Necessary conditions for the well-posedness of Schrödinger type equations in Gevrey spaces

Michael Dreher

Faculty of Mathematics and Computer Sciences, Freiberg University of Mining and Technology, Agricolastraße 1, 09596 Freiberg, Germany

Received 13 December 2002; accepted 2 March 2003

Abstract

We discuss evolution operators of Schrödinger type which have a non-self-adjoint lower order term and give a necessary condition for the Cauchy problem to such operators to be well-posed in Gevrey spaces. Under an additional assumption, this necessary condition is sharp.

MSC: 37K05; 37L50

1. Introduction

We study necessary conditions under which the following Cauchy problem of Schrödinger type,

\[ Lu = \left( i \partial_t + \Delta + \sum_{j=1}^{n} b_j(x) \partial_{x_j} + c(x) \right) u = f(t,x), \quad u(0,x) = \varphi(x), \quad (1.1) \]

is well-posed in Gevrey spaces \( G^s, 1 < s < \infty \). Here \( G^s = \lim_{\rho \to 0} G^s_\rho \), and \( G^s_\rho \) is the Hilbert space \( G^s_\rho = \{ v \in L^2(\mathbb{R}^n); \| v \|_{s,\rho} = \| \exp(\rho \langle \xi \rangle^{1/s}) \hat{v}(\xi) \|_{L^2} < \infty \} \), where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \) and \( \hat{v} \) is the usual Fourier transform of \( v \) with respect to \( x \in \mathbb{R}^n \).

Definition 1.1. We say that the Cauchy problem for the operator \( L \) is forward \( G^s \) well-posed if for every \( T > 0 \) and every \( \varrho_0 > 0 \) there are constants \( C = C(T, \varrho_0) \) and \( \varrho > 0 \) such

E-mail address: dreher@math.tu-freiberg.de (M. Dreher).
that for every $\varphi \in G^s_{\vartheta_0}$, $f \in C([0, T], G^s_{\vartheta_0})$ there is a unique solution $u \in C([0, T], G^s_{\vartheta_0})$ to (1.1) with

$$\|u(t, \cdot)\|_{s, \vartheta_0} \leq C\|\varphi\|_{s, \vartheta_0} + C\int_0^t \|f(\tau, \cdot)\|_{s, \vartheta_0} d\tau, \quad 0 \leq t \leq T.$$  

If the coefficients $b_j$ are purely imaginary valued, then $L = i\partial_t + A_0 + A_1$, where $A_0$ is a self-adjoint operator, and $A_1$ is a bounded operator. It is then known how to derive \textit{a priori} estimates of a solution $u$ to (1.1) in the space $L^2(\mathbb{R}^n)$, or Sobolev spaces $H^s(\mathbb{R}^n)$, or Gevrey spaces $G^s_{\vartheta}$; and the well-posedness of this Cauchy problem follows by functional analytic arguments. The situation is more delicate when $\Re b_j \neq 0$. For example, the Cauchy problem for the operator $L = i\partial_t + \partial_x^2 + \partial_x$ is neither well-posed in $L^2(\mathbb{R}^n)$ nor in $G^s$, $1 < s < \infty$, as can be shown by an explicit representation of the solution via Fourier transform with respect to $x$, see also [15]. Generally, well-posedness requires a certain decay of $\Re b_j(x)$ at infinity.

Therefore, we propose the following condition:

\textbf{Condition 1.} There is a constant $M = M(d_0)$ such that

$$\sup_{x \in \mathbb{R}^n, \theta \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(x + 2\theta \omega)\omega_j d\theta \right| \leq M(1 + |\sigma|)^{d_0}, \quad \forall \sigma \in \mathbb{R}.$$  

We assume that the coefficients $b_j$ and $c$ belong to Gevrey spaces $G^s_{L\infty}$, $G^s_{L\infty}$:

$$\|\partial_x^a b_j(\cdot)\|_{L^\infty} \leq C^{1 + |a|} |a|^{s_0}, \quad \forall a,$$

$$\|\partial_x^a c(\cdot)\|_{L^\infty} \leq C^{1 + |a|} |a|^{s_1}, \quad \forall a.$$  

(1.2)

The first of our main results is the following:

\textbf{Theorem 1.} Let (1.2) be satisfied, and let $d_0$ be a number with $d_0 > 3/(s + 1)$ and $d_0 > 2/(s + 1) - s_0$. Then Condition 1 with this $d_0$ is necessary for the $G^s$ well-posedness of the Cauchy problem (1.1).

Sufficient conditions for the $G^s$ well-posedness of the Cauchy problem for the operator $L = i\partial_t + \Delta + \sum_{j=1}^n b_j(t, x)\partial_{x_j} + c(t, x)$ were given in [2], namely $\Re b_j(t, x) = o((x)|1/2 - 1)).$ In case of the model operator $L = i\partial_t + \Delta + (x)^{d-1}\partial_t$ with $x \in \mathbb{R}^1$, and $0 < d < 1$, the Cauchy problem is therefore well-posed if $d < 1/s$. On the other hand, Theorem 1 implies ill-posedness for $d > 3/(s + 1)$ only.

This gap can be closed if we suppose that the pseudodifferential symbol of the vector field $\sum \Re b_j(x)D_j$ decays not too rapidly in a certain conic set:
Condition 2 (Slow decay). There are $x_0 \in \mathbb{R}^n$, $\omega_0 \in S^{n-1}$ (unit sphere), and $\varepsilon_0 > 0, c_0 > 0$ such that

$$- \sum_{j=1}^{n} \Re b_j(x + \tau \omega') \omega_j \geq 2c_0(\tau)^{d_0-1},$$

for all $\tau \geq 0$, $|x - x_0| < \varepsilon_0$, and all $\omega, \omega' \in S^{n-1}$ with $|\omega - \omega_0| < \varepsilon_0, |\omega' - \omega_0| < \varepsilon_0$.

In case of this slow decay condition, the following second main result can be proved:

**Theorem 2.** Suppose (1.2) with $s_b < s$ and Condition 2. Then $d_0 \leq 1/s$ is necessary for the $G^s$ well-posedness of the Cauchy problem (1.1).

A necessary condition for $H^\infty$ well-posedness was given in [7]:

$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^{n} \Re b_j(x + 2\theta \omega) \omega_j d\theta \right| \leq M \log(1 + |\sigma|) + N, \quad \forall \sigma \in \mathbb{R}.$$  

This condition is sufficient in the case of one space dimension; and it is sufficient in the cases of two or more space dimensions if one supposes certain relations on derivatives of the coefficients $b_j$, see [8].

The investigation of an operator with variable coefficients in the principal part, $L = i\partial_t + \sum_{j,k} a_{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_{j} b_j(x) \partial_{x_j} + c(x)$, where $a(x, \xi) = \sum_{j,k} a_{jk}(x) \xi_j \xi_k \geq c_0 |\xi|^2$, $c_0 > 0$, requires the introduction of the bicharacteristic strip $(X, P) = (X, P)(t, x, p)$, which is the solution to the Hamilton–Jacobi equations,

$$\partial_t X_j = \partial P_j a(X, P), \quad \partial_t P_j = -\partial X_j a(X, P), \quad (X, P)(0, x, p) = (x, p).$$

Then a necessary condition for the $H^\infty$ well-posedness is

$$\sup_{x, \omega} \left| \int_0^\sigma \sum_{j=1}^{n} \Re b_j(\theta, x, \omega) P_j(\theta, x, \omega) d\theta \right| \leq M \log(1 + |\sigma|) + N, \quad \forall \sigma \in \mathbb{R},$$

under an additional non-trapping condition. For details, see [5].

Sufficient conditions for $H^s$ well-posedness were proved in [3,4,11]. In [9] and [14], the following necessary condition for $L^2$ well-posedness was shown:

$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^{n} \Re b_j(\theta, x, \omega) P_j(\theta, x, \omega) d\theta \right| \leq M, \quad \forall \sigma \in \mathbb{R}.$$  

This condition is also sufficient, see [10].

Schrödinger type equations with a lower order term of order strictly less than 1 were investigated in [1]; and sufficient conditions for $G^s$ well-posedness were proved.

The challenging question of necessary conditions for the $G^s$ well-posedness of Schrödinger type equations with variable coefficients in the principal part will be answered in a forthcoming publication.
The paper is organized as follows. Theorem 1 and Theorem 2 will be proved simultaneously; and the both cases will be called Case I and Case II, respectively. Before we sketch the method of the proofs, we need a lemma (whose proof is below).

**Lemma 1.1.** Assume that $0 < d_0 < 1$ and that Condition 1 is violated. Then, for each $k \in \mathbb{N}$, there are $x_k \in \mathbb{R}^n$, $\sigma_k \in \mathbb{R}_+$, $\omega_k \in S^{n-1}$ with the property that

$$
- \int_0^{\sigma_k} \sum_{j=1}^{\infty} \eta b_j(x_k + 2\sigma_k \omega_k) \omega_k, j d\theta = k(1 + \sigma_k)^{d_0},
$$

$$
- \int_0^{\sigma_k} \sum_{j=1}^{\infty} \eta b_j(x_k + 2\sigma_k \omega_k) \omega_k, j d\theta \geq kd_0(1 + \sigma_k)^{d_0-1}, 0 \leq \sigma \leq \sigma_k,
$$

where $\sigma_k$ tends to infinity for $k \to \infty$.

This lemma gives us a sequence \{${\sigma_k}$\}_k tending to infinity in Case I. In Case II, we choose this sequence arbitrarily, but still approaching infinity. Now we fix special initial data, $\varphi_k(x) = \varphi(x - x_k)$ (in Case I), and $\varphi_k(x) = \varphi(x - x_0)$ (in Case II), where \( \varphi \in G^s \) is determined by $\tilde{\varphi}(\xi) = (\xi)^{-(n+1)/2} \exp(-\varrho_0 |\xi|^{1/s})$. Assuming that (1.1) is $G^s$ well-posed, there is a unique solution $u_k \in C^1([0, T], G^s)$ of

$$
Lu_k = 0, \quad u_k(0, x) = \varphi_k(x). \tag{1.3}
$$

Next we define a seminorm $E_k(t)$ for the function $u_k(t, \cdot)$.

Let $h = h(x) \in G^{s_0}$ (with $s_0 > 1$ very close to 1) be a function with

$$
h(x) = \begin{cases} 
0 & |x| \geq 1, \\
1 & |x| \leq 1/2, \\
0 & h(x) \leq 1,
\end{cases} \tag{1.4}
$$

(A thorough representation of Gevrey functions can be found, e.g., in [13, Volume 3].) We choose the pseudodifferential symbols

$$
w_k(t, x, \xi) = h \left( \frac{x - x_k - 2\sigma_k \delta_1 \omega_k}{\sigma_k} \right) \xi^2 \left( \frac{\xi - \sigma_k^d \omega_k}{\sigma_k^2} \right) \tag{Case I},
$$

$$
w_k(t, x, \xi) = h \left( \frac{x - x_0 - 2\sigma \omega_0}{\sigma} \right) \xi^2 \left( \frac{\xi - \sigma \omega_0}{\sigma^2} \right) \tag{Case II},
$$

where $0 < \varepsilon \ll s_0, \delta_1 = 1 - d_0$, and $\delta_2, \delta_3$ are certain positive constants determined later.

We are going to employ the multi-index notation: for $\alpha \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$
\hat{g}_\alpha = \frac{\partial^{|\alpha|}}{\partial \xi_1 \cdots \partial \xi_n}, \quad D^\alpha_y = (-i)^{|\alpha|} \hat{g}_\alpha, \quad i^2 = -1.
$$

For multi-indices $\alpha, \beta \in \mathbb{N}^n$, we specify

$$
w_k^{(ap)}(t, x, \xi) = \hat{g}_\alpha \hat{g}_\beta h(y, y) \hat{g}_\eta \hat{g}_\eta h(y, y)(y = \sigma_1 (x - x_k - 2\sigma_1 \delta_1 \omega_k), \eta = \sigma^{-d_0}(\xi - \delta_1 \omega_k)),
$$

$$
w_k^{(ap)}(t, x, \xi) = \hat{g}_\alpha \hat{g}_\beta h(y, y) \hat{g}_\eta \hat{g}_\eta h(y, y)(y = 2\sigma \omega_0, \eta = \sigma^{-d_0}(\xi - \sigma \omega_0)).
$$
in Case I, Case II, respectively. These cut-off symbols are supported near the bicharacter-
istic strip associated to the principal symbol \( a(x, \xi) = |\xi|^2 \). With some positive constant \( \kappa \) to be defined later, we set \( N \ni \mathcal{N} = \lfloor \sigma \kappa / s \rceil_k \), choose \( s_1 > s_0 \), and define the seminorm

\[
E_k(t) = \sum_{|\alpha| \leq N, |\beta| \leq N} (\alpha! \beta!)^{-\lfloor 1 \rfloor} R_{a_k}^{(\alpha \beta)}(t, x, D_x) u_k(t, x) \mid_{L^2(\mathbb{R}^n)}.
\]

(1.5)

In Sections 3 and 4, estimates of \( E_k \) from above and below will be derived, which contradict for large \( \sigma_k \) if we choose \( \delta_1, \delta_2, \delta_3, \kappa, \varepsilon \) suitably. This implies that the assumed well-posedness of the Cauchy problem (1.1) does not hold, completing the proofs of the Theorems 1 and 2.

**Remark 1.1.** Instead of Theorem 2, we will actually prove the following (equivalent) result: let (1.2) and Condition 2 be satisfied, and suppose that the constant \( d_0 \) of the slow decay condition satisfies

\[
\frac{1}{s} < d_0 < \frac{1}{s} + \left(1 - \frac{s_0}{s}\right).
\]

(1.6)

Then the Cauchy problem for the operator \( L \) is not \( G^s \) well-posed.

In the following, \( C \) and \( c \) denote generic large and small positive constants, which do neither depend on multi-indices nor \( \sigma_k \).

2. Tools and preliminaries

By \( S^0 \) we denote the usual space of pseudodifferential symbols, i.e., all functions \( p = p(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) such that \( |\partial_\alpha^a \partial_\beta^b p(x, \xi)| \leq C_{a\beta} \), for all \( (x, \xi) \in \mathbb{R}^{2n} \) and all \( \alpha, \beta \in \mathbb{N}^n \). The topology of the locally convex space \( S^0 \) is given by the seminorms

\[
|p|_{\ell} = \max_{|\alpha| \leq \ell, |\beta| \leq \ell} \sup_{(x, \xi) \in \mathbb{R}^{2n}} |\partial_\alpha^a \partial_\beta^b p(x, \xi)|.
\]

Each symbol \( p \in S^0_0 \) defines a pseudodifferential operator \( P : \mathcal{S} \rightarrow \mathcal{S} \) (Schwartz space of rapidly decreasing functions) by

\[
(Pu)(x) = \int e^{ix\xi} p(x, \xi) \hat{u}(\xi) \, d\xi,
\]

where we have introduced the convenient notation \( d\xi = (2\pi)^{-n} \, d\xi_1 \ldots d\xi_n \).

**Theorem 3** (Calderon–Vainilcourt). Let \( p \in S^0_0 \). The operator \( P \) can then be continuously extended to a bounded operator on \( L^2(\mathbb{R}^n) \),

\[
\|Pu\|_{L^2} \leq C|p|_{\ell_0} \|u\|_{L^2},
\]

(2.1)

where \( C \) and \( \ell_0 \) depend on the space dimension \( n \) only.
Let $p_1, p_2 \in S^{0}_{0,0}$, and define the oscillating integral

$$q(x, \xi) = \int \int e^{-iy\eta} p_1(x, \xi + \eta) p_2(x + y, \xi) \, dy \, d\eta$$

$$= \lim_{\epsilon \to 0} \int \int e^{-iy\eta} h(\epsilon y) h(\epsilon \eta) p_1(x, \xi + \eta) p_2(x + y, \xi) \, dy \, d\eta,$$

which is independent of the choice of the cut-off function $h$ satisfying (1.4). Then

$$Q(x, Dx) = P_1(x, Dx) P_2(x, Dx)$$

as a composition of mappings; we also write $q(x, \xi) = p_1(x, \xi) \circ p_2(x, \xi)$. Moreover, the symbol $q(x, \xi)$ allows the following expansion:

\[ q(x, \xi) = \sum_{|\gamma| = 0}^{N-1} \frac{1}{\gamma!} \left( \frac{1 - \theta}{\gamma} \right)^{N-1} q_{0,\gamma}(x, \xi) \, d\theta, \]

\[ q_{0,\gamma}(x, \xi) = \int \int e^{-iy\eta} (D^\gamma_{\xi} p_1(x, \xi + \theta \eta))(\partial^\gamma_x p_2(x + y, \xi)) \, dy \, d\eta. \]

For each $l_0 \in \mathbb{N}$, there is a constant $l_1 \in \mathbb{N}$ such that the seminorms of the remainder term $q_{0,\gamma}$ can uniformly in $\theta$ and $N$ be estimated by

\[ |q_{0,\gamma}|_{l_0} \leq C(l_0) |\partial^\gamma_{\xi} p_1|_{l_1} |\partial^\gamma_x p_2|_{l_1}. \]

**Proof.** This is Theorem 3.1 of Chapter 2, and Lemma 2.2 of Chapter 7 of [12].

The next estimate can be proved easily by means of Sobolev embedding theorem and Plancherel’s formula.

**Lemma 2.1.** If $v \in G^s$, then there is a constant $C$ with $|\partial^\alpha x v(x)| \leq C^{1+|\alpha|} |\alpha|^s$, for all $x \in \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n$.

The next lemma provides estimates of $u_k^{(\alpha \beta)}$ and gives a precise meaning to the statement that $u_k^{(\alpha \beta)}$ is supported near the bicharacteristic strip of the symbol $a(x, \xi) = |\xi|^2$.

**Lemma 2.2.** Let $0 \leq t < \infty$. If $(t, x, \xi) \in \text{supp } u_k^{(\alpha \beta)}$, then

\[ |x - x_k - 2t \sigma^\delta_k o_k| \leq \sigma^\delta_k, \quad |\xi - \sigma^\delta_k o_k| \leq \sigma^\delta_k \quad (\text{Case I}), \]

\[ |x - x_0 - 2t \sigma_k o_0| \leq \varepsilon (2t \sigma_0), \quad |\xi - \sigma_k o_0| \leq \sigma^\delta_k \quad (\text{Case II}). \]

Let $\alpha, \beta, \gamma, \delta, \mu$ be multi-indices. Then there is a constant $C = C(l, s, \varepsilon)$ with
We set \( F_k(t) \) and have \( F_k(0) = 0, F_k(\tau_1) = 2k(1 + \tau_k)^{d_0} \).
By continuity of \( F_k \), there is a number \( t_k, 0 < t_k < \tau_k \), such that
\[
k(1 + \tau_k - t_k)^{d_0} \geq F_k(\tau_k) - F_k(t_k), \quad t_k < t < \tau_k.
\]

Now we set \( x_k = y_k + 2t_k \omega_k, \sigma_k = \tau_k - t_k \), and obtain
\[
- \int_0^\tau \sum_{j=1}^n \Re b_j(y_k + 2\theta \omega_k) \omega_k, j \, d\theta = - \int_0^\tau \sum_{j=1}^n \Re b_j(y_k + 2(t_k + \theta) \omega_k) \omega_k, j \, d\theta
\]
\[
= F_k(t_k + \sigma_k) - F_k(t_k) = k(1 + \sigma_k)^{d_0}.
\]

From \( b_j \in L^\infty \) we then conclude that \( \sigma_k \to \infty \). In the same way we get
\[
- \int_0^\sigma \sum_{j=1}^n \Re b_j(x_k + 2\theta \omega_k) \omega_k, j \, d\theta = F_k(\tau_k + \sigma_k) - F_k(t_k + \sigma) \geq k(1 + \sigma)^{d_0} - k(1 + \sigma_k - \theta)^{d_0}
\]
\[
= kd_0\sigma(1 + \theta)^{d_0-1} \geq kd_0\sigma(1 + \sigma_k)^{d_0-1}
\]
for \( 0 \leq \sigma \leq \sigma_k \) and some \( \sigma_k - \sigma < \theta < \sigma_k \).

3. Estimate from above
We write the seminorm \( E_k(t) \) from (1.5) as
\[
E_k(t) = \sum_{|\alpha| \leq N, \beta \leq N} E_{k\alpha\beta}(t),
\]
and gain the following estimates from above if (1.1) is \( G^s \) well-posed:

**Proposition 3.1.** Let the Cauchy problem (1.1) be \( G^s \) well-posed in the sense of Definition 1.1. We then have, for arbitrary \( N \) and every \( 0 \leq t \leq T \),

\[
E_{k\alpha\beta}(t) \leq C\sigma_k^{s_0} C^{\alpha+\beta}(\alpha!\beta!)^{s_0-s_1},
\]

(3.1)

\[
E_k(t) \leq C\sigma_k^C.
\]

(3.2)

**Proof.** The well-posedness of (1.1) yields

\[
||u_k||_{L^2} \leq ||u_k||_{s,\varrho} \leq C||\varphi_k||_{s,\varrho} = \text{const},
\]

(3.3)
due to the choice of \( \varphi_k \). From (2.1), and (2.5), (2.6) we then obtain (3.1); which implies (in conjunction with \( s_1 > s_0 \)) (3.2). \( \Box \)

4. Estimate from below

**Proposition 4.1.** Let the Cauchy problem (1.1) be well-posed in the sense of Definition 1.1, and \( N = [\sigma_k^{s_1}] \).

(Case I) Suppose \( 1 - d_0 = \delta_1 < 1, \ 1 < s_0 < s_1 < 2, \) and

\[
\kappa \leq \delta_2 - (1 - d_0),
\]

(4.1)

\[
\kappa \leq \delta_3 - \delta_2 - \delta_1 - (1 - d_0),
\]

(4.2)

\[
\frac{s_1(s_1 + s_1 - 1)}{s_1} \quad \text{<} \quad \delta_2.
\]

(4.3)

Then we have, for sufficiently large \( \sigma_k \),

\[
E_k(\sigma_k^{1-\delta_1}) \geq \exp(c\sigma_k^{d_0})(c\sigma_k^C \exp(-2\sigma_k^{b_3} ) - C \exp(-\sigma_k^{s_1})).
\]

(4.4)

(Case II) Let \( s \) be sufficiently small, and assume the following conditions:

\[
\delta_2 = 1 + \kappa - d_0,
\]

(4.5)

\[
\frac{\kappa}{s_1} > 2\kappa - d_0,
\]

(4.6)

\[
1 > d_0 > \kappa > \frac{\kappa}{s_1} > \frac{1}{s_1},
\]

(4.7)

\[
\kappa\{s_1(s_1 + s_1 - 1)\} < \delta_2.
\]

(4.8)

Then there is a constant \( T_0, \ 0 < T_0 \leq T \), such that for large \( \sigma_k \):

\[
E_k(T_0) \geq \exp(c\sigma_k^{d_0})(c\sigma_k^C \exp(-2\sigma_k^{b_3} ) - C \exp(-\sigma_k^{s_1})).
\]

(4.9)

The proof is split into the Lemmas 4.1–4.4. For simplicity of notation, we set

\[
v_k(\alpha\beta)(t, x) = W_k(\alpha\beta)(t, x, D_x)u_k(t, x).
\]

(4.10)
Then we have, due to (1.3),

\[ Lf_k^{(\alpha \beta)} = f_k^{(\alpha \beta)} = [L, W_k^{(\alpha \beta)}]u_k. \]  

(4.11)

We introduce the notation

\[ b(x, \xi) = -\sum_{j=1}^n \Re b_j(x) \xi_j, \quad B(x, D_x) = -\sum_{j=1}^n \Re b_j(x) D_{x_j}, \]  

(4.12)

and can deduce that

\[ \|v_k^{(\alpha \beta)}\|_{L^2} \|\partial \|_{L^2} = \Re \langle \partial \rangle v_k^{(\alpha \beta)} + \sum_{j=1}^n \Re (ib_j \partial_{x_j} v_k^{(\alpha \beta)}) + \Re \langle ic \rangle v_k^{(\alpha \beta)} \]  

(4.13)

where we have exploited Garding’s inequality.

**Lemma 4.1.** Let the Cauchy problem (1.1) be well-posed in the sense of Definition 1.1, \( N = [\sigma_k^{N/4}] \), and \( \sigma_k \) large.

(Case I) Assuming (4.2), we have the estimate

\[ \|f_k^{(\alpha \beta)}\|_{L^2} \leq C \sigma_k^{N/4} \sum_{j=1}^n \|v_k^{(\alpha+2\epsilon_j, \beta)}\|_{L^2} + C \sigma_k^{-N/4} \sum_{j=1}^n \|v_k^{(\alpha+\epsilon_j \beta)}\|_{L^2} \]  

\[ + C \sum_{|\gamma|=1} \|C \sigma_k^{-\delta_2}\|_{L^2} \sum_{|\gamma|=1} \|v_k^{(\alpha+\epsilon_j \beta+\gamma)}\|_{L^2} \]  

\[ + C^{N}(\sigma_k^{N/4}) \alpha_{N-\delta_2}^{1/4} \sigma_k^{-N/4} \sum_{j=1}^n \|v_k^{(\alpha+\epsilon_j \beta+\gamma)}\|_{L^2} \]  

where \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 at the \( j \)-th position.

(Case II) For \( \delta_2 < 1 \), the following estimate holds:

\[ \|f_k^{(\alpha \beta)}\|_{L^2} \leq C \|v_k^{(\alpha+2\epsilon_j, \beta)}\|_{L^2} + C \|v_k^{(\alpha+\epsilon_j \beta)}\|_{L^2} \]  

\[ + C \sum_{|\gamma|=1} \|C \sigma_k^{-\delta_2}\|_{L^2} \sum_{|\gamma|=1} \|v_k^{(\alpha+\epsilon_j \beta+\gamma)}\|_{L^2} \]  

\[ + C^{N}(\sigma_k^{N/4}) \alpha_{N-\delta_2}^{1/4} \sigma_k^{-N/4} \sum_{j=1}^n \|v_k^{(\alpha+\epsilon_j \beta+\gamma)}\|_{L^2} \]  

where \( \epsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 at the \( j \)-th position.
Proof. (Case I) The right-hand side $f_k^{(a\beta)}$ of (4.11) is represented by

$$f_k^{(a\beta)} = [L, W_k^{\alpha\beta}] u_k$$

$$= [i \partial_t + \Delta, W_k^{\alpha\beta}] u_k + \sum_{j=1}^{n} [b_j(x) \partial_{x_j}, W_k^{\alpha\beta}] u_k + [c(x), W_k^{\alpha\beta}] u_k,$$

$$= [i \partial_t + \Delta, W_k^{\alpha\beta}] u_k = 2 \sigma_{ij} \sum_{j=1}^{n} \left( \partial_{x_j} - i \sigma_{k\gamma} \omega_k \right) v_k^{(a+e_j, \beta)} - \sigma_{2l} \sum_{j=1}^{n} v_k^{(a+2e_j, \beta)}.$$

Theorem 4 gives us the expansion

$$\text{symb} \left( \left[ b_j(x) D_{x_j}, W_k^{\alpha\beta} \right] \right) (t, x, \xi) = b_j(x) \left( D_{x_j} w_k^{\alpha\beta} \right) (t, x, \xi)$$

$$- \sum_{|\gamma|=1}^{N-1} \frac{1}{\gamma!} \left( \partial_{\xi}^\gamma b_j(x) \xi_j \right) \left( D_{x_j}^\gamma w_k^{\alpha\beta} \right) (t, x, \xi) - r_{kNj}^{(\alpha\beta)} (t, x, \xi).$$

$$r_{kNj}^{(\alpha\beta)} (t, x, \xi) = N \sum_{|\gamma|=N}^{1} \frac{1}{\gamma!} r_{kNj}^{(\alpha\beta)} (t, x, \xi).$$

$$r_{kNj}^{(\alpha\beta)} (t, x, \xi) = -N \sum_{|\gamma|=N}^{1} \frac{1}{\gamma!} r_{kNj}^{(\alpha\beta)} (t, x, \xi).$$

$$= -\theta \int_{\xi} e^{-i\eta} \left( D_{\xi}^\gamma w_k^{(a\beta)} (t, x, \xi + \theta \eta) \right) \left( D_{x_j} \partial_{x_j} b_j(x + y) \right) dy d\eta$$

$$+ \int_{\xi} e^{-i\eta} \left( (\xi_j + \theta \eta_j) D_{\xi}^\gamma w_k^{(a\beta)} (t, x, \xi + \theta \eta) \right) \left( \partial_{x_j} b_j(x + y) \right) dy d\eta.$$
For the estimate of the remaining terms, we define cut-off functions $\chi_k(\xi)$,

$$\chi_k(\xi) = h(42^{-1} \sigma_k^{1-|d_0-\delta_1|}(\xi - \sigma_k^{\delta_1} \omega_k)),$$

and observe that

$$\xi \in \text{supp } \chi_k \Rightarrow |\xi - \sigma_k^{\delta_1} \omega_k| \leq 42 \sigma_k^{\delta_1-1-|d_0|},$$

(4.14)

$$\xi \in \text{supp}(1 - \chi_k) \Rightarrow |\xi - \sigma_k^{\delta_1} \omega_k| \geq 84 \sigma_k^{\delta_1+1+|d_0|},$$

(4.15)

The supports of $(1 - \chi_k)$ and $w_k^{(\alpha \beta)}$ are disjoint, by (2.3) and (4.14). We can write

$$\left(\frac{\partial^\gamma b_j(x) \xi_j}{\alpha \beta}\right) \left(D^{\gamma}_x w_k^{(\alpha \beta)}(t, x, \xi)\right) = \underbrace{K_1 + K_2 + K_3}_{= \left(\frac{\partial^\gamma b_j(x) \xi_j}{\alpha \beta}\right)} + \left(\frac{\partial^\gamma b_j(x) \xi_j}{\alpha \beta}\right) \chi_k(\xi) \circ \left(D^{\gamma}_x w_k^{(\alpha \beta)}(t, x, \xi)\right) - \left(\frac{\partial^\gamma b_j(x)}{\alpha \beta}\right) \left(D_x D^{\gamma}_x w_k^{(\alpha \beta)}(t, x, \xi)\right).$$

Due to Theorem 4, the pseudodifferential symbol $K_1$ can be expanded as

$$K_1(t, x, \xi) = 0 + (N - |\gamma|) \sum_{|\mu| = N - |\gamma|}^{\text{supp}} \int \frac{(1 - \theta)^{N-|\gamma|}-1}{|\mu|!} K_{10\mu}(t, x, \xi) d\theta.$$

Then (1.2), (2.5), (4.2), and (4.15) give us the estimates

$$\left|K_{10\mu}(t, \cdot, \cdot)\right|_{b_0} \leq C \left|\frac{\partial^\mu \partial^\gamma b_j \xi_j}{\alpha \beta}\right| \left(D^{\gamma}_x w_k^{(\alpha \beta)}(t, x, \xi)\right) \leq C N ((\alpha! \beta! \gamma! \mu! \sigma_k^{\delta_1+|d_1|+|\delta_1|+|d_0-\delta_1|})^{N-|\gamma|} \sigma_k^{N-|\gamma|-|\gamma|-|d_0-\delta_1|-1}) \leq C N ((\alpha! \beta! \gamma! \mu! \sigma_k^{\delta_1+|d_1|+|\delta_1|+|d_0-\delta_1|})^{N-|\gamma|} \sigma_k^{N-|\gamma|-|\gamma|-|d_0-\delta_1|-1}).$$

For the estimate of $K_2$, we make use of $|\xi| \leq 2 \sigma_k^{\delta_1}$ on supp $\chi_k$, and get

$$\left\|K_2(t, x, D_x) w_k(t, x)\right\|_{L^2} \leq C |\gamma| |\alpha \beta| \sigma_k^{\delta_1-|d_1|} \sigma_k^{\delta_1-|d_1|} \|w_k^{(\alpha \beta + \gamma)}\|_{L^2}.$$

Similarly,

$$\left\|K_3(t, x, D_x) w_k(t, x)\right\|_{L^2} \leq C |\gamma| |\alpha \beta| \sigma_k^{\delta_1-|d_1|} \sigma_k^{\delta_1-|d_1|} \|w_k^{(\alpha + \gamma, \beta + \gamma)}\|_{L^2}.$$
The term \( \sigma_k^{ij}(\partial_{x_i} - i\sigma_k^{ij}\omega_{k,j})v_k^{(\alpha+\varepsilon_j,\beta)} \) can be estimated similarly as \( K_1 \) and \( K_2 \) above (with \( \gamma = 0 \)), leading to
\[
\| \sigma_k^{ij}(\partial_{x_i} - i\sigma_k^{ij}\omega_{k,j})v_k^{(\alpha+\varepsilon_j,\beta)} \|_{L^2} \leq C \sigma_k^{\delta_j+\delta_2}v_k^{(\alpha+\varepsilon_j,\beta)} \|_{L^2} + C^N(\alpha!\beta!)^0\sigma_k^C(N^{2n-1}\sigma_k^{\delta_2-x})^N \|u_k\|_{L^2}. \tag{4.16}
\]

This completes the proof in Case I.

(Case II) Now one part of the right-hand side \( f_k^{(\alpha\beta)} \) is given by
\[
[i\partial_t + \Delta, W_k^{(\alpha\beta)}]u_k = 2(2\tau\alpha_k)^{-1}\sum_{j=1}^n (\partial_{x_j} - i\sigma_k\omega_{k,j})v_k^{(\alpha+\varepsilon_j,\beta)}
- (2\tau\alpha_k)^{-2}\sum_{j=1}^n v_k^{(\alpha+2\varepsilon_j,\beta)}
- i(2\tau\alpha_k)^2\sum_{j=1}^n x_j - x_{0,j} - 2t\sigma_k\omega_{k,j}v_k^{(\alpha+\varepsilon_j,\beta)}.
\]

We choose the cut-off function \( \chi_k(\xi) = h(42^{-1}\sigma_k^{-\delta_2}(\xi - \sigma_k\omega_k)) \), and the rest of the proof runs similarly as above. \( \square \)

Now we estimate the next term of the right-hand side of (4.13).

**Lemma 4.2.** (Case I) Under the assumptions of Lemma 4.1 and \( 1 - \delta_0 \leq \delta_1 \),
\[
\| (B(x, D_x)v_k^{(\alpha\beta)}(t,x), v_k^{(\alpha\beta)}(t,x)) \|
\geq (B(v_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k) - C\sigma_k^{\delta_3-1}\delta_k)
\| v_k^{(\alpha\beta)}(t,\cdot) \|^2_{L^2}
- C^N(\alpha!\beta!)^0\sigma_k^C(\mathbb{N}^{(2n-1)}\sigma_k^{\delta_3-2})^N \| v_k^{(\alpha\beta)}(t,\cdot) \|^2_{L^2}, \ 0 \leq t < \infty.
\]

(Case II) If \( \delta_2 < 1, \sigma_k \) is large enough and \( \varepsilon > 0 \) is small enough, then
\[
\| (B(x, D_x)v_k^{(\alpha\beta)}(t,x), v_k^{(\alpha\beta)}(t,x)) \|
\geq (c_0\sigma_k(t\alpha_k)^{\delta_0-1} - C) \| v_k^{(\alpha\beta)}(t,\cdot) \|^2_{L^2}
- C^N(\alpha!\beta!)^0\sigma_k^C(\mathbb{N}^{(2n-1)}\sigma_k^{\delta_2-2})^N \| v_k^{(\alpha\beta)}(t,\cdot) \|^2_{L^2}, \ 0 \leq t < \infty.
\]

**Proof.** (Case I) We split the operator \( B(x, D_x) \) from (4.12) into three parts:
\[
B(x, D_x) = I_1 + I_2 + I_3 = (B(x, D_x) - B(x, \sigma_k^{\delta_3}\omega_k))
+ (B(x, \sigma_k^{\delta_3}\omega_k) - B(v_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k))
+ B(v_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k).
\]

Utilizing the idea from estimate (4.16), and (3.3), we find
\[
\| I_1 v_k^{(\alpha\beta)} \|^2_{L^2} \leq C \sigma_k^{\delta_2} v_k^{(\alpha\beta)} \|_{L^2} + C^N(\alpha!\beta!)^0\sigma_k^C(N^{2n-1}\sigma_k^{\delta_2-x})^N.
\]
On $\text{supp} \, v_k^{(\alpha \beta)}(t, \cdot)$ we have $|x - x_k - 2t\sigma_k^{\delta_5} \omega_k| < \sigma_k^{-\delta_1}$, see Lemma 2.2. Therefore,

$$\|I_2v_k^{(\alpha \beta)}\|_{L^2} \leq C\sigma_k^{\delta_3-\delta_1}\|v_k^{(\alpha \beta)}\|_{L^2}.$$  

By (4.2) and $1 - d_0 \leq \delta_1$, we may estimate $\sigma_k^{\delta_2} \leq \sigma_k^{\delta_3-1+d_0}$ and $\sigma_k^{\delta_3-\delta_1} \leq \sigma_k^{\delta_1-1+d_0}$.

(Case II) Choosing the cut-off functions $\chi_k(\xi) = h(42^{-1}\sigma_k^{-\delta_2}(\xi - \sigma_k\omega_k))$ and $\psi_k(t, x)$

$$= h(\epsilon^{-1}42^{-1}(2t\sigma_k)^{-1}(x - x_0 - 2t\sigma_k\omega_k)),$$

we can split

$$b(x, \xi) = I_1(t, x, \xi) + I_2(t, x, \xi) + I_3(t, x, \xi) = c_0\sigma_k(t\sigma_k)^{d_0-1}$$

$$+ (b(x, \xi) - c_0\sigma_k(t\sigma_k)^{d_0-1})\psi_k(t, x)\chi_k(\xi)$$

$$+ (b(x, \xi) - c_0\sigma_k(t\sigma_k)^{d_0-1})(1 - \psi_k(t, x)\chi_k(\xi)).$$

Let $(t, x, \xi) \in \text{supp} \, \psi_k(\cdot, \cdot)\chi_k(\cdot)$, and $\epsilon$ sufficiently small. Then Condition 2 yields

$$\|b(x, \xi)\| \leq -\epsilon |\xi| \sum_{j=1}^{n} N_b f_j \left( x + |x - x_0| \cdot \frac{x - x_0}{|x - x_0|} \right) \frac{\xi n}{|\xi|}$$

$$\geq 2\epsilon |\xi| c_0 |x - x_0|^{d_0-1} \geq c_0\sigma_k(t\sigma_k)^{d_0-1}.$$  

Moreover, $I_2(t, \cdot, \cdot) \in S_{1,0}^I$, and its symbol estimates are uniform in $t$ and $k$. Then Garding’s inequality gives the uniform in $t$ and $k$ estimate

$$\|I_2v, v\| \leq -C\|v\|_{L^2}^2.$$  

Finally, the supports of $I_3$ and $w_k^{(\alpha \beta)}$ are disjoint, according to the choice of $\chi_k$, $\psi_k$, and (2.4). This completes the proof, see the estimate of $K_1$ in the proof of Lemma 4.1. \hfill $\Box$

**Lemma 4.3.** Let the Cauchy problem (1.1) be $G^k$ well-posed, $N = [\sigma_k^{\ell_1}]$, where $\sigma_k$ is large, and $1 < s_0 < s_1$ with $s_1$ very close to 1.

(Case I) Suppose $\delta_1 = 1 - d_0$, and (4.1)–(4.3). Then the seminorm $E_k$ satisfies, for $0 \leq t \leq T$, the estimate

$$\partial_t E_k(t) \geq \left( B(x_k + 2\sigma_k^{\delta_1} \omega_k, \sigma_k^{\delta_1} \omega_k) - C\sigma_k^{\delta_1-1+d_0} \right) E_k(t) - C\sigma_k^{C-C^N}. \quad (4.17)$$

(Case II) Under the assumptions of the Lemmas 4.1 and 4.2, the seminorm $E_k$ satisfies the following inequality:

$$\partial_t E_k(t) \geq \left( c_0\sigma_k(t\sigma_k)^{d_0-1} - C \sigma_k^{\delta_2 + \delta_5} - C \sigma_k^{\delta_2 + \delta_5} - C \sigma_k^{\delta_2 + \delta_5} \right) E_k(t) - C\sigma_k^{C-C^N}, \quad 0 \leq t \leq T. \quad (4.18)$$

**Proof.** (Case I) We employ (4.13), and the Lemmas 4.1, 4.2:

$$\partial_t \|v_k^{(\alpha \beta)}\|_{L^2} \geq \left( B(x_k + 2\sigma_k^{\delta_1} \omega_k, \sigma_k^{\delta_1} \omega_k) - C\sigma_k^{\delta_1-1+d_0} \right) \|v_k^{(\alpha \beta)}\|_{L^2}$$

$$- C\sigma_k^{\delta_2} \sum_{j=1}^{n} \|v_k^{(\alpha+2\epsilon_j, \beta)}\|_{L^2} - C\sigma_k^{\delta_2} \sum_{j=1}^{n} \|v_k^{(\alpha+\epsilon_j, \beta)}\|_{L^2}.$$
\[
- C \sum_{|\gamma|=1}^{N-1} (C_{\sigma_k}^{\delta_k})_{\gamma} |\gamma|! \gamma^{\tau_{N-1}} - 1 \\
\times \left( |\sigma_k^{\delta_k}| v_k^{(\alpha, \beta + \gamma)} \right)_{L^2} + |\sigma_k^{\delta_k}| v_k^{(\alpha + \gamma, \beta + \gamma)} |L^2| \\
- C^N (\alpha! \beta!) \gamma \nu (C_{\sigma_k}^{\gamma})_{\gamma} |\gamma|! (\alpha + \gamma)! |\gamma|! (\beta + \gamma)! |\gamma|! E_k(\beta + \gamma) \\
= (1 - d_0) E_k \\
- C \sigma_k^{\delta_k} \sum_{|\gamma|=1}^{N-1} (C_{\sigma_k}^{\delta_k})_{\gamma} |\gamma|! |\gamma|! \gamma^{\tau_{N-1}} - 1 \\
\times \left( (\beta + \gamma)! |\gamma|! E_k(\beta + \gamma) \right) - C^N \sum_{|\gamma|=1}^{N-1} (C_{\sigma_k}^{\gamma})_{\gamma} |\gamma|! (\alpha! \beta!)^{N-1} - 1.
\]

The last double-sum on the right is bounded, due to \( s_1 > s_0 \).

Let us discuss all these terms one after the other. Recalling that \( N^{s_1} \sim \sigma_k^s \), we get from the assumptions (4.1), (4.2) the inequalities

\[ 2(1 - d_0) + 2 \kappa \leq \delta_2 - 1 + d_0, \quad (1 - d_0) + \delta_2 + \kappa \leq \delta_3 - 1 + d_0. \]

According to Proposition 3.1,

\[ E_k(\beta + \gamma) \leq C^N (\beta^{s_0 - \delta_2} \sigma_k^{-(s_0 - \delta_2) N^{s_1}}), \quad N + 1 \leq |\alpha| \leq N + 2. \]

By Stirling’s formula, \((\beta + \gamma)!/|\beta|! \leq (2N)^{|\gamma|} \leq (C_{\sigma_k}^{\gamma})_{|\gamma|} |\gamma|!\) if \(|\beta| + |\gamma| \leq 2N\). For \( 1 \leq |\gamma| \leq N \), we conclude that

\[ (C_{\sigma_k}^{\delta_k})_{|\gamma|} |\gamma|! \gamma^{\tau_{N-1}} - 1 \leq (C_{\sigma_k}^{\delta_k - \kappa(s_0 + \delta_3 - 1)/s_1 + \kappa})_{|\gamma|} = (C_{\sigma_k}^{\gamma})_{|\gamma|}, \quad (4.19) \]

where \( \gamma_1 = \delta_2 - \kappa (s_0 + s_3 - 1)/s_1 > 0 \), due to (4.3). There is a \( \Gamma_0 = \Gamma_0(\delta_2, \kappa, \gamma_1) \) with

\[ (C_{\sigma_k}^{\gamma})_{|\gamma|} \leq C_{\sigma_k}^{\delta_k - \kappa} \gamma |\gamma| \]

for \(|\gamma| \geq \Gamma_0 \) and \( \sigma_k \) large. If \( 1 \leq |\gamma| \leq \Gamma_0 \), we can neglect the factor \( \gamma^{\tau_{N-1}} - 1 \) and get

\[ (C_{\sigma_k}^{\gamma})_{|\gamma|} \gamma^{\tau_{N-1}} - 1 \leq C_{\sigma_k}^{\delta_k - \kappa} \gamma |\gamma|. \]
In case $|\beta + \gamma| > N$, we have (according to (3.1) and (4.19))

$$
(C\sigma_k^{-\delta_1})^{\beta_1(\beta_1^2+1)} \left(\frac{\beta_1(\beta_1+\gamma)}{\delta_1}\right) c_k(\beta_1+\gamma) \\
 \leq (C\sigma_k^{-\epsilon_1})^{\beta_1(\beta_1^2+1)} \alpha_1(\beta_1+\gamma)^{\beta_1-\gamma_1} \leq C^N N^{\gamma_0-\gamma_1} \leq C^N \sigma_k^{-\epsilon N_0}.
$$

From the assumptions (4.1), (4.2) it can be deduced that

$$
\delta_3 + (\kappa - \delta_2) \leq \delta_3 - 1 + d_0, \\
(1 - d_0) + \kappa + (\kappa - \delta_2) \leq \kappa \leq \delta_3 - 1 + d_0.
$$

Summing up, we can conclude that

$$
\partial_t E_k(t) \geq \left( B \left( x_k + 2\sigma_0 \omega_k, \sigma_0 \omega_k \right) - C\sigma_k^{-\delta_3-1+d_0} \right) E_k(t) - C \sigma_k^{-\epsilon} N^k.
$$

This completes the proof of (4.17).

(Case II) The proof is similar, therefore we drop it. $\blacksquare$

We write (4.17) and (4.18) in the form

$$
\partial_t E_k(t) \geq A_k(t) E_k(t) - R_k(t).
$$

(4.20)

The following lemma is an analog to Lemma 1.1.

Lemma 4.4. (Case II) Suppose (4.5), (4.7), and let $\delta_2$ be any number with $0 < \delta_2 < d_0 - \kappa$. Then there is a constant $T_0$, $0 < T_0 \leq T$, such that the function $A_k = A_k(t)$ of (4.18) and (4.20) has the following properties:

$$
\int_0^t A_k(\tau) \, d\tau \geq -C \sigma_k^{-\epsilon}, \quad 0 \leq t \leq T_0,
$$

$$
\int_0^{T_0} A_k(\tau) \, d\tau \geq C \sigma_k^{-d_0}.
$$

Proof. By computation and (4.5),

$$
\int_0^t A_k(\tau) \, d\tau = \int_0^{\sigma_k t} \frac{c_0(\tau) \gamma_0}{(\gamma_0+1)} - C \gamma_0 \sigma_k^{\gamma_0+1} \tau - C \gamma_0 \sigma_k^{\gamma_0+1} \tau \, d\tau \\
\geq \int_0^{\sigma_k t} \frac{c_0}{d_0} \left( 1 + \sigma_k t \right) - C \sigma_k^{\gamma_0+(\kappa-1)} \ln \left( 1 + \sigma_k t \right) - C \sigma_k^{d_0}.
$$

We distinguish two cases.

Case (a): $0 \leq \sigma_k t \leq 42$. Then we have, by (4.7),

$$
\int_0^t A_k(\tau) \, d\tau \geq -C \sigma_k^{\gamma_0+(\kappa-1)} - C \sigma_k^{\gamma_0+(\kappa-42)} - C \geq -C \sigma_k^{-\epsilon_2}.
$$
Case (β): $42 \leq \sigma_k t \leq \sigma_k T$. Using $\ln(1+r) \leq C_\gamma r^\gamma$ for each $\gamma > 0$, we obtain
\[
\int_0^t A_k(\tau) \, d\tau \geq C_1 t^d_0 \sigma_k^d_0 - C_\gamma \sigma_k^{\gamma x+(\kappa-1)} - C \sigma_k^{\kappa-\varepsilon_2} - C_2 \sigma_k^{d_0}.
\]
It remains to choose $T_0 > 0$ with $C_2 t \leq (1/2) C_1 t^d_0$ for $0 \leq t \leq T_0$.

Now we are in a position to estimate $E_k$ from below.

**Proof of Proposition 4.1.** From Gronwall’s Lemma and (4.20) it follows that
\[
E_k(T_0) \geq \exp \left( \int_0^{T_0} A_k(\tau) \, d\tau \right) \left( E_k(0) - \int_0^{T_0} \exp \left( - \int_0^\tau A_k(\sigma) \, d\sigma \right) R_k(\tau) \, d\tau \right).
\]

Recalling Lemma 4.3, we find
\[
0 \leq R_k(\tau) \leq C \exp(-2\sigma_k^{k/s_1}) \leq C \exp(-2\sigma_k^{k/s_1}).
\]

In Case I, we choose $T_0 = t_k = \sigma_1^{1-\delta_3} < T$. Then Lemma 1.1 yields
\[
\int_0^{T_0} A_k(\tau) \, d\tau \geq (k-C) \sigma_0^d_0,
\]
\[
- \int_0^\tau A_k(\sigma) \, d\sigma \leq 0, \quad 0 \leq \tau \leq t_k.
\]

In Case II, the needed estimates of $A_k$ are given in Lemma 4.4. Since $\kappa/s_1 > 2\kappa - d_0$, we may choose $0 < \varepsilon_2 < d_0 - \kappa$ such that $\kappa/s_1 > \kappa - \varepsilon_2$, which ensures
\[
\exp \left( - \int_0^\tau A_k(\sigma) \, d\sigma \right) R_k(\tau) \leq C \exp(-\sigma_k^{k/s_1}).
\]

Next we consider $E_k(0)$. In Case I, we have
\[
E_k(0) \geq \left\| W_k(0, x, \xi) u_k(0, x) \right\|_{L_2}^2
\]
\[
= \sigma_k^d \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \hat{h} \left( \frac{\xi - \eta}{\sigma_k^{\delta_1}} \right) \frac{\hat{\varphi}(\eta)}{\sigma_k^{\delta_2}} \right|^2 \hat{h}(\eta) \, d\eta \, d\xi.
\]

We fix $\hat{\varphi}(\xi) = (\xi^{-\alpha(\alpha+1)/2} \exp(-\partial_0(\xi)^{1/s_1})$ and choose $h$ in such a way that $\hat{h}(0) > 0$ and $\hat{h}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. The existence of such functions $h$ can be proved by means of
the methods of [6, Chapter 1]. Then we get a lower estimate of $E_k(0)^2$ by restricting the domains of integration. We set

$$G_{1k} = \{ \eta : |\eta - \sigma_k^3/\delta_k| \leq \sigma_k^2/4, \,(\sigma_k^3 - \sigma_k^2/4)\omega_k - \eta| \leq 1 \},$$

$$G_{2k} = \{ \xi : \exists \eta \in G_{1k}, \,|\xi - \eta| \leq 1 \}$$

and obtain

$$E_k(0)^2 \geq c \sigma_k^{-C} \int_{\xi \in G_{2k}} \int_{\eta \in G_{1k}} |\hat{\phi}(\eta)|^2 d\xi \geq c \sigma_k^{-C} \exp(-4\varrho_0 \sigma_3^{1/3}).$$

Similarly, $E_k(0)^2 \geq c \sigma_k^{-C} \exp(-4\varrho_0 \sigma_3^{1/3})$ in Case II.

Summing up, we obtain (4.4) and (4.9), and Proposition 4.1 is proved. \(\square\)

5. The choice of parameters

**Proof of Theorem 1.** The estimates from Proposition 3.1 and 4.1 can be combined in the following way:

$$C \sigma_k^C \geq E_k(\sigma_k^{1-\epsilon}) \geq \exp(ck \sigma_k^{d_0})(c \sigma_k^{-C} \exp(-C_1 \sigma_k^{g_3/s}) - C \exp(-\sigma_k^{g_3/s})). \quad (5.1)$$

Assume now

$$\frac{\delta_3}{s} < \frac{\kappa}{s_1}, \quad \frac{\delta_3}{s} \leq d_0. \quad (5.2)$$

Then the right-hand side of (5.1) is positive for large $\sigma_k$. If $\sigma_k$ becomes even larger, then the right-hand side becomes bigger than the left-hand side, because $d_0 \geq \delta_3/s$. That is the desired contradiction.

It remains to show how to choose all constants so that the constraints $d_1 = 1 - d_0$ and (4.1), (4.2), (4.3), (5.2) are satisfied.

In order to be able to choose $d_0$ small, we should choose $\delta_3$ as small as possible. Therefore, we fix $\kappa = \delta_3 - \delta_2 = 2(1 - d_0)$, and choose $s_1$ very close to 1. Then this system is solvable if and only if

$$\kappa \leq \delta_3 - 3(1 - d_0) - \kappa, \quad \frac{\kappa(s_0 + s_1 - 1)}{s_1} < \delta_3 - \kappa < 2(1 - d_0),$$

and (5.2) hold, which are equivalent to

$$\frac{\delta_3}{s} < \frac{\kappa}{2} \left( \frac{\delta_3 - 3(1 - d_0)}{s} \right), \quad \frac{\kappa s_0 + 2s_1 - 1}{s_1} < \delta_3 - 2(1 - d_0), \quad \frac{\delta_3}{s} \leq d_0.$$  

This system has a solution $\kappa$ if

$$\frac{2\delta_3}{s} + 3(1 - d_0) < \delta_3, \quad \delta_3 \leq d_0 s, \quad \frac{\delta_3}{s} (s_0 + 1) < 2(1 - d_0),$$

which is equivalent to

$$3(1 - d_0) < \delta_3 (1 - 2/s), \quad \delta_3 \leq d_0 s, \quad 2(1 - d_0) < \delta_3 (1 - (s_0 + 1)/s).$$
which has a solution $\delta_3$ if and only if

$$3(1 - d_0) < d_0(s - 2), \quad 2(1 - d_0) < d_0(s - s_b - 1).$$

These are the conditions of Theorem 1. $\blacksquare$

**Proof of Theorem 2.** In order to prove the ill-posedness of (1.1), we have to satisfy the constraints (4.5), (4.6), (4.7), and (4.8).

Eliminating $\delta_2$ we find

$$\frac{\kappa}{s_1} > 2\kappa - d_0, \quad 1 > d_0 > \kappa > \frac{1}{s}, \quad \frac{1 - d_0}{s_b - 1} > \frac{\kappa}{s_1}$$

which has a solution $\kappa/s_1$ if and only if

$$\frac{1 - d_0}{s_b - 1} > \frac{1}{s}, \quad 1 > d_0 > \kappa > \frac{1}{s}, \quad \frac{1 - d_0}{s_b - 1} > 2\kappa - d_0.$$ 

And this system has a solution $\kappa$ if and only if

$$\frac{1 - d_0}{s_b - 1} > \frac{1}{s}, \quad 1 > d_0 > \kappa > \frac{1}{s}, \quad d_0 + \frac{1 - d_0}{s_b - 1} > \frac{2}{s}$$

which is equivalent to (1.6). $\blacksquare$

**Acknowledgements**

The author is indebted to Prof. Kajitani and would like to thank him for his advice and encouragement. Moreover, the author thanks Prof. Doi for discussions which influenced the final version of this paper.

**References**


