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Local Solutions of Weakly Parabolic Quasilinear Differential Equations

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Abstract. We consider a quasilinear parabolic boundary value problem, the elliptic part of which degenerates near the boundary. In order to solve this problem, we approximate it by a system of linear degenerate elliptic boundary value problems by means of semidiscretization with respect to time. We use the theory of degenerate elliptic operators and weighted Sobolev spaces to find a–priori–estimates for the solutions of the approximating problems. These solutions converge to a local solution, if the step size of the time–discretization goes to zero. It is worth pointing out that we do not require any growth conditions on the nonlinear coefficients and right–hand side, since we are able to prove L^∞–estimates.

1. Introduction

We want to prove, by means of the Rothe method, the local existence of a solution of the weakly parabolic quasilinear initial boundary value problem

$$u_t(x,t) + A_{t,u}u(x,t) = f(x,t,u) \quad \text{in } Q, \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \Gamma, \tag{1.2}$$

$$u(x,0) = U_0(x) \quad \text{in } \Omega. \tag{1.3}$$

Let us denote by $\Omega \subset \mathbb{R}^N$ a bounded domain with boundary $\partial\Omega \in C^1$, $T > 0$, $I = [0,T]$, $Q = \Omega \times I$, $\Gamma = \partial\Omega \times I$ and

$$A_{t,v}u = -\sum_{i,k=1}^{N} \frac{\partial}{\partial x_i}\left(\varrho^{\mu}(x) b_{ik}(x,t,v)\frac{\partial u}{\partial x_k}\right) + \sum_{i=1}^{N} \varrho^{(\nu+\mu)/2}(x) a_i(x,t,v)\frac{\partial u}{\partial x_i} + \varrho^{\nu}(x) b_0(x,t,v)u.$$

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The function ϱ describes the degeneration and fulfills some additional conditions. For discussions of semilinear weakly parabolic equations with degenerated coefficient of u_t we refer the reader to [Kač85], [Kač90], [Plu92] and [Web95].

Semilinear problems with degenerated elliptic part were studied in [DP96]. In this paper we used Hilbert space techniques to establish the a–priori–estimates. This method breaks down in the case of *quasilinear* problems. Therefore we use a–priori–estimates and existence results in weighted $W_p^2(\Omega)$–spaces of solutions of degenerated elliptic boundary value problems from [Tri78].

We emphasize that the coefficients and the right–hand side are defined only in a neighbourhood of the initial data. For this reason one has to prove L^∞–estimates of $u - U_0$. This is done by the technique of Moser and Alikakos, see [Mos60] and [Ali79]. To handle the quasilinear elliptic part we use some techniques from [Plu96].

2. Preliminaries

From now on, $\|.\|_p$ stands for the Lebesgue space norm and $(.,.)$ for the $L^2(\Omega)$ scalar product. We will write the spaces on continuous and L^p–integrable mappings $I \to V$ as $C(I, V)$, $L^p(I, V)$, respectively. We will use the letters C, c to denote positive constants which may have different values at different places, but are independent of h and p.

The function ϱ has to satisfy the conditions:

$$\varrho \in C^1(\Omega), \quad \varrho(x) \geq c > 0, \quad |\nabla \varrho(x)| \leq c_\varrho \varrho(x)^2, \tag{2.1}$$

$$\varrho(x) \to \infty \text{ uniformly, if } \operatorname{dist}(x, \partial\Omega) \to 0. \tag{2.2}$$

Example 2.1. There exists a weight function ϱ with

$$c_1 \operatorname{dist}(x, \partial\Omega) \leq \varrho(x)^{-1} \leq c_2 \operatorname{dist}(x, \partial\Omega).$$

For this example and the following definition of weighted Sobolev spaces cf. [Tri78].

Definition 2.2. Let $1 < p < \infty$, $s \geq 0$, $\mu + sp < \nu$.
If $s \in \mathbb{N}$, then we define

$$W_p^s(\Omega, \varrho^\mu, \varrho^\nu) := \big\{ f \in L^p_{\text{loc}}(\Omega) : \|f\|^p_{W_p^s(\Omega,\varrho^\mu,\varrho^\nu)} := \int_\Omega \sum_{|\alpha|=s} \varrho^\mu(x)\, |\partial^\alpha f(x)|^p + \varrho^\nu(x) |f(x)|^p \, dx < \infty \big\}.$$

If $s = [s] + \{s\}$ with $[s] \in \mathbb{N}$, $0 < \{s\} < 1$, then we define

$$W_p^s(\Omega, \varrho^\mu, \varrho^\nu) := \big\{ f \in L^p_{\text{loc}}(\Omega) : \|f\|^p_{W_p^s(\Omega,\varrho^\mu,\varrho^\nu)} := \int_{\Omega\times\Omega} \sum_{|\alpha|=[s]} \frac{\left| \varrho^{\frac{\mu}{p}}(x)\partial^\alpha f(x) - \varrho^{\frac{\mu}{p}}(y)\partial^\alpha f(y) \right|^p}{|x-y|^{N+\{s\}p}} \, dx\, dy + \int_\Omega \varrho^\nu(x) |f(x)|^p \, dx < \infty \big\}.$$

These spaces are reflexive Banach spaces, $C_0^\infty(\Omega)$ is a dense subset.
Now we are able to write down the assumptions. For fixed $R > 0$ we set

$$M_R(U_0) := \{(x,t,u) \in \mathrm{I\!R}^{N+2} : (x,t) \in \Omega \times [0,T], |u - U_0(x)| \leq R\},$$
$$B_R(U_0) := \{u \in L^\infty(\Omega) : \|u - U_0\|_\infty \leq R\}.$$

We assume

$$\nu > \mu + 2, \quad \mu < 0 < \nu, \quad \nu + \mu > 0, \quad \frac{2\nu}{\nu - \mu} > 1 + \frac{N}{p_0}, \tag{2.3}$$
$$U_0 \in W^2_{p_0}\left(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu}\right), \quad p_0 \geq 2, \quad p_0 > N, \tag{2.4}$$
$$\varrho^{-\mu} \in L^{N'}(\Omega), \quad \varrho^\nu \notin L^1(\Omega), \quad N' > N. \tag{2.5}$$

For $\xi \in \mathrm{I\!R}^N, (x,t,u) \in M_R(U_0)$ we suppose

$$\sum_{i,k=1}^N b_{ik}(x,t,u)\xi_i\xi_k \geq C_E|\xi|^2, \quad b_0(x,t,u) \geq C_E > 0, \tag{2.6}$$
$$|a_i(x,t,u)| \leq C_a \varrho^{-\delta}(x), \quad 0 < \delta < \frac{\nu}{2}. \tag{2.7}$$

Furthermore,

$$b_{ik}, \frac{\partial b_{ik}}{\partial x_j}, \frac{\partial b_{ik}}{\partial u}, b_0, \frac{\partial b_0}{\partial x_j}, \frac{\partial b_0}{\partial u}, a_i \in C(\overline{M_R(U_0)}). \tag{2.8}$$

For $t, t' \in [0,T], \quad u, u' \in B_R(U_0)$ we assume

$$\|b_{ik}(.,t,u(.)) - b_{ik}(.,t',u'(.))\|_{\beta_1} \leq C_b(|t-t'| + \|u-u'\|_\sigma), \tag{2.9}$$
$$\frac{1}{\beta_1} < \frac{1}{2}\left(\frac{2}{N} - \frac{1}{N'}\right), \tag{2.10}$$
$$\|b_0(.,t,u(.)) - b_0(.,t',u'(.))\|_{\beta_0} \leq C_b(|t-t'| + \|u-u'\|_\sigma), \tag{2.11}$$
$$\frac{1}{\beta_0} + \frac{1}{p_0} < \frac{2}{N} - \frac{1}{N'}, \tag{2.12}$$
$$\|a_i(.,t,u(.)) - a_i(.,t',u'(.))\|_{\alpha_1} \leq C_a(|t-t'| + \|u-u'\|_\sigma), \tag{2.13}$$
$$\frac{1}{\alpha_1} + \frac{1}{p_0} < \frac{2}{N} - \frac{1}{N'}, \tag{2.14}$$
$$\|f(.,t,u(.))\|_\varphi \leq C_f, \quad \varphi \geq p_0, \tag{2.15}$$
$$\|f(.,t,u(.)) - f(.,t',u'(.))\|_\varphi \leq C_f(|t-t'| + \|u-u'\|_\sigma), \tag{2.16}$$
$$\frac{p_0}{2\sigma} > \frac{N'+1}{2N'} - \frac{1}{N}, \quad \sigma \geq \max(\beta_1, \beta_0, \alpha_1, \varphi). \tag{2.17}$$

We can drop (2.10), (2.12) and (2.14), if b_{ik}, b_0, a_i do not depend on u.
We are looking for weak solutions of (1.1), (1.2), (1.3), i.e. functions with

$$\int_{I^*} (u_t, v)\ dt + \int_{I^*} a_{t,u}(u,v)\, dt = \int_{I^*} (f(x,t,u), v)\ dt \tag{2.18}$$

for all $v \in L^1(I^*, W^1_{p_0'}(\Omega))$ $(p_0^{-1} + {p_0'}^{-1} = 1)$, where

$$a_{t,w}(u,v) = \sum_{i,k=1}^{N} \int_\Omega \varrho^\mu b_{ik}(x,t,w) \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_i}\, dx$$
$$+ \sum_{i=1}^{N} \int_\Omega \varrho^{(\mu+\nu)/2} a_i(x,t,w) \frac{\partial u}{\partial x_i} v\, dx + \int_\Omega \varrho^\nu b_0(x,t,w) uv\, dx.$$

We will prove

Theorem 2.3. *Under the assumptions* (2.1), ... , (2.17) *there exists a* T^*, $0 < T^* \le T$, *such that problem* (2.18) *has a uniquely determined solution*

$$u \in C(I^*, C^1(\Omega)) \cap L^\infty\left(I^*, W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})\right), \quad u_t \in L^\infty(I^*, L^{p_0}(\Omega)).$$

Remark 2.4. The regularity of u guarantees that u is even a solution of (1.1). The boundary condition (1.2) follows from $u\varrho^\nu \in L^{p_0}(\Omega)$ and $\varrho^\nu \notin L^1(\Omega)$.

Remark 2.5. Let us show first that it is possible to fulfill the assumptions of this theorem: We choose ϱ as in Example 2.1. Let $\nu = 2$. We fix $p_0 \ge 2$, $p_0 > N$, such that the last inequality of (2.3) is valid for all $-N^{-1} < \mu < 0$. Furthermore,

$$\beta_1 = \beta_0 = \alpha_1 = \varphi = \sigma = p_0 > N.$$

The inequalities (2.10), (2.12), (2.14), (2.17) are satisfied, if $N' > N$ is sufficiently large. Finally, we choose μ with $-{N'}^{-1} < \mu < 0$. Obviously, the other assumptions can be fulfilled, too.

We need some tools.

Theorem 2.6. (*see [Tri78], Theorem 3.4.2 and Remark 3.4.2/1*)
Let $0 \le s_1, s_2 < \infty$, $1 < p_1, p_2 < \infty$, $\nu_i \ge \mu_i + s_i p_i$, $(i = 1, 2)$ *and*

$$(\mu_1 - \nu_1)s_2 p_2 = (\mu_2 - \nu_2)s_1 p_1.$$

Furthermore, let $0 < \theta < 1$ *and*

$$s = (1-\theta)s_1 + \theta s_2, \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$
$$\frac{\nu}{p} = (1-\theta)\frac{\nu_1}{p_1} + \theta\frac{\nu_2}{p_2}, \qquad \frac{\mu-\nu}{sp} = \frac{\mu_1-\nu_1}{s_1p_1} = \frac{\mu_2-\nu_2}{s_2p_2}.$$

(*If* $s_1 = 0$, $s_2 > 0$, *then set* $\mu_1 = \nu_1$ *and* $\frac{\mu-\nu}{sp} = \frac{\mu_2-\nu_2}{s_2p_2}$.)
If s_1, s_2, s *are integers, then the interpolation*

$$\left[W^{s_1}_{p_1}(\Omega, \varrho^{\mu_1}, \varrho^{\nu_1}), W^{s_2}_{p_2}(\Omega, \varrho^{\mu_2}, \varrho^{\nu_2})\right]_\theta = W^s_p(\Omega, \varrho^\mu, \varrho^\nu)$$

holds.

If s_1, s_2 are integers, but s is not, then

$$\left(W_{p_1}^{s_1}\left(\Omega,\varrho^{\mu_1},\varrho^{\nu_1}\right),W_{p_2}^{s_2}\left(\Omega,\varrho^{\mu_2},\varrho^{\nu_2}\right)\right)_{\theta,p}=W_p^s\left(\Omega,\varrho^{\mu},\varrho^{\nu}\right).$$

Theorem 2.7. (*see [Dre96], Satz 3.2.25*) *We assume (2.3), $p\geq p_0>N$ and $p_0\geq 2$. Then we have the continuous embedding $W_p^2\left(\Omega,\varrho^{p\mu},\varrho^{p\nu}\right)\subset C_0^1(\overline{\Omega})$, where $C_0^1(\overline{\Omega})$ denotes the set of all $C^1(\overline{\Omega})$–functions with vanishing boundary values.*

Lemma 2.8. (*see [Dre96], Lemma 3.2.6*)
Let $\varrho^{-\mu N'}\in L^1(\Omega)$, $N'>N$, $N'\in\mathbb{R}$. Then the embeddings

$$W_2^1\left(\Omega,\varrho^{\mu},\varrho^{0}\right)\subset W^1_{\frac{2N'}{N'+1}}(\Omega)\subset L^{\frac{2N'}{N'-1}}(\Omega)$$

are continuous.

Lemma 2.9. *Let $2N'>N$, $N\geq 2$, $N'\in\mathbb{R}$,*

$$r_1:=\frac{2N'}{N'+1},\quad 2\leq s<\frac{r_1N}{N-r_1},\quad 1\leq q\leq r_1.$$

Then for every $\varepsilon>0$ there exists $C_\varepsilon>0$, such that for all $u\in W_2^1\left(\Omega,\varrho^{\mu},\varrho^{0}\right)$

$$\|u\|_s^{2\alpha}\leq\varepsilon\|u\|^2_{W_2^1(\Omega,\varrho^{\mu},\varrho^0)}+C_\varepsilon\|u\|_q^{2\beta}$$

holds, where

1. *If $0<\alpha<1$, then $0<\beta\leq\overline{\beta}<\alpha$ and $C_\varepsilon\sim\varepsilon^{-\frac{\alpha-\beta}{1-\alpha}}$,*
2. *If $\alpha=1$, then $\beta=1$ and $C_\varepsilon\sim\varepsilon^{-\sigma}$, $\overline{\sigma}\leq\sigma<\infty$,*
3. *If $1<\alpha<\overline{\alpha}$, then $\alpha<\overline{\beta}\leq\beta<\infty$ and $C_\varepsilon\sim\varepsilon^{-\frac{\beta-\alpha}{\alpha-1}}$,*

with

$$\overline{\theta}:=\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{q}-\frac{1}{r_1}+\frac{1}{N}},\quad\overline{\sigma}:=\frac{\overline{\theta}}{1-\overline{\theta}},\quad\overline{\alpha}:=\frac{1+\overline{\sigma}}{\overline{\sigma}}=\frac{1}{\overline{\theta}},\quad\overline{\beta}:=\frac{\alpha}{1+(1-\alpha)\overline{\sigma}}.$$

If $q=1$, then $\beta\neq\overline{\beta}$ and $\sigma\neq\overline{\sigma}$.

This lemma and its proof are modifications of a similar result in [Plu96]. For details we refer the reader to [Dre96].

The next lemma is a straight-forward generalization of a result in [Plu96], which we need for the Moser technique, we drop the proof.

For positive λ_1,λ_2 let $Q_{\lambda_1,\lambda_2}(t):=t^{\lambda_1}$ if $0\leq t\leq 1$ and $Q_{\lambda_1,\lambda_2}(t):=t^{\lambda_2}$ if $t>1$.

Lemma 2.10. *Let $(m_\nu),(\beta_{1,\nu}),(\beta_{2,\nu}),(p_\nu)$ be sequences of nonnegative real numbers with*

$$0<\beta_{1,\nu}\leq\beta_{2,\nu}\leq 1,\quad\prod_{\nu=1}^{\infty}\beta_{1,\nu}=\beta_1>0,\quad\prod_{\nu=1}^{\infty}\beta_{2,\nu}=\beta_2>0,$$
$$p_\nu=p_0\lambda^\nu,\quad\lambda>1,\quad p_0\geq 1.$$

We suppose

$$m_\nu^{p_\nu} \le C_0 p_\nu^{C_1} t \left(m_{\nu-1}^{p_\nu} + m_{\nu-1}^{\beta_{1,\nu} p_\nu} + m_{\nu-1}^{\beta_{2,\nu} p_\nu} \right), \quad \forall \nu = 1, 2, \ldots, \quad 0 \le t \le T.$$

Then

$$\limsup_{\nu \to \infty} m_\nu \le c Q_{\gamma_1, \gamma_2}(t) m_0^{\tilde{\beta}},$$

where $\tilde{\beta} = \prod_{\nu=1}^{\infty} \tilde{\beta}_\nu$,

$$\tilde{\beta}_\nu := \begin{cases} \beta_{1,\nu} & : m_{\nu-1} < 1, \\ 1 & : m_{\nu-1} \ge 1, \end{cases} \qquad \gamma_2 = \frac{1}{p_0(\lambda - 1)}, \quad \gamma_1 = \beta_1 \gamma_2.$$

Now we list some results about a class of degenerated elliptic differential operators, see [Tri78] and [Dre96].

Definition 2.11. Let $\nu > \mu + 2$, $\kappa_l = \frac{1}{2}(\nu(2 - l) + \mu l)$, $l = 0, 1, 2$. We consider the class $\mathfrak{A}_{\mu,\nu}^{(2)}$ of differential operators

$$Au = \sum_{l=0}^{1} \sum_{|\alpha| = 2l} \varrho^{\kappa_{2l}}(x) b_\alpha(x) \partial^\alpha u + \sum_{|\beta| < 2} \varrho^{\kappa_{|\beta|}}(x) a_\beta(x) \partial^\beta u.$$

We assume

$$b_\alpha \in C^1(\overline{\Omega}) \quad \|b_\alpha\|_{C^1(\overline{\Omega})} < \infty, a_\beta \in C(\overline{\Omega}), \quad \|a_\beta\|_C < \infty, \tag{2.19}$$

$$- \sum_{|\alpha| = 2} b_\alpha(x) \xi^\alpha \ge C_E |\xi|^2 \quad \forall (\xi, x) \in \mathbb{R}^N \times \Omega, \quad b_{(0,\ldots,0)}(x) \ge C_E > 0,$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad |a_\beta(x)| \le \varepsilon \text{ if } \operatorname{dist}(x, \partial\Omega) < \delta. \tag{2.20}$$

The next two theorems give an a–priori–estimate and an existence result.

Theorem 2.12. *Let* $A \in \mathfrak{A}_{\mu,\nu}^{(2)}$, $\nu \ge 0$, $1 < p < \infty$. *Then there exists a* $\lambda_0 \le 0$, *such that for* $\lambda \le \lambda_0$, $u \in W_p^2(\Omega, \varrho^{p\mu}, \varrho^{p\nu})$

$$C_1 \|u\|_{W_p^2(\Omega, \varrho^{p\mu}, \varrho^{p\nu})} \le \|Au - \lambda u\|_p \le C_2 \|u\|_{W_p^2(\Omega, \varrho^{p\mu}, \varrho^{p\nu})}$$

holds, where $C_i = C_i(\lambda)$ *and* C_1 *depends on* C_E, $\|b_\alpha\|_{C^1}$ *and* $\|a_\beta\|_C$.

Theorem 2.13. *Let* $A \in \mathfrak{A}_{\mu,\nu}^{(2)}$, $\nu \ge 0$, $1 < p < \infty$. *Then there exists a* $\lambda_0 < 0$, *such that for* $\lambda \le \lambda_0$ *the operator* $A - \lambda E$ *gives an isomorphic mapping from* $W_p^2(\Omega, \varrho^{p\mu}, \varrho^{p\nu})$ *onto* $L^p(\Omega)$.

3. A–priori–estimates

We will use the Rothe method to solve problem (1.1), (1.2), (1.3). We choose $n \in \mathbb{N}$, $h = \frac{T}{n}$, $t_j = jh$, $(j = 0, \dots, n)$ and consider a sequence $(u_j) \subset W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$ with

$$\frac{1}{h}(u_j - u_{j-1}) + A_{t_j,u_{j-1}} u_j = f(x, t_j, u_{j-1}), \tag{3.1}$$
$$u_0 = U_0.$$

We assume $u_{j-1} \in B_R(U_0) \cap W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$. By Theorem 2.7 we have $u_{j-1} \in C^1_0(\overline{\Omega})$. Hence we can write

$$A'_{t_j,u_{j-1}} u_j - \frac{1}{h} u_j = f(., t_j, u_{j-1}) + \frac{1}{h} u_{j-1},$$

where $A'_{t_j,u_{j-1}}$ stands for $A_{t_j,u_{j-1}}$ after application of the product rule to the principal part. From (2.8) and $u_{j-1} \in C^1_0(\overline{\Omega})$ we deduce that $A'_{t_j,u_{j-1}} \in \mathfrak{A}^{(2)}_{\mu,\nu}$. Theorem 2.13 shows that $u_j \in W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$, if $-h^{-1} \le \lambda_0$.

It remains to prove that there exists a $T^* > 0$ such that $u_j \in B_R(U_0)$ if $t_j \le T^*$.

We introduce the notation

$$\delta u_j = \frac{1}{h}(u_j - u_{j-1}), \quad a_j(u, v) = a_{t_j,u_{j-1}}(u, v), \quad A_j u = A_{t_j,u_{j-1}} u,$$
$$f_j(x) = f(x, t_j, u_{j-1}(x))$$

and consider the variational problems

$$(\delta u_j, v) + a_j(u_j, v) = (f_j, v) \quad \forall v \in C^1_0(\overline{\Omega}). \tag{3.2}$$

We will use some estimates for the bilinear form $a_j(u, v)$.

Theorem 3.1. *Let $u, v \in W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$, $p \ge 2$ and $w = |u|^{\frac{p-2}{2}} u$. Then it holds with $\frac{1}{s} > \frac{N'+1}{2N'} - \frac{1}{N}$:*

$$a_j(u, u) \ge c_1 \|u\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)} - c_1' \|u\|^2_2, \tag{3.3}$$

$$a_j(u, |u|^{p-2}u) \ge \frac{c_2}{p} \|w\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)} - \frac{c_2'}{p} \|w\|^2_2, \tag{3.4}$$

$$|a_j(v, |u|^{p-2}u)| \le C \|v\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} \|w\|_s^{\frac{p-2}{p}} \|w\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}, \tag{3.5}$$

$$|a_j(v, u)| \le C \|v\|_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)} \|u\|_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)}, \tag{3.6}$$

$$|a_j(v, |u|^{p-2}u) - a_m(v, |u|^{p-2}u)| \tag{3.7}$$
$$\le C(|t_j - t_m| + \|u_{j-1} - u_{m-1}\|_\sigma) \|v\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} \|w\|_s^{\frac{p-2}{p}} \|w\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}.$$

Proof. We have

$$a_j(u, u) \ge C_E \sum_{i=1}^N \int_\Omega \varrho^\mu \left(\frac{\partial u}{\partial x_i}\right)^2 dx + \sum_{i=1}^N \int_\Omega \varrho^{(\mu+\nu)/2} a_i \frac{\partial u}{\partial x_i} u \, dx + C_E \int_\Omega \varrho^\nu u^2 dx.$$

From (2.7) and the Cauchy Schwarz inequality we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\varrho^{(\mu+\nu)/2}|a_i|\left|\frac{\partial u}{\partial x_i}\right||u|\,dx \le C_a\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega,\varrho^{\mu})}\|u\|_{L^2(\Omega,\varrho^{\nu-2\delta})}.$$

From

$$\left(L^2(\Omega,\varrho^0),W_2^1(\Omega,\varrho^{\mu},\varrho^{\nu})\right)_{\frac{\nu-2\delta}{\nu},2}=W_2^{\frac{\nu-2\delta}{\nu}}\left(\Omega,\varrho^{\frac{\nu-2\delta}{\nu}\mu},\varrho^{\nu-2\delta}\right)$$

we conclude $\|u\|_{L^2(\Omega,\varrho^{\nu-2\delta})}\le c\,\|u\|^{\theta}_{W_2^1(\Omega,\varrho^{\mu},\varrho^{\nu})}\|u\|_2^{1-\theta}$. Application of Young's inequality gives (3.3).

Now we prove (3.4). We have

$$\begin{aligned}\nabla|u|^{p-2}u &= (p-1)|u|^{p-2}\nabla u=\frac{2(p-1)}{p}|w|^{\frac{p-2}{p}}\nabla w,\\ \nabla w &= \frac{p}{2}|u|^{\frac{p-2}{2}}\nabla u,\quad |u|=|w|^{\frac{2}{p}},\end{aligned}$$

thus

$$\sum_{i,k=1}^{N}\int_{\Omega}\varrho^{\mu}b_{ik}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_i}|u|^{p-2}u\,dx\ge\frac{2C_E}{p}\sum_{i=1}^{N}\int_{\Omega}\varrho^{\mu}\left(\frac{\partial w}{\partial x_i}\right)^2dx.$$

Furthermore,

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\varrho^{(\mu+\nu)/2}|a_i|\left|\frac{\partial u}{\partial x_i}\right||u|^{p-1}\,dx\le\frac{C}{p}\sum_{i=1}^{N}\int_{\Omega}\varrho^{(\mu+\nu)/2-\delta}\left|\frac{\partial w}{\partial x_i}\right||w|\,dx\\ &\le\frac{\varepsilon}{p}\|w\|^2_{W_2^1(\Omega,\varrho^{\mu},\varrho^{\nu})}+\frac{C_\varepsilon}{p}\|w\|_2^2.\end{aligned}$$

This gives (3.4).

Now we show (3.7). It holds

$$\begin{aligned}&|a_j(v,|u|^{p-2}u)-a_m(v,|u|^{p-2}u)|\\ &\le\sum_{i,k=1}^{N}\underbrace{\int_{\Omega}\varrho^{\mu}|b_{ik}(x,t_j,u_{j-1})-b_{ik}(x,t_m,u_{m-1})|\left|\frac{\partial v}{\partial x_k}\right|\left|\frac{\partial}{\partial x_i}|u|^{p-2}u\right|dx}_{I_1}\\ &+\sum_{i=1}^{N}\underbrace{\int_{\Omega}\varrho^{(\mu+\nu)/2}|a_i(x,t_j,u_{j-1})-a_i(x,t_m,u_{m-1})|\left|\frac{\partial v}{\partial x_i}\right||u|^{p-1}\,dx}_{I_2}\\ &+\underbrace{\int_{\Omega}\varrho^{\nu}|b_0(x,t_j,u_{j-1})-b_0(x,t_m,u_{m-1})|\,|v|\,|u|^{p-1}\,dx.}_{I_3}\end{aligned}$$

We use Hölder's inequality with $\frac{1}{\beta_1}+\frac{1}{s_1}=\frac{1}{2}$, $\frac{1}{\alpha_1}+\frac{1}{p_0}+\frac{1}{s_2}=1$, $\frac{1}{\beta_0}+\frac{1}{p_0}+\frac{1}{s_3}=1$ and conclude

$$I_1 \leq C \int_\Omega \underbrace{|b_{ik}(x,t_j,u_{j-1}) - b_{ik}(x,t_m,u_{m-1})|}_{\beta_1} \underbrace{\left|\frac{\partial v}{\partial x_k}\right|}_{\infty} \underbrace{|w|^{\frac{p-2}{p}}}_{s_1} \underbrace{\left|\frac{\partial w}{\partial x_i}\right|}_{2} \varrho^{\frac{\mu}{2}}\, dx$$

$$\leq C(|t_j - t_m| + \|u_{j-1} - u_{m-1}\|_\sigma) \|v\|_{C^1} \|w\|^{\frac{p-2}{p}}_{s_1 \frac{p-2}{p}} \|w\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)},$$

$$I_2 \leq \int_\Omega \underbrace{|a_i(x,t_j,u_{j-1}) - a_i(x,t_m,u_{m-1})|}_{\alpha_1} \underbrace{\varrho^{(\mu+\nu)/2}\left|\frac{\partial v}{\partial x_i}\right|}_{p_0} \underbrace{|w|^{2\frac{p-1}{p}}}_{s_2}\, dx$$

$$\leq C(|t_j - t_m| + \|u_{j-1} - u_{m-1}\|_\sigma) \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_0}(\Omega,\varrho^{p_0(\mu+\nu)/2})} \|w\|^{2\frac{p-1}{p}}_{2s_2 \frac{p-1}{p}},$$

$$I_3 \leq \int_\Omega \underbrace{|b_0(x,t_j,u_{j-1}) - b_0(x,t_m,u_{m-1})|}_{\beta_0} \underbrace{|\varrho^\nu v|}_{p_0} \underbrace{|w|^{2\frac{p-1}{p}}}_{s_3}\, dx$$

$$\leq C(|t_j - t_m| + \|u_{j-1} - u_{m-1}\|_\sigma) \|v\|_{L^{p_0}(\Omega,\varrho^{\nu p_0})} \|w\|^{2\frac{p-1}{p}}_{2s_3 \frac{p-1}{p}}.$$

According to (2.10), (2.12), (2.14) we have

$$s := \max(s_1, 2s_2, 2s_3) < \left(\frac{N'+1}{2N'} - \frac{1}{N}\right)^{-1}.$$

From

$$\left[L^{p_0}(\Omega,\varrho^{p_0\nu}), W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})\right]_{\frac{1}{2}} = W^1_{p_0}\left(\Omega,\varrho^{p_0(\mu+\nu)/2},\varrho^{p_0\nu}\right)$$

and Theorem 2.7 we obtain (3.7). The proofs of (3.5) and (3.6) are left to the reader. □

The main result of this section is the following

Theorem 3.2. *There exist constants $K > 0$, $h_0 > 0$, $0 < T^* \leq T$, such that for all $h \leq h_0$ and all $t_j \in I^* = [0, T^*]$*

$$u_j \in B_R(U_0), \quad \|u_j\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} \leq K.$$

holds.

We need some lemmata.

Lemma 3.3. *There exist $h_0 > 0, C > 0$, such that:*
If $u_0, u_1, \ldots, u_{i-1} \in B_R(U_0)$, then $\|u_i\|_2 \leq C$.

Proof. We choose $v = u_j$ in (3.2) and obtain

$$(u_j - u_{j-1}, u_j) + ch\|u_j\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)} \leq h\|f_j\|_2\|u_j\|_2 + Ch\|u_j\|^2_2.$$

Summing up $(j = 1, \dots, i)$ and applying Young's inequality gives

$$\|u_i\|_2^2 - \|u_0\|_2^2 + ch \sum_{j=1}^{i} \|u_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le C_f' + Ch \sum_{j=1}^{i} \|u_j\|_2^2 .$$

Using the discrete version of Gronwall's Lemma completes the proof. □

For details of the following proof we refer the reader to [DP96], Theorem 3.1.

Lemma 3.4. *Let* $0 \le k \le j$, $u_1, \dots, u_{j-1} \in B_R(U_0)$, $\|u_k\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} \le K$. *Then there exists a continuous, monotone increasing function* $M(t)$ *with* $M(0) = 0$, *such that*

$$\|u_j - u_k\|_\infty \le M(t_j - t_k),$$

if $h \le h_0$. *The function* M *does not depend on* k, *but on* K.

Proof. Let $z_i = u_i - u_k$. Then we have $\delta z_i = \delta u_i$ and

$$(\delta z_i, v) + a_i(z_i, v) = (f_i, v) - a_i(u_k, v) \quad \forall v \in C_0^1(\overline{\Omega}).$$

We choose $p \ge 2$ and write $v = |z_i|^{p-2} z_i$, $w_i = |z_i|^{\frac{p-2}{2}} z_i$. Then

$$\begin{aligned} &\frac{1}{h}\left(z_i, |z_i|^{p-2} z_i\right) - \frac{1}{h}\left(z_{i-1}, |z_i|^{p-2} z_i\right) + \frac{c_2}{p} \|w_i\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \\ &\le \|f_i\|_\varphi \left\| |z_i|^{p-1} \right\|_{\varphi'} + CK \|w_i\|_s^{\frac{p-2}{p}} \|w_i\|_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} + \frac{c_2'}{p} \|w_i\|_2^2 . \end{aligned}$$

Since $u_{i-1} \in B_R(U_0)$ we have $\|f_i\|_\varphi \le C_f$. Multiplying with p and applying Young's inequality give

$$\begin{aligned} &\frac{1}{h}\left(\|w_i\|_2^2 - \|w_{i-1}\|_2^2\right) + c_2 \|w_i\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} - \varepsilon K^2 \|w_i\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} \\ &\le Cp \|w_i\|_{2\varphi'}^{2\frac{p-1}{p}} + C_\varepsilon p^2 \|w_i\|_s^{2\frac{p-2}{p}} + c_2' \|w_i\|_2^2 . \end{aligned}$$

Since $\varphi \ge p_0 > N$ and $\frac{1}{s} > \frac{N'+1}{2N'} - \frac{1}{N}$ we can apply Lemma 2.9 to estimate the norms on the right–hand side. Thus,

$$\|w_i\|_2^2 - \|w_{i-1}\|_2^2 + ch \|w_i\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le Cp^{\sigma_M} h \left(\|w_i\|_q^{2\beta_1(p)} + \|w_i\|_q^{2\beta_2(p)} + \|w_i\|_q^2 \right),$$

where $1 < q < r_1$, σ_M, $\overline{\sigma}$ do not depend on p, h and $\beta_1(p) = \frac{p-1}{p+\overline{\sigma}}$, $\beta_2(p) = \frac{p-2}{p+2\overline{\sigma}}$.

Now let $l = 0, 1, 2 \dots$ $p_l = 2\left(\frac{2}{q}\right)^l$, $\beta_{1,l} = \beta_1(p_l) = \frac{p_l - 1}{p_l + \overline{\sigma}}$, $\beta_{2,l} = \beta_2(p_l) = \frac{p_l - 2}{p_l + 2\overline{\sigma}}$ and $m_{j,l} = \max_{k \le i \le j} \|z_i\|_{p_l}$.

Summing up $(i = k+1, \dots, j)$ gives

$$m_{j,l}^{p_l} \le C t_j p_l^{\sigma_M} \left(m_{j,l-1}^{p_l \beta_{1,l}} + m_{j,l-1}^{p_l \beta_{2,l}} + m_{j,l-1}^{p_l} \right).$$

From Lemma 2.10 we deduce

$$\limsup_{l\to\infty} m_{j,l} \le cQ_{\gamma_1,\gamma_2}(t_j) m_{j,0}^{\beta_0}.$$

We have $m_{j,0} \le \max_{k\le i\le j} \|u_i\|_2 + \|u_k\|_2 \le C(t_j) \le C$ due to Lemma 3.3, thus

$$\|u_j - u_k\|_\infty \le CQ_{\gamma_1,\gamma_2}(t_j) =: M(t_j).$$

□

Corollary 3.5. *There exists a $0 < T^* \le T$, so that $u_j \in B_R(U_0)$ if $t_j \le T^*$.*

Proof. We use the previous lemma with $k = 0$. We can choose T^* so that $M(T^*) \le R$ and $T^* \le T$. □

Finally, we need an estimate of $\|\delta u_j\|_{p_0}$ for the proof of Theorem 3.2.

Lemma 3.6. *Let $K > 0$. Then exists a constant $h_0 > 0$ and positive constants c_1, c_2, c_3, γ (indepedent of h, K), such that:*
If $h \le h_0$ and $\|u_j\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} \le K$ for all $0 \le j \le i-1$, then

$$\|\delta u_j\|_{p_0}^{p_0} \le \frac{1}{1-hc_2K^\gamma}(c_1 + c_3 t_j)\exp\left(t_{j-1}\frac{c_2K^\gamma}{1-hc_2K^\gamma}\right) =: S_K(h, t_{j-1})$$

holds for $1 \le j \le i$.

Proof. We have for every $v \in C^1_0(\overline{\Omega})$ and $j = 2, \dots, i$

$$(\delta u_j - \delta u_{j-1}, v) + ha_j(\delta u_j, v) = (f_j - f_{j-1}, v) + a_{j-1}(u_{j-1}, v) - a_j(u_{j-1}, v).$$

Let $v = |\delta u_j|^{p_0-2}\delta u_j$ and $w_j = |\delta u_j|^{\frac{p_0-2}{2}}\delta u_j$. From (3.4), (2.16) and (3.7) we conclude

$$\begin{aligned}
&\|w_j\|_2^2 - \|w_{j-1}\|_2^2 + ch\|w_j\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)} \\
&\le C_f h(1 + \|\delta u_{j-1}\|_\sigma)\|w_j\|_{2\varphi'}^{2\frac{p_0-1}{p_0}} + Ch\|w_j\|_2^2 \\
&+ C_a' h(1 + \|\delta u_{j-1}\|_\sigma)K\|w_j\|_s^{\frac{p_0-2}{p_0}}\|w_j\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}.
\end{aligned}$$

We apply Young's inequality,

$$\underbrace{(1 + \|\delta u_{j-1}\|_\sigma)}_{p_0}\underbrace{\|w_j\|_{2\varphi'}^{2\frac{p_0-1}{p_0}}}_{\frac{p_0}{p_0-1}} \le C(1 + \|\delta u_{j-1}\|_\sigma^{p_0}) + C\|w_j\|_{2\varphi'}^2,$$

$$\begin{aligned}
&\underbrace{(1 + \|\delta u_{j-1}\|_\sigma)}_{p_0}\underbrace{K\|w_j\|_s^{\frac{p_0-2}{p_0}}}_{\frac{2p_0}{p_0-2}}\underbrace{\|w_j\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}}_{2} \\
&\le \varepsilon\|w_j\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^0)} + C\|\delta u_{j-1}\|_\sigma^{p_0} + C_\varepsilon K^\gamma\|w_j\|_s^2 + C,
\end{aligned}$$

and deduce

$$\begin{aligned}&\|w_j\|_2^2 - \|w_{j-1}\|_2^2 + ch\,\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)}\\ &\le Ch(1+\|\delta u_{j-1}\|_\sigma^{p_0} + \|w_j\|^2_{2\varphi'} + K^\gamma\,\|w_j\|_s^2 + \|w_j\|_2^2).\end{aligned}$$

We show in a similar way as in the proof of the previous lemma

$$\begin{aligned}&\|w_j\|^2_{2\varphi'} \le \varepsilon\,\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} + C_\varepsilon\,\|w_j\|_q^2\,, \quad 1<q<r_1,\\ &K^\gamma\,\|w_j\|_s^2 \le \varepsilon\,\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} + C_\varepsilon K^{\gamma'}\,\|w_j\|_q^2\,,\end{aligned}$$

where C_ε does not depend on K. Thus,

$$\|w_j\|_2^2 - \|w_{j-1}\|_2^2 + ch\,\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le Ch(1+\|\delta u_{j-1}\|_\sigma^{p_0} + K^{\gamma'}\,\|w_j\|_2^2).$$

We have $\|\delta u_{j-1}\|_\sigma^{p_0} = \|w_{j-1}\|^2_{\frac{2\sigma}{p_0}}$. Summing up $(j=2,\dots,i)$ yields

$$\|w_i\|_2^2 - \|w_1\|_2^2 + ch\sum_{j=2}^{i}\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le Ct_i + Ch\sum_{j=1}^{i-1}\|w_j\|^2_{\frac{2\sigma}{p_0}} + CK^{\gamma'}h\sum_{j=2}^{i}\|w_j\|_2^2\,.$$

From (2.17) and Lemma 2.9 we see $\|w_j\|^2_{\frac{2\sigma}{p_0}} \le \varepsilon\,\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} + C\,\|w_j\|_2^2$, hence

$$\|w_i\|_2^2 + ch\sum_{j=2}^{i}\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le Ct_i + \varepsilon h\,\|w_1\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} + \|w_1\|_2^2 + CK^{\gamma'}h\sum_{j=2}^{i}\|w_j\|_2^2\,.$$

It remains to estimate $\|w_1\|_2^2 + h\,\|w_1\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)}$. We take $v = |\delta u_1|^{p_0-2}\delta u_1$, $w_1 = |\delta u_1|^{\frac{p_0-2}{2}}\delta u_1$ and obtain

$$\|\delta u_1\|_{p_0}^{p_0} + ha_1(\delta u_1, |\delta u_1|^{p_0-2}\delta u_1) = \big(f_1, |\delta u_1|^{p_0-2}\delta u_1\big) - a_1(U_0, |\delta u_1|^{p_0-2}\delta u_1).$$

We have

$$\begin{aligned}&-a_1(U_0,v) = -(A_1U_0, v)\\ &\le \|A_1U_0\|_{p_0}\,\|v\|_{p_0'} \le C\,\|U_0\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})}\,\|\delta u_1\|_{p_0}^{p_0-1} \le C_\varepsilon + \varepsilon\,\|\delta u_1\|_{p_0}^{p_0}\,.\end{aligned}$$

Taking account of $\varphi \ge p_0$, we have

$$\big(f_1, |\delta u_1|^{p_0-2}\delta u_1\big) \le \|f_1\|_{p_0}\,\big\||\delta u_1|^{p_0-1}\big\|_{\frac{p_0}{p_0-1}} \le C_\varepsilon\,\|f_1\|_{p_0}^{p_0} + \varepsilon\,\|w_1\|_2^2\,.$$

This gives with (3.4)

$$(1-2\varepsilon)\,\|w_1\|_2^2 + ch\,\|w_1\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le C + C_1'h\,\|w_1\|_2^2\,,$$

and so $\|w_1\|_2^2 + h\,\|w_1\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le C$, if $h \le h_0$. Hence we conclude

$$\|w_i\|_2^2 + ch\sum_{j=1}^{i}\|w_j\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \le c_1 + c_3t_i + c_2K^\gamma h\sum_{j=1}^{i}\|w_j\|_2^2\,.$$

The constants c_1, c_2, c_3 do not depend on K, i, h. By Gronwall's Lemma we obtain (if $1 - c_2 K^\gamma h_0 > 0$)

$$\|w_i\|_2^2 \leq \frac{1}{1 - c_2 K^\gamma h} (c_1 + c_3 t_i) \exp\left(t_{i-1} \frac{c_2 K^\gamma}{1 - c_2 K^\gamma h} \right).$$

□

Remark 3.7. It holds $S_K(h, t_{i-1}) \leq S_K(h_0, t_{i-1})$. $S_K(h_0, 0)$ is uniformly bounded with respect to h_0 and K, if $1 - c_2 K^\gamma h_0 \geq \frac{1}{2}$.

The next lemma will complete the proof of Theorem 3.2.

Lemma 3.8. *There exist constants $K, 0 < T^* \leq T, h_0$, such that:*

1. *It holds $\|U_0\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} < K$.*
2. *If $h \leq h_0$, $u_0, \dots, u_{j-1} \in B_R(U_0)$, $\|u_l\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \leq K$ $(l = 0, 1, \dots, j-1)$ and $t_j < T^*$, then $\|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \leq K$.*

Proof. From (3.1) we see

$$\begin{aligned}
Bu :=& -\sum_{i,k=1}^N \varrho^\mu b_{ik}(x, t_j, U_0) \frac{\partial^2 u_j}{\partial x_i \partial x_k} - \sum_{i,k=1}^N \mu \varrho^{\mu-1} \frac{\partial \varrho}{\partial x_i} b_{ik}(x, t_j, u_{j-1}) \frac{\partial u_j}{\partial x_k} \\
&+ \sum_{i=1}^N \varrho^{(\mu+\nu)/2} a_i(x, t_j, u_{j-1}) \frac{\partial u_j}{\partial x_i} + \varrho^\nu b_0(x, t_j, U_0) u_j \\
&= f(x, t_j, u_{j-1}) - \delta u_j + \sum_{i,k=1}^N \varrho^\mu \frac{\partial b_{ik}}{\partial u} \frac{\partial u_{j-1}}{\partial x_i} \frac{\partial u_j}{\partial x_k} \\
&+ \sum_{i,k=1}^N \varrho^\mu (b_{ik}(x, t_j, u_{j-1}) - b_{ik}(x, t_j, U_0)) \frac{\partial^2 u_j}{\partial x_i \partial x_k} \\
&+ \varrho^\nu (b_0(x, t_j, U_0) - b_0(x, t_j, u_{j-1})) u_j.
\end{aligned}$$

Theorem 2.12 gives

$$\begin{aligned}
C_1 \|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \leq& \|f(x, t_j, u_{j-1})\|_{p_0} + \|\delta u_j\|_{p_0} + \sum_{i,k=1}^N \left\| \varrho^\mu \frac{\partial b_{ik}}{\partial u} \frac{\partial u_{j-1}}{\partial x_i} \frac{\partial u_j}{\partial x_k} \right\|_{p_0} \\
&+ \sum_{i,k=1}^N \left\| \varrho^\mu (b_{ik}(x, t_j, u_{j-1}) - b_{ik}(x, t_j, U_0)) \frac{\partial^2 u_j}{\partial x_i \partial x_k} \right\|_{p_0} \\
&+ \|\varrho^\nu (b_0(x, t_j, U_0) - b_0(x, t_j, u_{j-1})) u_j\|_{p_0} + |\lambda| \|u_j\|_{p_0},
\end{aligned}$$

where C_1 does not depend on K, since the coefficients of order 2 and 0 of the operator B are independent of u_{j-1}.

We estimate the items on the right-hand side:

Since $p_0 \le \varphi$ and $u_{j-1} \in B_R(U_0)$ we have $\|f(x,t_j,u_{j-1})\|_{p_0} \le C$.
It holds $\|\delta u_j\|_{p_0} \le (S_K(h_0,t_{j-1}))^{\frac{1}{p_0}}$. Furthermore,

$$\left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial u_{j-1}}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right\|_{p_0} \le \left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial u_{j-1}}{\partial x_i}\frac{\partial (u_j-U_0)}{\partial x_k}\right\|_{p_0}$$
$$+\left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial (u_{j-1}-U_0)}{\partial x_i}\frac{\partial U_0}{\partial x_k}\right\|_{p_0} + \left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial U_0}{\partial x_i}\frac{\partial U_0}{\partial x_k}\right\|_{p_0}.$$

We have

$$\left\|\frac{\partial b_{ik}}{\partial u}\right\|_\infty \le C, \quad \left\|\frac{\partial u_{j-1}}{\partial x_i}\right\|_\infty \le \|u_{j-1}\|_{C^1} \le C\,\|u_{j-1}\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} \le CK.$$

The Interpolation Theorem 2.6 gives with $C_{U_0} = \|U_0\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})}$

$$\left\|\frac{\partial (u_j-U_0)}{\partial x_k}\right\|_{L^{p_0}(\Omega,\varrho^{p_0\mu})} \le C\left(\|u_j-U_0\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})}\right)^{\frac{1}{2}}\|u_j-U_0\|_{p_0}^{\frac{1}{2}}$$
$$\le \varepsilon\,\|u_j\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} + \varepsilon C_{U_0} + \frac{C}{\varepsilon}M(t_j).$$

In a similar way we conclude

$$\left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial (u_{j-1}-U_0)}{\partial x_i}\frac{\partial U_0}{\partial x_k}\right\|_{p_0} \le C\,\|u_{j-1}-U_0\|_{W^1_{p_0}(\Omega,\varrho^{p_0\mu/2},\varrho^{p_0\nu/2})}$$
$$\le C(K+C_{U_0})^{\frac{1}{2}}\,\|u_{j-1}-U_0\|_{p_0}^{\frac{1}{2}}$$
$$\le C(K+C_{U_0})M(t_{j-1})+1.$$

Finally,

$$\left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial U_0}{\partial x_i}\frac{\partial U_0}{\partial x_k}\right\|_{p_0} \le C(C_{U_0}).$$

The result is

$$\sum_{i,k=1}^N \left\|\varrho^\mu \frac{\partial b_{ik}}{\partial u}\frac{\partial u_{j-1}}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right\|_{p_0}$$
$$\le \varepsilon KC\,\|u_j\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} + \varepsilon KCC_{U_0} + \frac{CK}{\varepsilon}M(t_j) + C(K+C_{U_0})M(t_j) + C.$$

From (2.8) and the mean value theorem we obtain

$$\sum_{i,k=1}^N \left\|\varrho^\mu(b_{ik}(x,t_j,u_{j-1}) - b_{ik}(x,t_j,U_0))\frac{\partial^2 u_j}{\partial x_i \partial x_k}\right\|_{p_0}$$
$$+\,\|\varrho^\nu(b_0(x,t_j,U_0) - b_0(x,t_j,u_{j-1}))u_j\|_{p_0}$$
$$\le CM(t_{j-1})\,\|u_j\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})}.$$

Finally,

$$|\lambda| \|u_j\|_{p_0} \leq C \|u_j - U_0\|_\infty + C \|U_0\|_\infty \leq CM(t_j) + C.$$

Thus,

$$\begin{aligned} &C_1 \|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \\ &\leq C_2 + (S_K(h_0, t_{j-1}))^{\frac{1}{p_0}} + \varepsilon K C_3 \|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} + \varepsilon K C_3 C_{U_0} \\ &+ \frac{C_4 K}{\varepsilon} M(t_j) + C_5(K + C_6) M(t_j) + C_7 M(t_j) \|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})}. \end{aligned}$$

We set $\varepsilon = \frac{C_1}{4KC_3}$ and choose $t_j \leq T_1^* \leq T$ with $C_7 M(T_1^*) \leq \frac{C_1}{4}$, which implies

$$C_1 \|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \leq C_8 + (S_K(h_0, t_{j-1}))^{\frac{1}{p_0}} + C_9 M(t_j)(1 + K + K^2).$$

The constants $C_1, \ldots, C_9$ are independent of K, h, t_j. Without loss of generality we may assume $h_0 \leq 1$, hence

$$S_K(h_0, 0) = \frac{c_1 + c_3 h_0}{1 - c_2 K^\gamma h_0} \leq \frac{c_1 + c_3}{1 - c_2 K^\gamma h_0}.$$

We fix K such that $K > 1$, $C_{U_0} < K$ and

$$C_8 + \left(2 \frac{c_1 + c_3}{\frac{1}{2}}\right)^{\frac{1}{p_0}} < \frac{C_1}{4} K.$$

Now we choose $1 \geq h_0 > 0$ with $1 - c_2 K^\gamma h_0 \geq \frac{1}{2}$. Finally, we determine $0 < T^* \leq T_1^*$, such that

$$\begin{aligned} &S_K(h_0, T^*) \leq \frac{c_1 + c_3 T^*}{\frac{1}{2}} \exp\left(T^* \frac{c_2 K^\gamma}{1 - c_2 K^\gamma h_0}\right) < 4(c_1 + c_3), \\ &C_9 M(T^*)(1 + K + K^2) \leq \frac{C_1}{4} K. \end{aligned}$$

We thus get $\frac{C_1}{2} \|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \leq \frac{C_1}{2} K$, hence $\|u_j\|_{W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})} \leq K$. □

Induction completes the proof of Theorem 3.2.

4. Convergence and Existence

We define the piecewise linear and piecewise constant interpolations

$$\begin{aligned} u^n(x, t) &:= \frac{t_i - t}{t_i - t_{i-1}} u_{i-1}(x) + \frac{t - t_{i-1}}{t_i - t_{i-1}} u_i(x), \quad (t_{i-1} < t \leq t_i), \\ \overline{u}^n(x, t) &:= \begin{cases} u_i(x) & : t_{i-1} < t \leq t_i, \\ 0 & : t \leq 0. \end{cases} \end{aligned}$$

We have proved

$$\begin{aligned}\|u_t^n(.,t)\|_{p_0} &\le C,\\ \|u^n(.,t)-\overline{u}^n(.,t)\|_{p_0} &\le Ch_n,\\ \|\overline{u}^n(.,t)-\overline{u}^n(.,t-h_n)\|_\infty &\le Ch_n^\gamma,\\ \|u^n(.,t)\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})} &\le K.\end{aligned}$$

We show a first convergence result:

Theorem 4.1. *There exists a function $u\in C(I^*,L^{p_0}(\Omega))$, such that*

$$u^n\to u \text{ in } C(I^*,L^{p_0}(\Omega)).$$

Proof. It holds $u_t^n+\overline{A}^n\overline{u}^n=\overline{f}^n$, where

$$\overline{A}^n=A_{t_j,u_{j-1}},\quad \overline{f}^n=f(.,t_j,u_{j-1}(.)),\quad (t_{j-1}<t\le t_j).$$

Considering two different subdivisions, this clearly forces

$$(u^n-u^m)_t+\overline{A}^n(\overline{u}^n-\overline{u}^m)=\overline{f}^n-\overline{f}^m+(\overline{A}^m-\overline{A}^n)\overline{u}^m.$$

Let $u^{nm}=u^n-u^m$, $\overline{u}^{nm}=\overline{u}^n-\overline{u}^m$, $\overline{v}^{nm}=|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}$, $\overline{w}^{nm}=|\overline{u}^{nm}|^{\frac{p_0-2}{2}}\overline{u}^{nm}$. The variational formulation of the above equation implies

$$\begin{aligned}&\big((u^n-u^m)_t,|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\big)+\overline{a}^n(\overline{u}^{nm},|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm})\\ &=\Big(\overline{f}^n-\overline{f}^m,|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\Big)+(\overline{a}^m-\overline{a}^n)(\overline{u}^m,|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}).\end{aligned}$$

From (3.4) and

$$\begin{aligned}&\Big|\Big(\overline{f}^n-\overline{f}^m,|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\Big)\Big|\\ &\quad\le\Big\|\overline{f}^n-\overline{f}^m\Big\|_\varphi\|\overline{u}^{nm}\|^{p_0-1}_{\varphi'(p_0-1)}\\ &\quad\le C(h_n+h_m+\|\overline{u}^n(t-h_n)-\overline{u}^m(t-h_m)\|_\sigma)\,\|\overline{u}^{nm}\|^{p_0-1}_{\varphi'(p_0-1)}\\ &\quad\le C((h_n+h_m)^{p_0}+\|\overline{u}^n(t-h_n)-\overline{u}^m(t-h_m)\|^{p_0}_\sigma)+C\,\|\overline{u}^{nm}\|^{p_0}_{\varphi'(p_0-1)},\\ &\|\overline{u}^{nm}\|^{p_0}_{\varphi'(p_0-1)}=\|\overline{w}^{nm}\|^2_{2\varphi'\frac{p_0-1}{p_0}}\\ &\quad\le\varepsilon\,\|\overline{w}^{nm}\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}+C_\varepsilon\,\|\overline{w}^{nm}\|^2_2,\\ &\big|(\overline{a}^m-\overline{a}^n)(\overline{u}^m,|\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm})\big|\\ &\quad\le C(h_n+h_m+\|\overline{u}^n(t-h_n)-\overline{u}^m(t-h_m)\|_\sigma)\,\|\overline{u}^m\|_{W^2_{p_0}(\Omega,\varrho^{p_0\mu},\varrho^{p_0\nu})}\times\\ &\qquad\times\|\overline{w}^{nm}\|_s^{\frac{p_0-2}{p_0}}\,\|\overline{w}^{nm}\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}\\ &\quad\le C_\varepsilon\Big((h_n+h_m)^{p_0}+\|\overline{u}^n(t-h_n)-\overline{u}^m(t-h_m)\|^{p_0}_\sigma+\|\overline{w}^{nm}\|^2_s\Big)\\ &\qquad+\varepsilon\,\|\overline{w}^{nm}\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^0)},\\ &\|\overline{w}^{nm}\|^2_s\le\varepsilon\,\|\overline{w}^{nm}\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}+C_\varepsilon\,\|\overline{w}^{nm}\|^2_2\end{aligned}$$

we deduce that

$$\begin{aligned}&\left(u_t^{nm}, |\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\right) + c\,\|\overline{w}^{nm}\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)} \\ &\le C\left((h_n+h_m)^{p_0} + \|\overline{u}^n(t-h_n) - \overline{u}^m(t-h_m)\|_\sigma^{p_0} + \|\overline{w}^{nm}\|_2^2\right).\end{aligned} \tag{4.1}$$

Using an idea of Lemma 6 of [Plu88], we obtain

$$\begin{aligned}&\frac{d}{dt}\|u^{nm}\|_{p_0}^{p_0} = p_0\left(u_t^{nm}, |u^{nm}|^{p_0-2}u^{nm}\right) \\ &\le p_0\left(u_t^{nm}, |\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\right) + p_0\,\|u_t^{nm}\|_{p_0}\left\||u^{nm}|^{p_0-2}u^{nm} - |\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\right\|_{p_0'} \\ &\le p_0\left(u_t^{nm}, |\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\right) \\ &\quad + p_0\,\|u_t^{nm}\|_{p_0}(p_0-1)\left(\|u^{nm}\|_{p_0} + \|\overline{u}^{nm}\|_{p_0}\right)^{p_0-2}\|u^{nm} - \overline{u}^{nm}\|_{p_0}.\end{aligned}$$

From this and $\|u^{nm} - \overline{u}^{nm}\|_{p_0} \le C(h_n+h_m)$ we conclude

$$\frac{d}{dt}\|u^{nm}\|_{p_0}^{p_0} \le p_0\left(u_t^{nm}, |\overline{u}^{nm}|^{p_0-2}\overline{u}^{nm}\right) + C\,\|u^{nm}\|_{p_0}^{p_0} + C(h_n+h_m)^{p_0} + C(h_n+h_m)^{\frac{p_0}{2}}.$$

Integrating this inequality over $[0, t_0]$ and (4.1) yield

$$\begin{aligned}&\|u^{nm}(.,t_0)\|_{p_0}^{p_0} + c\int_0^{t_0}\|\overline{w}^{nm}(t)\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)}\,dt \\ &\le C\int_0^{t_0}\|u^{nm}(.,t)\|_{p_0}^{p_0}\,dt + C\int_0^{t_0}\|\overline{w}^{nm}(.,t)\|_2^2\,dt + C(h_n+h_m)^{\frac{p_0}{2}} \\ &+ C\int_0^{t_0}\|\overline{u}^n(t-h_n) - \overline{u}^m(t-h_m)\|_\sigma^{p_0}\,dt.\end{aligned}$$

From $\|\overline{w}^{nm}\|_2^2 = \|\overline{u}^{nm}\|_{p_0}^{p_0} \le C((h_n+h_m)^{p_0} + \|u^{nm}\|_{p_0}^{p_0})$ and

$$\begin{aligned}&\|\overline{u}^n(t-h_n) - \overline{u}^m(t-h_m)\|_\sigma^{p_0} \\ &\le C(\|\overline{u}^n(t-h_n) - \overline{u}^n(t)\|_\infty^{p_0} + \|\overline{u}^n(t) - \overline{u}^m(t)\|_\sigma^{p_0} + \|\overline{u}^m(t) - \overline{u}^m(t-h_m)\|_\infty^{p_0}) \\ &\le C\left((h_n+h_m)^{p_0\gamma} + \|\overline{w}^{nm}\|^2_{\frac{2\sigma}{p_0}}\right) \\ &\le C\left((h_n+h_m)^{p_0\gamma} + \varepsilon\,\|\overline{w}^{nm}\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^0)} + \|u^{mn}\|_{p_0}^{p_0} + (h_n+h_m)^{p_0}\right)\end{aligned}$$

we obtain

$$\begin{aligned}&\|u^{nm}(.,t_0)\|_{p_0}^{p_0} + c\int_0^{t_0}\|\overline{w}^{nm}(t)\|^2_{W_2^1(\Omega,\varrho^\mu,\varrho^\nu)}\,dt \\ &\le C\int_0^{t_0}\|u^{nm}(.,t)\|_{p_0}^{p_0}\,dt + C\left((h_n+h_m)^{\frac{p_0}{2}} + (h_n+h_m)^{p_0\gamma} + (h_n+h_m)^{p_0}\right).\end{aligned}$$

The application of Gronwall's Lemma completes the proof. □

Now we can show more convergence properties.

Theorem 4.2. *There exists a function*

$$u \in C(I^*, C_0^1(\overline{\Omega})) \cap L^\infty\left(I^*, W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})\right),$$
$$u_t \in L^\infty(I^*, L^{p_0}(\Omega)),$$

such that

$$u^n \to u \text{ in } C(I^*, C_0^1(\overline{\Omega})), \tag{4.2}$$
$$u^n \rightharpoonup^* u \text{ in } L^\infty(I^*, W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})), \tag{4.3}$$
$$u_t^n \rightharpoonup^* u_t \text{ in } L^\infty(I^*, L^{p_0}(\Omega)). \tag{4.4}$$

Proof. To simplify notation we write $W_\theta = W^{s_\theta}_{p_0}(\Omega, \varrho^{\mu_\theta p_0}, \varrho^{\nu_\theta p_0})$, where

$$0 < \theta < 1, \quad s_\theta = 2(1-\theta), \quad \nu_\theta = \nu(1-\theta), \quad \frac{\mu_\theta - \nu_\theta}{s_\theta} = \frac{\mu - \nu}{2}.$$

We have $\left(W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu}), L^{p_0}(\Omega)\right)_{\theta,p_0} = W_\theta$ and $s_\theta \to 2$, $\nu_\theta \to \nu$, $\mu_\theta \to \mu$ if $\theta \to 0$. From this interpolation, $\|u^n - u^m\|_{L_\infty(I^*, W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu}))} \le C$ and $\|u^n - u^m\|_{p_0} \to 0$ we obtain

$$u^n \to u \text{ in } C(I^*, W_\theta).$$

We have $W_\theta \subset C_0^1(\overline{\Omega})$, if $\theta > 0$ is sufficiently small. This gives (4.2).

Since $W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$, $L^{p_0}(\Omega)$ are reflexive Banach spaces and the limits are unique, we have (4.3) and (4.4). □

Theorem 4.3. *This function u is a solution of* (2.18).

Proof. We have

$$\int_{I^*} (u_t^n, v)\, dt + \int_{I^*} \overline{a}^n(\overline{u}^n, v)\, dt = \int_{I^*} \left(\overline{f}^n, v\right)\, dt.$$

Fix $v \in L^1(I^*, W^1_{p_0'}(\Omega))$. From (4.4) we deduce $\int_{I^*} (u_t^n, v)\, dt \to \int_{I^*} (u_t, v)\, dt$. We apply (4.2), $\overline{u}^n \to u$ in $L^\infty(I^*, C_0^1(\overline{\Omega}))$ and the Theorem of Lebesgue, which completes the proof. □

Theorem 4.4. *There exists at most one solution of* (2.18) *in* $C(I^*, C_0^1(\overline{\Omega})) \cap L^\infty(I^*, W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$ *with* $u_t \in L^\infty(I^*, L^{p_0}(\Omega))$.

Proof. Let u_1, u_2 be two solutions, $u = u_1 - u_2$,

$$v(t) = \begin{cases} |u(t)|^{p_0-2}u(t) & : t \le t_0, \\ 0 & : t > t_0, \end{cases} \quad w(t) = \begin{cases} |u(t)|^{(p_0-2)/2}u(t) & : t \le t_0, \\ 0 & : t > t_0. \end{cases}$$

We have

$$\int (u_t, v)\, dt + \int a_1(u, v)\, dt = \int (f_1 - f_2, v)\, dt + \int a_2(u_2, v) - a_1(u_2, v)\, dt.$$

From

$$a_1(u,v) \geq c_1 \|w\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)} - c_1' \|w\|_2^2,$$
$$(f_1 - f_2, v) \leq C \|u\|_\sigma \|w\|_{2p_0'}^{2\frac{p_0-1}{p_0}},$$
$$|a_2(u_2,v) - a_1(u_2,v)| \leq C \|u\|_\sigma \|w\|_s^{\frac{p_0-2}{p_0}} \|w\|_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}$$

we conclude

$$\|u(t_0)\|^{p_0}_{p_0} + c\int_0^{t_0} \|w\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^\nu)}\, dt$$
$$\leq \varepsilon \int_0^{t_0} \|w\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}\, dt + C\int_0^{t_0}\left(\|w\|^2_{\frac{2\sigma}{p_0}} + \|w\|^2_s + \|u\|^{p_0}_{p_0}\right) dt$$
$$\leq \varepsilon' \int_0^{t_0} \|w\|^2_{W^1_2(\Omega,\varrho^\mu,\varrho^0)}\, dt + C\int_0^{t_0} \|u\|^{p_0}_{p_0}\, dt.$$

Gronwall's Lemma now implies $\|u(t_0)\|_{p_0} = 0$. □

5. Counter Example

The assumptions $b_0(x,t,u) \geq C_E > 0$ and $\varrho^\nu \notin L^1(\Omega)$ of Theorem 2.3 have the surprising consequence, that the coefficient of u diverges strongly to infinity as x goes to the boundary. These conditions come from the theory of weighted Sobolev spaces and degenerated elliptic operators.

The following example shows that these assumptions can not be weakened.

Let $N = 1$, $\Omega = (-1,1)$, $\varrho(x) = (1-x^2)^{-1}$, $\mu = -\frac{9}{20}$, $\nu = \frac{8}{5}$, $p_0 = 2$. We consider

$$u_t - (\varrho^\mu(x)u_x)_x + \varrho^\nu(x)(1-x^2)^{\frac{3}{5}}u =$$
$$\frac{11}{10}t + t(1-x^2)^{-\frac{9}{20}} + (1-x^2)^{\frac{11}{20}} + \frac{12}{5}(1-x^2)^{\frac{13}{20}} - \frac{78}{25}x^2(1-x^2)^{-\frac{7}{20}} + (1-x^2)^{\frac{1}{5}}$$

with the data

$$u(-1,t) = u(1,t) = 0, \quad u(x,0) = (1-x^2)^{\frac{6}{5}}.$$

All assumptions but the second of (2.6) are satisfied. The solution is $u(x,t) = t(1-x^2)^{\frac{11}{20}} + (1-x^2)^{\frac{6}{5}}$. However, $u(.,t) \notin W^2_{p_0}(\Omega, \varrho^{p_0\mu}, \varrho^{p_0\nu})$ if $t > 0$.

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