exists in this parallelepipedon, cf. Theorem 5.3.1. This solution exists in the whole domain of dependence if the equation is linear, cf. Corollary 5.3.4.

$C^\infty$ **regularity:** We consider a quasilinear Cauchy problem, whose coefficients, right-hand side, weight functions and initial data are $C^\infty$. Let us be given a Sobolev solution in some domain of dependence. Then this solution is $C^\infty$ in this domain, cf. Theorem 5.4.2.

### 5.1. Definition of Domains of Dependence

We come to the definition of a domain of dependence, see also [2].

We consider the Cauchy problem

$$\begin{aligned} D_t^m u + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) &= f(x,t), \\ u(x,0) = \varphi_0(x), \dots, D_t^{m-1} u(x,0) &= \varphi_{m-1}(x) \end{aligned} \tag{5.1}$$

for $(x,t) \in \Omega_0 \times [0,T] \subset \mathbb{R}^n_x \times \mathbb{R}_t$; $\Omega_0$ is an open and bounded domain with smooth boundary.

We suppose Condition 3.1 and

$$\sigma \in C_b^\infty(\Omega_0), \tag{5.2}$$

$$a_{j,\alpha} \in \begin{cases} C_b^1(\overline{\Omega_0} \times [0,T]) & : j + |\alpha| = m, \\ C_b^0(\overline{\Omega_0} \times [0,T]) & : j + |\alpha| < m. \end{cases} \tag{5.3}$$

The principal part of the operator on the left is

$$P_{m,\sigma}(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t)\sigma(x)^{|\alpha|} D_x^\alpha D_t^j.$$

To this operator we assign the strictly hyperbolic operator

$$P_{m,1}(x,t,D_x,D_t) = D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t) D_x^\alpha D_t^j.$$

The domain of dependence $\Omega$ over a bounded domain $\Omega_0 \subset \mathbb{R}^n$ has to satisfy the following conditions. At first,

$$\Omega = \Omega' \cap \{(x,t) : t \geq 0\}, \quad \Omega' \Subset \mathbb{R}^{n+1}, \quad \Omega' \text{ open}, \tag{5.4}$$

$$\Omega_0 = \Omega' \cap \{(x,t) : t = 0\}. \tag{5.5}$$

Next, the projections $\pi : (x,t) \mapsto x$ of the level sets $\Omega_{t_0} = \Omega' \cap \{(x,t) : t = t_0\}$ shall become "smaller" for increasing $t_0$,

$$\pi\Omega_{t_1} \Subset \pi\Omega_{t_0} \quad 0 \leq t_0 < t_1 \leq T. \tag{5.6}$$

The set $\Omega$ can be exhausted with hypersurfaces $S_r$,

$$\Omega = \bigcup_{0 \leq r < r^*} S_r = \bigcup_{0 \leq r < r^*} \{(x,t) : g(x,t) = r\}. \tag{5.7}$$

We suppose $g \in C_b^\infty(\Omega_0 \times [0,T])$ and

$$\frac{\partial g}{\partial t} > 0 \quad \text{in } \Omega. \tag{5.8}$$

Furthermore, we assume that each hypersurface $S_r$ intersects with the initial domain $\Omega_0$ and

$$\left(\Omega_0 \cap \bigcup_{0 \leq r \leq r_0} S_r\right) \Subset \left(\Omega_0 \cap \bigcup_{0 \leq r \leq r_1} S_r\right) \quad 0 \leq r_0 < r_1 < r^*. \tag{5.9}$$

Finally, we need a connection between the slope of the normal vector to $S_r$ at the point $(x,t)$ and the largest characteristic root of $P_{m,\sigma}$ at that point. Let $Q_m$ be the principal part of a hyperbolic in direction $\tau$ operator. Then the largest characteristic root $\lambda_{max}(x,t;Q_m)$ of $Q_m$ at the point $(x,t)$ is defined by

$$\lambda_{max}(x,t;Q_m) = \sup\{|\tau| : Q_m(x,t,\tau,\xi) = 0, |\xi| = 1\}.$$

The slope of the normal vector and the largest characteristic roots satisfy

$$\lambda_{max}(x,t;P_{m,\sigma}) \left| \frac{\nabla_x g(x,t)}{g_t(x,t)} \right| < 1. \tag{5.10}$$

This condition can be interpreted in the way that the polynomial $P_{m,\sigma}$ is weakly hyperbolic at the point $(x,t)$ in the normal direction of $S_r$.

For technical reasons we assume the following condition:

(5.11) The domain $\Omega_0$ has the $J$–extension property defined below.

**Definition 5.1.1.** We say that a domain $\Omega_0$ has the $J$–extension property if for every small $\varepsilon > 0$ there is an operator $\mathcal{E}_\varepsilon : C_b(\Omega_0) \to C_b(\mathbb{R}^n)$ such that

- $(\mathcal{E}_\varepsilon u)(x) = u(x)$ for $x \in \Omega_0$, $\operatorname{dist}(x, \partial\Omega_0) > \varepsilon$,
- there is a $C^\infty$ mapping $\Psi : \mathbb{R}^n \to \Omega_0 \cap \{x\colon \operatorname{dist}(x,\partial\Omega_0) > \varepsilon\}$ such that $|(\mathcal{E}_\varepsilon u)(x) - u(\Psi(x))| < \varepsilon$ for all $x \in \mathbb{R}^n$.

*Example.* Every star–shaped domain has the $J$–extension property.

**Definition 5.1.2.** A set $\Omega$ is called a *domain of dependence over* $\Omega_0$ *for the operator* $P_{m,\sigma}$ if the conditions (5.4)–(5.11) are satisfied.

*Example* (Characteristic cone). The characteristic cone $K(B)$ for the ball $B = B(x^*, d)$ in the initial plane is defined by

$$K(B) = \left\{(x,t) : |x - x^*| < d - \lambda'_{max,\sigma} t, \quad 0 \le t < \frac{d}{\lambda'_{max,\sigma}}\right\}$$

with

$$\lambda'_{max,\sigma} = \|\sigma\|_{L^\infty(B)} \sup \left\{\lambda_{max}(x,t;P_{m,1}) : \quad (x,t) \in \overline{B} \times [0,T]\right\}.$$

In this section we will also examine the quasilinear Cauchy problem

$$\begin{aligned} &D_t^m u + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta \sigma(x)^{|\beta|} D_t^k u\}) D_x^\alpha D_t^j \left(\sigma(x)^{|\alpha|} u\right) \\ &\qquad = f(x,t,\{D_x^\beta \sigma(x)^{|\beta|} D_t^k u\}|_{k+|\beta|\le m-1}), \\ &u(x,0) = \varphi_0(x), \dots, D_t^{m-1} u(x,0) = \varphi_{m-1}(x). \end{aligned} \tag{5.12}$$

This equation will be written as $P^{(u)}_{m,\sigma} u = f^{(u)}$.

If $u$ solves this Cauchy problem, then one can define a domain of dependence $\Omega^{(u)}$, which itself depends on $u$, since the coefficients of the principal part depend on $u$. For this Cauchy problem we will assume almost the same conditions as in Section 3, see (3.27), (3.28) and Remark 3.2.1:

$$a_{j,\alpha} \in C^1([0,T], C^{s_c}(\Omega_0 \times \mathbb{R}^{n_0})), \quad s_c > \frac{n}{2} + 1, \tag{5.13}$$

$$\varphi_j \in H^{s_c+m-1-j}(\Omega_0), \tag{5.14}$$

$$f \in C([0,T], C^{s_c}(\Omega_0 \times \mathbb{R}^{n_0})). \tag{5.15}$$

And we modify Condition 3.2 in an obvious way:

**Condition 5.1.** *The roots* $\tau_j(x,t,v,\xi)$ *of*

$$\tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,v)\xi^\alpha \tau^j = 0$$

*are real and distinct,* $|\tau_j(x,t,v,\xi) - \tau_i(x,t,v,\xi)| \ge c|\xi|$, $c > 0$, $i \ne j$, *for all* $(x,t,v,\xi) \in \Omega_0 \times [0,T] \times \mathbb{R}^{n_0} \times \mathbb{R}^n$.

### 5.2. Uniqueness for Linear Equations

There is another way to define domains of dependence. A set $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^n$ is called *domain of dependence over* $\Omega_0 \subset \mathbb{R}^n$ for some hyperbolic operator if the vanishing of data in $\Omega_0$ and the vanishing of the right–hand side in $\Omega$ imply the vanishing of the solution to the Cauchy problem in $\Omega$. The following theorem states that the two definitions are in concordance.

**Theorem 5.2.1** (Uniqueness)**.** *We suppose* (5.2), (5.3) *and Condition 3.1. Let* $\Omega \subset \Omega_0 \times [0,T]$ *be a domain of dependence over* $\Omega_0$ *for the operator* $P_{m,\sigma}$*. Assume* $\varphi_0 \equiv \cdots \equiv \varphi_{m-1} \equiv 0$ *in* $\Omega_0$ *and* $f \equiv 0$ *in* $\Omega$*. Then* $u \equiv 0$ *in* $\Omega$ *for every solution* $u$ *of* (5.1) *with*

$$D_t^{m-|\alpha|} D_x^\alpha \sigma^{|\alpha|} u \in C(\Omega), \quad |\alpha| \leq m.$$

*Proof.* Let us sketch the proof. The solution is defined in $\Omega$, the coefficients $a_{j,\alpha}$ are defined in $\overline{\Omega_0} \times [0,T]$. In a first step we extend $u$ and $a_{j,\alpha}$ to the domain $\Omega_0 \times (-\infty, 0]$; the function $u$ vanishes there. This gives a new Cauchy problem with solution $u$ and vanishing initial values. In a second step we transform the variables. The domain $\Omega$ is mapped to some domain $\tilde{\Omega} = \{(y,r) : y \in \Omega_0, g(y,0) \leq r < r^*\}$, and $v$, the image of $u$, has compact support with respect to the spatial variable $y$ if the time variable is frozen. We extend the transformed coefficients $\tilde{a}_{j,\alpha}$ and the weight $\sigma$ to $\mathbb{R}^n \times [0,T]$, apply an energy estimate, and obtain $v \equiv 0$. This will prove the theorem.

The coefficients $a_{j,\alpha}$ satisfy Condition 3.1, (5.3) and are defined in $\Omega_0 \times [0,T]$. For $0 < t \leq \varepsilon$ we set $a_{j,\alpha}(x,-t) = 2a_{j,\alpha}(x,0) - a_{j,\alpha}(x,t)$. Then we change $a_{j,\alpha}(x,t)$ appropriately for $-\varepsilon \leq t \leq -\varepsilon/2$, set $a_{j,\alpha}(x,t) = a_{j,\alpha}(x,-\varepsilon)$ for $-\infty < t < -\varepsilon$, and conclude that $a_{j,\alpha} \in C_b^1(\Omega_0 \times (-\infty, T])$ for $j + |\alpha| = m$ and $a_{j,\alpha} \in C_b^0(\Omega_0 \times (-\infty, T])$ for $j + |\alpha| < m$. The Condition 3.1 is true in this domain if $\varepsilon$ is small.

The same method can be used to extend the derivative $g_t(x,t)$ of the function $g(x,t)$ to $\Omega_0 \times (-\infty, T]$. The result is $g \in C^\infty(\Omega_0 \times (-\infty, T])$, $g_t \in C_b^\infty(\Omega_0 \times (-\infty, T])$ $g_t(x,t) = g_t(x,-\varepsilon)$, $t \leq -\varepsilon$. We see that the function $g$ takes arbitrarily small values and conclude that

$$\Omega \cup Z = \Omega \cup (\Omega_0 \times (-\infty, 0]) = \bigcup_{-\infty < r < r^*} S_r.$$

We extend the solution $u$ by $u(x,t) = 0$, $(x,t) \in Z$, the letter $Z$ stands for "zero". Then $u$ solves (5.1) in $Z$ with homogeneous data for $t = -1$, $f \equiv 0$ in $Z$, and

$$D_t^{m-|\alpha|} D_x^\alpha \sigma^{|\alpha|} u \in C(\Omega \cup Z), \quad |\alpha| \leq m.$$

We apply a Holmgren type transform to change the variables,

$$\begin{aligned} &y = x, \quad r = g(x,t), \quad \nabla_x = \nabla_y + (\nabla_x g)\partial_r, \quad \partial_t = (\partial_t g)\partial_r, \\ &v(y,r) = u(x,t). \end{aligned}$$

The dual variables fulfill

$$\xi = \eta + (\nabla_x g)\varrho = \eta + c(y,r)\varrho, \quad \tau = g_t \varrho = c_0(y,r)\varrho.$$

The domain $\Omega$ is mapped to the set

$$\tilde{\Omega} = \{(y,r) : y \in \Omega_0, g(y,0) \leq r < r^*\},$$

and $\tilde{Z}$ is the image of $Z$:

$$\tilde{Z} = \{(y,r) : y \in \Omega_0, -\infty < r \leq g(y,0)\}.$$

It is easy to verify that $D_r^{m-|\alpha|} D_y^\alpha \sigma(y)^{|\alpha|} v(y,r) \in C(\tilde{\Omega} \cup \tilde{Z})$ for $|\alpha| \leq m$.

The function $v$ is a solution of

$$D_r^m v + \sum_{j+|\alpha|\le m, j<m} \tilde{a}_{j,\alpha}(y,r) D_y^\alpha D_r^j \left(\sigma(y)^{|\alpha|} v\right) = 0,$$
$$v(y,0) = \cdots = D_r^{m-1} v(y,0) = 0, \quad (y,r) \in \Omega_0 \times [0, r^*),$$

$\tilde{a}_{j,\alpha}(y,r)$ are the transformed coefficients. We have to check whether this Cauchy problem is weakly hyperbolic and satisfies the Levi conditions. We remember the definition of (strict) hyperbolicity, see [23]:

**Definition 5.2.2.** A differential operator $Q(z, D_z) = \sum_{|\alpha|=m} a_\alpha(z) D_z^\alpha$ is called *hyperbolic* at the point $z_0$ in the direction $N \neq 0$ if

- $Q(z_0, N) \neq 0$,
- $Q(z_0, \tau N + \zeta) = 0$ has only real roots $\tau$ for every $\zeta \neq 0$.

A differential operator $Q(z, D_z)$ is called *strictly hyperbolic* at $(z_0, N)$ if it is hyperbolic at $(z_0, N)$ and if $Q(z_0, \tau N + \zeta) = 0$ has $m$ real and distinct roots $\tau$ for every $\zeta \perp N$, $\zeta \neq 0$.

By definition, the operator $P_{m,\sigma}(x, t, D_x, D_t)$ is hyperbolic in the direction $N = (1, 0, \ldots, 0) \in \mathbb{R}^{1+n}$ and the operator $P_{m,1}(x, t, D_x, D_t)$ is strictly hyperbolic in this direction $N$. The symbol of the principal part $\tilde{P}_{m,\sigma}$ of the transformed operator is

$$\tilde{P}_{m,\sigma}(y, r, \eta, \varrho) = P_{m,\sigma}(x, t, \eta + c\varrho, c_0\varrho).$$

To be able to apply the results of Section 3, we have to verify that the operator $\tilde{P}_{m,\sigma}$ is hyperbolic in the direction $\tilde{N}$ and that the operator $\tilde{P}_{m,1}$ is strictly hyperbolic in the direction $\tilde{N}$. Here $\tilde{N} = (1, 0, \ldots, 0) \in \mathbb{R}^1_\varrho \times \mathbb{R}^n_y$ is the normal direction of the hypersurfaces $r = \text{const}$.

It can be seen that

$$\tilde{P}_{m,\sigma}(y, r, 0, 1) = P_{m,\sigma}(x, t, c, c_0) = |c|^m P_{m,\sigma}\left(x, t, \frac{c}{|c|}, \frac{c_0}{|c|}\right).$$

From (5.10) and the definition of $\lambda_{max}(x, t; P_{m,\sigma})$ we then conclude that

$$\tilde{P}_{m,\sigma}(y, r, \tilde{N}) = \tilde{P}_{m,\sigma}(y, r, 0, 1) \neq 0.$$

In the next step we show that the equation $\tilde{P}_{m,\sigma}(y, r, \eta, \varrho) = 0$ has $m$ real roots $\varrho_1, \ldots, \varrho_m$ for every $\eta \neq 0$. It holds

$$\tilde{P}_{m,\sigma}(y, r, \eta, \varrho) = P_{m,\sigma}(x, t, \eta + c\varrho, c_0\varrho) = c_0^m P_{m,1}\left(x, t, \frac{\sigma}{c_0}(\eta + c\varrho), \varrho\right).$$

If $\sigma(x) = 0$, then the only roots are $\varrho_1 = \cdots = \varrho_m = 0$. If $\sigma(x) \neq 0$, then

$$\tilde{P}_{m,\sigma}(y, r, \eta, \varrho) = \sigma^m P_{m,1}\left(x, t, \eta + c\varrho, \frac{c_0}{\sigma}\varrho\right).$$

The polynomial $P_{m,1}(x,t,\eta,c_0\varrho/\sigma)$ is strictly hyperbolic in the direction $N$. From (5.10) we gather

$$\left|\frac{\sigma(x)c(x,t)}{c_0(x,t)}\right| < \frac{1}{\lambda_{max}(x,t;P_{m,1})},$$

and Proposition 6.1 reveals the strict hyperbolicity of the polynomial $P_{m,1}(x,t,\eta+c\varrho,(c_0/\sigma)\varrho)$ in the direction $N+\sigma c/c_0$. We get that $\tilde{P}_{m,\sigma}(y,r,\eta,\varrho)=0$ has $m$ real roots, which are distinct if $\sigma(y)\neq 0$.

Exploiting Proposition 6.1 once more, and making use of

$$\tilde{P}_{m,1}(y,r,\eta,\varrho) = P_{m,1}(x,t,\eta+\sigma c\varrho, c_0\varrho),$$

it is easy to verify that $\tilde{P}_{m,1}$ is strictly hyperbolic.

We come back to the function $v$. Our aim is to show that $v\equiv 0$ in $\tilde{\Omega}$, by the aid of Remark 3.1.5 and Gronwall's lemma. But first the coefficients $\tilde{a}_{j,\alpha}$ and the weight $\sigma$ have to be extended to the whole $\mathbb{R}^n_x\times[0,T]$, $\mathbb{R}^n_x$, respectively, since they are only defined on $\Omega_0\times(-\infty,T]$ and $\Omega_0$.

We fix some arbitrary $0<r_0<r^*$ and will show $v(y,r)=0$ in $\Omega_0\times[0,r_0]$. The function $v(y,r)$ vanishes for $y$ close to $\partial\Omega_0$; hence we are allowed to change the coefficients $\tilde{a}_{j,\alpha}$ and the weight $\sigma$ there. We extend these functions by

$$\tilde{a}_{j,\alpha}(y,t) = (\mathcal{E}_\varepsilon\tilde{a}_{j,\alpha})(y,t), \quad \sigma(y) = (\mathcal{E}_\varepsilon\sigma)(y), \quad y\notin\Omega_0,$$

where $\mathcal{E}_\varepsilon$ is the extension operator of Definition 5.1.1. Condition 3.1 is satisfied everywhere in $\mathbb{R}^n_y\times[0,r_0]$ if $\varepsilon$ is small enough.

We get coefficients $a^*_{j,\alpha}\in C^1_b(\mathbb{R}^n\times[0,r_0])$ for $j+|\alpha|=m$ and $a^*_{j,\alpha}\in C^0_b(\mathbb{R}^n\times[0,r_0])$ for $j+|\alpha|<m$. The function $v$ can be extended by zero, and solves

$$D^m_t v + \sum_{j+\alpha=m,j<m} a^*_{j,\alpha}(y,r)D^\alpha_y D^j_r(\sigma(y)^{|\alpha|}v) = 0, \quad (y,r)\in\mathbb{R}^n\times[0,r_0],$$

$$v(y,0) = \cdots = D^{m-1}_r v(y,0) = 0, \quad y\in\mathbb{R}^n.$$

Remark 3.1.5 and Gronwall's lemma yield $v(y,r)=0$ for $y\in\mathbb{R}^n$, $0\le r\le r_0$. Since $r_0<r^*$ can be chosen arbitrarily, we have $v=0$ in $\tilde{\Omega}$, hence $u=0$ in $\Omega$. The theorem is proved. □

## 5.3. Existence and Uniqueness for Solutions of Quasilinear Equations

**Theorem 5.3.1** (Local existence). *We suppose* (5.13)–(5.15) *and Condition 5.1. Let* $Q_0=\Pi^n_{i=1}[a_i,b_i]$ *be a rectangular parallelepiped (RP for short),* $Q_0\Subset\Omega_0$. *Then a constant* $0<T_0\le T$ *and a solution* $u$ *of* (5.12) *exist with*

$$D^j_t\sigma^{m-1-j}u\in C([0,T_0],H^{s_c+m-1-j}(Q_0)), \quad j=0,\ldots,m-1.$$

*Proof.* We take a cut–off function $\varphi(x)$ which is supported in a neighborhood of $Q_0$ and is identical to 1 on $Q_0$. Then we replace the functions $\sigma(x)$, $f(x,t,\{V_{k,\beta}\})$, $\varphi_j(x)$ by $\varphi(x)\sigma(x)$, $\varphi(x)f(x,t,\{V_{k,\beta}\})$ and $\varphi(x)\varphi_j(x)$.

Let $Q$ be an RP with $\operatorname{supp}\varphi\Subset Q$. Using suitable "reflections" (see the proof of Theorem 5.2.1), we can extend the coefficients $a_{j,\alpha}$ from $Q\times[0,T]\times\mathbb{R}^{n_0}$ to the larger set $Q' \times [0,T] \times \mathbb{R}^{n_0}$, $Q'$ being an $RP$ with twice the edge lengths of $Q$ which can be regarded as a torus. We get a Cauchy problem on $Q'$. Theorem 3.2.2 shows that a solution $u$ exists with the desired smoothness on the torus $Q'$. This function is a solution on $Q_0 \times [0,T_0]$, since $\varphi \equiv 1$ on $Q_0$. □

So far, nothing has been said about the *uniqueness* of this solution $u$. This question is taken up in the next theorem.

**Theorem 5.3.2** (Local uniqueness). *Let the conditions of Theorem 5.3.1 be satisfied. Let $B_0 \Subset \Omega_0$ be a sufficiently small ball. Then a number $T_0 > 0$ and a cone $\Omega$ over $B_0$ exist such that a uniquely determined solution $u$ with*

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T_0], H^{s_c+m-1-j}(\Omega_t)), \quad j = 0, \dots, m-1,$$

*exists in $\Omega$. The notation $v \in \mathcal{C}([0,T], H^s(\Omega_t))$ means that*

- $\sup_{t\in[0,T]} \|v(\cdot,t)\|_{H^s(\Omega_t)} < \infty$,
- *for all $T'$, $\Omega_0'$ with $\Omega_0' \times [0,T'] \Subset \Omega$ it holds $v \in C([0,T'], H^s(\Omega'))$.*

*Here $H^s(\Omega_t)$ denotes the usual Sobolev space on the bounded domain $\Omega_t$.*

*Proof.* Theorem 5.3.1 shows that a small $T_0$ and a solution $u$ exist with

$$D_t^j \sigma^{m-1-j} u \in C([0,T_0], H^{s_c+m-1-j}(B_0)), \quad j = 0, \dots, m-1,$$

$$P_{m,\sigma}^{(u)} u = f^{(u)} \text{ in } B_0 \times [0,T_0], \quad D_t^j u(\cdot,t) = \varphi_j(\cdot) \text{ in } B_0.$$

We define the cone $\Omega = K(B_0)$ over $B_0$ as given in Example 5.1 with

$$P_{m,1}(x,t,\tau,\xi) = \tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(x,t,\{D_x^\beta \sigma^{|\beta|} D_t^k u\}) \xi^\alpha \tau^j.$$

Then it follows that

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T_0], H^{s_c+m-1-j}(K(B_0)_t)), \quad j = 0, \dots, m-1.$$

Let $U^*$ be another solution in $\Omega$, i.e.,

$$P_{m,\sigma}^{(u^*)} u^* = f^{(u^*)} \text{ in } B_0 \times [0,T_0], \quad D_t^j u^* = \varphi_j \text{ in } B_0.$$

The difference $u - u^*$ then solves

$$P_{m,\sigma}^{(u)}(u-u^*) = \sum_{k+|\beta|\le m-1} g_{k,\beta}(x,t) D_x^\beta (\sigma^{|\beta|} D_t^k (u-u^*))$$

with $g_{k,\beta} \in C(\Omega)$, and Theorem 5.2.1 applied to the homogeneous Cauchy problem reveals $u - u^* \equiv 0$ in $\Omega$. □

*Remark* 5.3.3. The techniques of Section 5.4 enable us to show that solutions to quasilinear Cauchy problems are unique not only locally, but also globally in domains of dependence. Since the proof is quite similar to that of Theorem 5.4.2, it is dropped.

If the equation is *linear*, we even have *global* existence:

**Corollary 5.3.4** (Global existence)**.** *Let us consider the Cauchy problem* (5.1)*. We suppose Condition 3.1 and*

$$\sigma \in C^\infty(\Omega_0^*), \quad \Omega_0^* \Supset \Omega_0,$$

$$a_{j,\alpha} \in \begin{cases} C^1([0,T], H^{s_c}(\Omega_0^*)) & : j+|\alpha| = m, \\ C([0,T], H^{s_c}(\Omega_0^*)) & : j+|\alpha| < m, \end{cases} \quad s_c > \frac{n}{2}+1,$$

$$\varphi_j \in H^{s_c+m-1-j}(\Omega_0^*), \quad f \in C([0,T], H^{s_c}(\Omega_0^*)).$$

*Let $\Omega$ be a domain of dependence for the operator $P_{m,\sigma}$ over the domain $\Omega_0$. Then a unique solution $u$ exists with*

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T], H^{s_c+m-1-j}(\Omega_t)).$$

*Proof.* We pick a cut–off function $\varphi(x)$ that is identical to 1 in a neighborhood of $\Omega_0$ and supported in $\Omega_0^*$. Then a cube $Q$ with $\operatorname{supp}\varphi \Subset Q$ is chosen. We replace the functions $\sigma(x)$, $f(x,t)$, $\varphi_j(x)$ by $\varphi(x)\sigma(x)$, $\varphi(x)f(x,t)$ and $\varphi(x)\varphi_j(x)$. The coefficients $a_{j,\alpha}(\cdot,t)$ are extended from $\Omega_0^*$ to $Q \setminus \Omega_0^*$ by the aid of the procedure from the end of the proof of Theorem 5.2.1. We acquire a linear Cauchy problem on a torus. Remark 3.2.4 gives us a solution defined in $Q \times [0,T]$ which has the desired smoothness. Theorem 5.2.1 shows that this solution is the only solution in $\Omega$. □

### 5.4. $C^\infty$ regularity

First, we show a *local* regularity result in $C^\infty$.

**Lemma 5.4.1** (Local $C^\infty$–regularity)**.** *Let $u$ be a function defined in $\Omega^{(u)}$ which is a domain of dependence over $\Omega_0$ for the operator $P_{m,\sigma}^{(u)}$. Assume that $u$ with*

$$D_t^j \sigma^{m-1-j} u \in \mathcal{C}([0,T], H^{s_c+m-1-j}(\Omega_t)), \quad j = 0, \dots, m-1,$$

*is a solution of* (5.12)*. The coefficients $a_{j,\alpha}$ and the right–hand side are supposed to be $C^\infty$ with respect to all their arguments. Finally, we assume that*

$$u \in C^\infty(\Omega^{(u)} \cap \{(x,t) : t \le t_0\})$$

*for some $t_0 \ge 0$. Let $B(x_0,d) \Subset \Omega_{t_0}$ be a ball. Then a number $t_1 = t_1(x_0,t_0,d) > t_0$ exists with*

$$u \in C^\infty(B(x_0,d) \times [t_0,t_1]), \quad B(x_0,d) \times [t_0,t_1] \Subset \Omega^{(u)}.$$

*Proof.* We apply the procedure given in the proof of Theorem 5.3.1 to extend the Cauchy problem from $B(x_0,d) \times [t_0,T]$ to the set $Q' \times [t_0,T]$, with $Q'$ being a torus. We get a quasilinear weakly hyperbolic Cauchy problem on a torus with $C^\infty$ coefficients and $C^\infty$ data. Theorem 3.2.3 gives us a local $C^\infty$ solution, and Theorem 5.3.2 shows that this solution is unique in some domain of dependence which contains some cylindrical set $B(x_0,d) \times [t_0,t_1]$ with small $t_1 - t_0$. This implies $u \in C^\infty(B(x_0,d) \times [t_0,t_1])$. □

This lemma is the key tool to prove the following theorem:

**Theorem 5.4.2** (Global $C^\infty$ regularity). *Let $u$ be a function defined in $\Omega^{(u)}$ which is a domain of dependence over $\Omega_0$ for the operator $P^{(u)}_{m,\sigma}$. We suppose that $u$ with*

$$D_t^j\sigma^{m-1-j}u \in \mathcal{C}([0,T], H^{s_c+m-1-j}(\Omega_t^{(u)})), \quad j = 0,\dots,m-1,$$

*is a solution of* (5.12). *Moreover, we assume $a_{j,\alpha}, f \in C_b^\infty(\Omega_0 \times [0,T] \times \mathbb{R}^{n_0})$, $\varphi_j \in C_b^\infty(\Omega_0)$. Then $u \in C^\infty(\Omega^{(u)})$.*

*Proof.* If $B(x_0, d) \Subset \Omega_0$ is a ball, Lemma 5.4.1 gives us a number $t_1(x_0, d)$ with

$$u \in C^\infty(B(x_0, d) \times [0, t_1(x_0, d)]).$$

This implies that a smooth function $h = h(x) > 0$ and a set $M_0$ exist with

$$M_0 = \{(x, t) : x \in \Omega_0, 0 \le t < h(x)\}, \quad u \in C^\infty(M).$$

The domain $\Omega^{(u)}$ can be exhausted with hypersurfaces $S_r$. We transform the variables in the same way as in the proof of Theorem 5.2.1. This results in a quasilinear weakly hyperbolic initial value problem for the function $v$,

$$\tilde{P}^{(v)}_{m,\sigma}v = \tilde{f}(y, r, \{D_y^\beta\sigma(y)^{|\beta|}D_r^k v\}), \quad y \in \Omega_0, \quad g(y,0) \le r < r^*,$$
$$v(y, g(y,0)) = \psi_0(y), \dots, D_r^{m-1}v(y, g(y,0)) = \psi_{m-1}(y).$$

Denoting the images of $M_0$ and $\Omega^{(u)}$ under under the transformation of variables by $\tilde{M}_0$ and $\tilde{\Omega}^{(u)}$, we know that $v$ is $C^\infty$ on the sets

$$\tilde{M}_0, \quad \tilde{M}(r_0) = \tilde{\Omega}^{(u)} \cap \{(y, r) \colon y \in \Omega_0, r \le r_0\},$$

for some $r_0 > 0$. The proof will be complete if we verify two properties:

**A:** If $v \in C^\infty(\tilde{M}(r))$, then $v \in C^\infty(\tilde{M}(r'))$ for some $r' > r$.

**B:** If $v \in C^\infty(\tilde{M}(r))$ for all $r$ with $r_0 \le r < r_1$, then $v \in C^\infty(\tilde{M}(r_1))$.

For **A**: The set $\partial\tilde{M}(r_0) = \tilde{\Omega}^{(u)} \cap \{(y, r) \colon y \in \Omega_0\}$ can be covered by $\tilde{M}_0$ and finitely many balls. For each ball, we apply Lemma 5.4.1.

For **B**: We cover the set $\partial\tilde{M}(r_1)$ by $\tilde{M}_0$ and finitely many balls $B(y_0, d)$. For each such ball, we consider a Cauchy problem with initial data prescribed at $B(y_0, d) \times \{r = r_1 - \varepsilon\}$, $\varepsilon > 0$ small. This Cauchy problem has the solution $v$, which is locally $C^\infty$. Proposition 3.2.8 and Remark 3.2.10 tell us that the life–span of this solution can not approach zero for $\varepsilon \to 0$. This shows that $v$ is $C^\infty$ on $B(y_0, d) \times \{r \le r_1\}$. □

## 6. Appendix

Here we provide some auxiliary results.

**Proposition 6.1** ([3], [4]). *Let the homogeneous differential operator $P(x, D)$ of order $m$ be strictly hyperbolic at the point $x_0$ in the direction $N$, $|N| = 1$. By $\lambda_{max}$ we denote the largest absolute value of the characteristic roots, i.e.,*

$$\lambda_{max} = \sup\{|\tau| : P(x_0, \tau N + \xi) = 0, |\xi| = 1, \xi \perp N\}.$$

*Then $P$ is strictly hyperbolic at $x_0$ in any direction $N+e$ with $N \perp e$, $|e|^{-1} > \lambda_{max}$.*

The following lemma is a generalization of Gronwall's lemma to differential inequalities with a singular coefficient.

**Lemma 6.2** ([24]). *Let $y(t) \in C([0,T]) \cap C^1((0,T))$ be a solution of the differential inequality*

$$y'(t) \leq K(t)y(t) + f(t), \quad 0 < t < T,$$

*where the functions $K(t)$ and $f(t)$ belong to $C(0,T)$. We assume for every $t \in (0,T)$ and every $\delta \in (0,t)$ that*

$$\int_0^\delta K(\tau)\,d\tau = \infty, \quad \int_\delta^T K(\tau)\,d\tau < \infty,$$

$$\lim_{\delta \to +0} \int_\delta^t \exp\left(\int_s^t K(\tau)\,d\tau\right) f(s)\,ds \quad \textit{exists},$$

$$\lim_{\delta \to +0} y(\delta) \exp\left(\int_s^t K(\tau)\,d\tau\right) = 0.$$

*Then it holds*

$$y(t) \leq \int_0^t \exp\left(\int_s^t K(\tau)\,d\tau\right) f(s)\,ds.$$

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