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Local Solutions of Weakly Parabolic Semilinear Differential Equations

By Michael Dreher of Freiberg and Volker Pluschke of Halle

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Abstract. Semilinear parabolic boundary value problems with degenerated elliptic part where the right–hand side depends on the solution are studied. We approximate the parabolic semilinear problem by a system of linear degenerate elliptic problems by the aid of semidiscretization in time. Using weighted Sobolev spaces one derives a–priori–estimates for the approximate solutions. These approximate solutions converge to a uniquely determined weak solution, if the time interval is sufficiently small. We point out that the nonlinear right–hand side is defined only in a neighbourhood of the initial data, therefore one has to prove L^∞–estimates for the solutions of the approximate problems.

1. Introduction

In this paper we will prove, by means of the Rothe method, the local existence of a solution of the weakly parabolic semilinear initial boundary value problem

$$u_t(x,t) + A_t u(x,t) = f(x,t,u) \quad \text{in } Q, \tag{1.1}$$
$$u(x,t) = 0 \quad \text{on } \Gamma, \tag{1.2}$$
$$u(x,0) = U_0(x) \quad \text{in } \Omega. \tag{1.3}$$

We denote by $\Omega \subset \mathbb{R}^N$ a bounded domain with boundary $\partial\Omega \in C^1$, $T > 0$, $I = [0,T]$, $Q = \Omega \times I$, $\Gamma = \partial\Omega \times I$ and

$$A_t u = -\sum_{i,k=1}^{N} \frac{\partial}{\partial x_i}\left(g(x) a_{ik}(x,t) \frac{\partial u}{\partial x_k}\right) + \sum_{i=1}^{N} a_i(x,t) \frac{\partial u}{\partial x_i},$$
$$g(x) > 0 \text{ a.e. in } \Omega, \quad g \in L^\infty(\Omega), \quad g^{-N'} \in L^1(\Omega) \text{ for some } N' > N.$$

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Semilinear weakly parabolic problems with degenerated coefficient of u_t were considered in [Plu92] and [Kač85], Chapter 6.1. In [Kač85], Chapter 6.2, elliptic–parabolic problems were studied. The parabolic problem degenerates into an elliptic one on a subset of positive measure. Such problems with nonlinear Neumann boundary condition were treated in [Web95], too. Parabolic systems with nonlinear degeneration were investigated in [Kač90] by means of Rothe's method.

There are different ways to prove convergence of the Rothe functions. The papers of Kacur [Kač90], [KL91] used compactness arguments. Here we follow an approach of Pluschke [Plu88], [Plu92], [Plu96], where the convergence is proved by an estimate of the time–derivative. Furthermore we will give estimates of the convergence order and the error of the approximations.

We point out that we omit global growth assumptions on the nonlinearity of the right-hand side. Only assumptions in a neighbourhood of the initial data are made. The required L^∞–estimates for $u - U_0$ will be derived by the technique of Moser and Alikakos, where these results are obtained by a limit process $p \to \infty$, see [Mos60] and [Ali79].

2. Preliminaries

In the following $\|.\|_p$ denotes the usual Lebesgue space norm and $(.,.)$ the $L^2(\Omega)$ scalar product. We will denote by $C(I,V)$, $L^p(I,V)$, $W_2^1(I,V)$ and $C^{0,1}(I,V)$ the spaces of continuous, L^p–integrable, W_2^1–differentiable and Lipschitz continuous mappings $I \to V$, respectively. $L^p_g(\Omega)$ is a weighted Lebesgue space with the norm $\|u\|_{p,g} = \left\|g^{1/p}u\right\|_p$ $(1 \le p < \infty)$. The weighted Sobolev space $W^1_{p,g}(\Omega)$ is defined in the following way:

$$u \in W^1_{p,g}(\Omega) \iff u \in W^1_1(\Omega), \quad \|u\|^p_{p,1,g} := \|u\|^p_p + \sum_{i=1}^N \|u_{x_i}\|^p_{p,g} < \infty.$$

The space $\overset{o}{W}{}^1_{p,g}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the $W^1_{p,g}(\Omega)$–norm. By C, c we denote positive constants which may have different values at different places, but are independent of h and p, if p is variable.

We study weak solutions and define the bilinear form

$$a_t(u,x) = \sum_{i,k=1}^N \int_\Omega g(x) a_{ik}(x,t) \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_i}\, dx + \sum_{i=1}^N \int_\Omega a_i(x,t) \frac{\partial u}{\partial x_i} v\, dx.$$

Let us suppose the following conditions:

$$
\begin{aligned}
&a_{ik}, a_i \in C^{0,1}(I, L^\infty(\Omega)),\\
&\sum_{i,k=1}^{N} a_{ik}(x,t)\xi_i\xi_k \geq C_E|\xi|^2 \quad C_E > 0 \quad \forall \xi \in \mathbb{R}^N \text{ a.e. in } Q,\\
&U_0 \in L^\infty(\Omega) \cap \overset{o}{W}{}^1_{2r,g}(\Omega) \quad (r > N, r > 2),\\
&\exists y_0 \in L^2(\Omega) \text{ with } (f(.,0,U_0), v) - a_0(U_0, v) = (y_0, v) \quad \forall v \in \overset{o}{W}{}^1_{2,g}(\Omega),\\
&B_R(U_0) := \{u \in L^\infty(\Omega) : \|u - U_0\|_\infty \leq R\}, \quad R > 0,\\
&f(.,t,u) : I \times B_R(U_0) \to L^r(\Omega) \text{ with}\\
&\|f(x,t,u(x)) - f(x,t',u'(x))\|_2 \leq C_f(|t-t'| + \|u-u'\|_2).
\end{aligned}
\tag{2.1}
$$

The aim of this paper is to prove

Theorem 2.1. *There exists $T^* > 0$ with the property that the problem* (1.1), (1.2), (1.3) *has a uniquely determined weak solution $u \in W_2^1([0,T^*], \overset{o}{W}{}^1_{2,g}(\Omega))$ with $u_t \in L^\infty([0,T^*], L^2(\Omega))$ and $u(t) \in B_R(U_0)$ for $t \leq T^*$. This solution fulfils the relation*

$$
\begin{aligned}
\int_{I^*} (u_t(x,t), v(x,t))\, dt + \int_{I^*} a_t(u(x,t), v(x,t))\, dt \\
= \int_{I^*} (f(x,t,u(x,t)), v(x,t))\, dt
\end{aligned}
\tag{2.2}
$$

for any $v \in L^2(I^*, \overset{o}{W}{}^1_{2,g}(\Omega))$.

We list some results which will be used later:

Lemma 2.2. *Let $1 \leq q < p$, $g^{\frac{q}{q-p}} \in L^1(\Omega)$. Then the embedding $L^p_g(\Omega) \subset L^q(\Omega)$ is continuous.*

Lemma 2.3. *Let $g^{-N'} \in L^1(\Omega)$, $N' > N$, $N' \in \mathbb{R}$. Then the embeddings*

$$
W^1_{2,g}(\Omega) \subset W^1_{\frac{2N'}{N'+1}}(\Omega) \subset L^{\frac{2N'}{N'-1}}(\Omega)
$$

are continuous.

The (short) proofs can be found in [Dre96].

Lemma 2.4.*[Nirenberg–Gagliardo Interpolation] Let $1 \leq q \leq p \leq s$ and $\frac{1}{p} < \frac{1}{s} + \frac{1}{N}$. Then there is a C, such that for all $u \in \overset{o}{W}{}^1_p(\Omega)$*

$$
\|u\|_s \leq C\, \|u\|^\theta_{p,1}\, \|u\|^{1-\theta}_q \quad \text{with } \overline{\theta} := \frac{\frac{1}{q} - \frac{1}{s}}{\frac{1}{q} - \frac{1}{p} + \frac{1}{N}} \leq \theta \leq 1.
$$

If $q = 1$, *then* $\overline{\theta} < \theta \leq 1$.

For details we refer the reader to [LSU67], pp. 80–84. As a corollary we have

Lemma 2.5. *Let* $2N' > N$, $N \geq 2$, $N' \in \mathbb{R}$,

$$r_1 := \frac{2N'}{N'+1}, \quad 2 \leq s < \frac{r_1 N}{N - r_1}, \quad 1 \leq q \leq r_1.$$

Then for every $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$, *such that for all* $u \in \overset{o}{W}{}^1_{2,g}(\Omega)$

$$\|u\|_s^{2\alpha} \leq \varepsilon \|u\|_{2,1,g}^2 + C_\varepsilon \|u\|_q^{2\beta}$$

holds, where

1. *If* $0 < \alpha < 1$, *then* $0 < \beta \leq \overline{\beta} < \alpha$ *and* $C_\varepsilon \sim \varepsilon^{-\frac{\alpha-\beta}{1-\alpha}}$,
2. *If* $\alpha = 1$, *then* $\beta = 1$ *and* $C_\varepsilon \sim \varepsilon^{-\sigma}$, $\overline{\sigma} \leq \sigma < \infty$,
3. *If* $1 < \alpha < \overline{\alpha}$, *then* $\alpha < \overline{\beta} \leq \beta < \infty$ *and* $C_\varepsilon \sim \varepsilon^{-\frac{\beta-\alpha}{\alpha-1}}$,

with

$$\overline{\theta} := \frac{\frac{1}{q} - \frac{1}{s}}{\frac{1}{q} - \frac{1}{r_1} + \frac{1}{N}}, \quad \overline{\sigma} := \frac{\overline{\theta}}{1 - \overline{\theta}}, \quad \overline{\alpha} := \frac{1 + \overline{\sigma}}{\overline{\sigma}} = \frac{1}{\overline{\theta}}, \quad \overline{\beta} := \frac{\alpha}{1 + (1-\alpha)\overline{\sigma}}.$$

If $q = 1$, *then* $\beta \neq \overline{\beta}$ *and* $\sigma \neq \overline{\sigma}$.

This lemma and its proof are modifications of a similar result in [Plu96]. For details we refer the reader to [Dre96].

The next lemma is a straight-forward generalisation of a result in [Plu96], we drop the proof.

For positive λ_1, λ_2 let $Q_{\lambda_1,\lambda_2}(t) := t^{\lambda_1}$ if $0 \leq t \leq 1$ and $Q_{\lambda_1,\lambda_2}(t) := t^{\lambda_2}$ if $t > 1$.

Lemma 2.6. *Let* $(m_\nu), (\beta_{1,\nu}), (\beta_{2,\nu}), (p_\nu)$ *be sequences of nonnegative real numbers with*

$$0 < \beta_{1,\nu} \leq \beta_{2,\nu} \leq 1, \quad \prod_{\nu=1}^{\infty} \beta_{1,\nu} = \beta_1 > 0, \quad \prod_{\nu=1}^{\infty} \beta_{2,\nu} = \beta_2 > 0,$$
$$p_\nu = p_0 \lambda^\nu, \quad \lambda > 1, \quad p_0 \geq 1.$$

We suppose

$$m_\nu^{p_\nu} \leq C_0 p_\nu^{C_1} t \left(m_{\nu-1}^{p_\nu} + m_{\nu-1}^{\beta_{1,\nu} p_\nu} + m_{\nu-1}^{\beta_{2,\nu} p_\nu} \right) \quad \forall \nu = 1, 2, \ldots, \quad 0 \leq t \leq T.$$

Then

$$\limsup_{\nu \to \infty} m_\nu \leq c Q_{\gamma_1,\gamma_2}(t) m_0^{\tilde{\beta}},$$

where $\tilde{\beta} = \prod_{\nu=1}^{\infty} \tilde{\beta}_\nu$,

$$\tilde{\beta_\nu} := \begin{cases} \beta_{1,\nu} & : m_{\nu-1} < 1, \\ 1 & : m_{\nu-1} \geq 1, \end{cases} \quad \gamma_2 = \frac{1}{p_0(\lambda - 1)}, \quad \gamma_1 = \beta_1 \gamma_2.$$

The following estimates play an important role in the sequel.

Lemma 2.7. *Let* $u, v \in \overset{o}{W}{}^1_{2,g}(\Omega) \cap L^\infty(\Omega)$*,* $p \geq 2$ *and* $w := |u|^{\frac{p-2}{2}} u$*. Then* $|u|^{p-2}u \in \overset{o}{W}{}^1_{2,g}(\Omega) \cap L^\infty(\Omega)$ *and*

$$\nabla w = \frac{p}{2} |u|^{\frac{p-2}{2}} \nabla u, \tag{2.3}$$

$$\nabla \left(|u|^{p-2} u \right) = (p-1)|u|^{p-2} \nabla u = \frac{2(p-1)}{p} |w|^{\frac{p-2}{p}} \nabla w, \tag{2.4}$$

$$a_t(u, |u|^{p-2}u) \geq \frac{c_1}{p} \|w\|^2_{2,1,g} - \frac{c_2}{p} \|w\|^2_2 , \tag{2.5}$$

$$|a_t(u, v) - a_{t'}(u, v)| \leq C|t - t'| \, \|u\|_{2,1,g} \|v\|_{2,1,g} , \tag{2.6}$$

$$|a_t(u, v)| \leq C \|u\|_{2,1,g} \|v\|_{2,1,g} . \tag{2.7}$$

We use the convention $|u(x)|^{p-2} = 1$*, if* $u(x) = 0$ *and* $p = 2$*.*

Proof. The proof that $|u|^{p-2}u \in \overset{o}{W}{}^1_{2,g}(\Omega)$ and the proofs of (2.3) and (2.4) can be found in [Dre96] and [Plu88], respectively. We have

$$\sum_{i,k=1}^{N} \int_\Omega g a_{ik} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_k} \left(|u|^{p-2} u \right) dx \geq C_E (p-1) \sum_{i=1}^{N} \int_\Omega g |u|^{p-2} \left(\frac{\partial u}{\partial x_i} \right)^2 dx$$
$$= \frac{4 C_E (p-1)}{p^2} \int_\Omega g |\nabla w|^2 \, dx \geq \frac{c}{p} \left(\|w\|^2_{2,1,g} - \|w\|^2_2 \right)$$

and

$$\sum_{i=1}^{N} \int_\Omega |a_i| \left| \frac{\partial u}{\partial x_i} \right| |u|^{p-1} \, dx \leq \frac{c}{p} \int_\Omega |\nabla w| \, |w| \, dx \leq \frac{c}{p} \|\nabla w\|_{\frac{2N'}{N'+1}} \|w\|_{\frac{2N'}{N'-1}} .$$

Lemma 2.3 leads to $\|\nabla w\|_{\frac{2N'}{N'+1}} \leq c \|w\|_{2,1,g}$. It holds $\left(\frac{2N'}{N'+1}\right)^{-1} < \left(\frac{2N'}{N'-1}\right)^{-1} + N^{-1}$. By Lemma 2.4 and 2.3, there is a θ with

$$\|w\|_{\frac{2N'}{N'-1}} \leq c \|w\|^\theta_{\frac{2N'}{N'+1},1} \|w\|_1^{1-\theta} \leq c \|w\|^\theta_{2,1,g} \|w\|_2^{1-\theta} \leq \varepsilon \|w\|_{2,1,g} + C_\varepsilon \|w\|_2 .$$

The application of Young's inequality yields (2.5). Furthermore,

$$\begin{aligned}
&|a_t(u,v) - a_{t'}(u,v)| \\
&\le \sum_{i,k=1}^{N} \int_\Omega g(x)|a_{ik}(x,t) - a_{ik}(x,t')| \left|\frac{\partial u}{\partial x_i}\right| \left|\frac{\partial v}{\partial x_k}\right| dx \\
&+ \sum_{i=1}^{N} \int_\Omega |a_i(x,t) - a_i(x,t')| \left|\frac{\partial u}{\partial x_i}\right| |v|\, dx \\
&\le c|t-t'| \sum_{i,k=1}^{N} \int_\Omega g(x) \left|\frac{\partial u}{\partial x_i}\right| \left|\frac{\partial v}{\partial x_k}\right| dx + c|t-t'| \int_\Omega |\nabla u|\,|v|\, dx.
\end{aligned}$$

These integrals can be estimated as above. This gives (2.6). The inequality (2.7) can be proved in a similar way. □

3. A–priori–estimates

We will use the Rothe method to attack problem (1.1) – (1.3). For this purpose we choose $n \in \mathbb{N}$, $h = \frac{T}{n}$, $t_j = jh$ $(j = 0, \ldots, n)$ and consider the variational problems $(j = 1, \ldots, n)$

$$\begin{aligned}
&(\delta u_j, v) + a_j(u_j, v) = (f_j, v) \quad \forall v \in \overset{o}{W}{}^1_{2,g}(\Omega), \qquad (3.1_j)\\
&u_j \in \overset{o}{W}{}^1_{2,g}(\Omega), \quad u_0 = U_0,
\end{aligned}$$

where $\delta u_j = \frac{1}{h}(u_j - u_{j-1})$, $f_j(x) = f(x, t_j, u_{j-1}(x))$, $a_j(.,.) = a_{t_j}(.,.)$.

(3.1) is a system of linear weakly elliptic boundary value problems, which can be solved step by step. We define the Rothe functions by piecewise linear and piecewise constant interpolation with respect to time,

$$u^n(x,t) = \begin{cases} u_{i-1}(x) + (t - t_{i-1})\delta u_i(x) & : t_{i-1} < t \le t_i, \quad i = 1, \ldots, n, \\ U_0(x) & : t \le 0 \end{cases}$$

$$\overline{u}^n(x,t) = \begin{cases} u_i(x) & : t_{i-1} < t \le t_i, \quad i = 1, \ldots, n, \\ U_0(x) & : t \le 0. \end{cases}$$

By (2.5) $(p = 2)$, (2.7) and the Theorem of Lax and Milgram, it is obvious that $u_j \in \overset{o}{W}{}^1_{2,g}(\Omega)$ exists and is uniquely determined, if $u_{j-1} \in B_R(U_0)$ and h is sufficiently small.

The following lemma gives a first a–priori–estimate.

Lemma 3.1. *There exist positive constants C and h_0 such that: If $h \le h_0$ and $u_0, u_1, \ldots, u_{i-1} \in B_R(U_0)$, then*

$$\|\delta u_j\|_2 \le C, \quad \|u_j\|_{2,1,g} \le C, \quad \sum_{l=1}^{j} h\, \|\delta u_l\|^2_{2,1,g} \le C$$

for all $1 \le j \le i$.

Proof. Testing (3.1_j) with $v = u_j$ leads to

$$(u_j - u_{j-1}, u_j) + h a_j(u_j, u_j) = h\,(f_j, u_j)\,,$$

hence

$$\frac{1}{2}\left(\|u_j\|_2^2 - \|u_{j-1}\|_2^2\right) + ch\,\|u_j\|_{2,1,g}^2 \le h\,\|f_j\|_2\,\|u_j\|_2 + Ch\,\|u_j\|_2^2\,.$$

Summing up ($j = 1, \ldots, l$) and applying Young's inequality we deduce that

$$\|u_l\|_2^2 + ch\sum_{j=1}^{l}\|u_j\|_{2,1,g}^2 \le Ch\sum_{j=1}^{l}\left(\|f_j\|_2^2 + \|u_j\|_2^2\right) + \|U_0\|_2^2\,.$$

The discrete version of Gronwall's Lemma (see [Kač85], p.29) shows that

$$\|u_l\|_2 \le C, \quad h\sum_{j=1}^{l}\|u_j\|_{2,1,g}^2 \le C, \tag{3.2}$$

if $h \le h_0$. Testing (3.1_j) and (3.1_{j-1}) with $v = \delta u_j$ and subtracting we conclude that

$$(\delta u_j - \delta u_{j-1}, \delta u_j) + h a_j(\delta u_j, \delta u_j) = h\,(\delta f_j, \delta u_j) - (a_j(u_{j-1}, \delta u_j) - a_{j-1}(u_{j-1}, \delta u_j)).$$

From Young's Inequality and (2.1), (2.5), (2.6) we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta u_j\|_2^2 - \|\delta u_{j-1}\|_2^2\right) + ch\,\|\delta u_j\|_{2,1,g}^2 \\
&\le C_f(h + \|u_{j-1} - u_{j-2}\|_2)\,\|\delta u_j\|_2 + h\,\|u_{j-1}\|_{2,1,g}\,\|\delta u_j\|_{2,1,g} + Ch\,\|\delta u_j\|_2^2 \\
&\le C_\varepsilon h\left(1 + \|\delta u_{j-1}\|_2^2\right) + 2\varepsilon h\,\|\delta u_j\|_{2,1,g}^2 + C_\varepsilon h\,\|u_{j-1}\|_{2,1,g}^2 + Ch\,\|\delta u_j\|_2^2\,,
\end{aligned}$$

hence

$$\|\delta u_j\|_2^2 - \|\delta u_{j-1}\|_2^2 + ch\,\|\delta u_j\|_{2,1,g}^2 \le Ch\left(1 + \|\delta u_{j-1}\|_2^2 + \|\delta u_j\|_2^2 + \|u_{j-1}\|_{2,1,g}^2\right).$$

Summing up ($j = 2, \ldots, l$) we see that

$$\begin{aligned}
&\|\delta u_l\|_2^2 - \|\delta u_1\|_2^2 + ch\sum_{j=2}^{l}\|\delta u_j\|_{2,1,g}^2 \qquad (3.3)\\
&\le Ct_l + Ch\sum_{j=1}^{l}\|\delta u_j\|_2^2 + Ch\sum_{j=1}^{l-1}\|u_j\|_{2,1,g}^2\,.
\end{aligned}$$

To estimate $\|\delta u_1\|_2$ and $h\,\|\delta u_1\|_{2,1,g}^2$, we choose $v = \delta u_1$ in (3.1_1):

$$\begin{aligned}
&\|\delta u_1\|_2^2 + h a_1(\delta u_1, \delta u_1) = (f_1, \delta u_1) - a_1(U_0, \delta u_1) \\
&= (f_0, \delta u_1) - a_0(U_0, \delta u_1) + h\,(\delta f_1, \delta u_1) + (a_0(U_0, \delta u_1) - a_1(U_0, \delta u_1)).
\end{aligned}$$

Since $(f_0, \delta u_1) - a_0(U_0, \delta u_1) = (y_0, \delta u_1)$, we conclude that

$$\begin{aligned}&\|\delta u_1\|_2^2 + ch\,\|\delta u_1\|_{2,1,g}^2\\ &\le C_\varepsilon\,\|y_0\|_2^2 + \varepsilon\,\|\delta u_1\|_2^2 + C_f h + Ch\,\|\delta u_1\|_2^2\\ &+ C_{a,\varepsilon} h\,\|U_0\|_{2,1,g}^2 + \varepsilon h\,\|\delta u_1\|_{2,1,g}^2 + Ch\,\|\delta u_1\|_2^2\,,\end{aligned}$$

hence $\|\delta u_1\|_2^2 + ch\,\|\delta u_1\|_{2,1,g}^2 \le C$, if ε and h are sufficiently small. From this estimate and (3.2), (3.3) it follows that

$$\|\delta u_l\|_2^2 + ch\sum_{j=1}^{l}\|\delta u_j\|_{2,1,g}^2 \le C + Ch\sum_{j=1}^{l}\|\delta u_j\|_2^2\,.$$

Gronwall's Lemma implies $\|\delta u_l\|_2 \le C$ and $h\sum_{j=1}^{l}\|\delta u_j\|_{2,1,g}^2 \le C$. To estimate $\|u_j\|_{2,1,g}$, we test (3.1_j) with $v = u_j$ and use $\|\delta u_j\|_2 \le C$, $\|u_j\|_2 \le C$. □

Our next step is to prove $u_j \in B_R(U_0)$ if $t_j \le T^* \le T$, T^* independent of h. To derive the desired L^∞–estimate of $u_j - U_0$, we insert test functions $v = |u - U_0|^{p-2}(u - U_0)$ and perform the limit process $p \to \infty$. But before doing this we must check whether these test functions are admissible, i.e. belong to $\overset{o}{W}{}^1_{2,g}(\Omega)$. We will do this by the aid of the first assertion of Lemma 2.7 and Lemma 3.3. The proof of this lemma is based on the following proposition from Ladyshenskaya [LSU67], Chapter 2, Theorem 5.1.

Proposition 3.2. *Let $u \in W^1_m(\Omega) \cap L^q(\Omega)$ such that there exists a constant K_0 with $\sup\operatorname{ess}_{\partial\Omega} u(x) \le K_0$. We assume for $K \ge K_0$*

$$\int_{A_K} |\nabla u|^m\,dx \le \gamma\left(\int_{A_K}(u-K)^l\,dx\right)^{\frac{m}{l}} + \gamma K^\alpha\,(\operatorname{mes} A_K)^{1-\frac{m}{N}+\varepsilon}\,,$$

where $l < \frac{Nm}{N-m}$, $\varepsilon > 0$, $m \le \alpha \le \varepsilon q + m$ and $A_K = \{x \in \Omega : u(x) \ge K\}$. Then $u \in L^\infty(\Omega)$.

Lemma 3.3. *Fix $h > 0$ and let $u \in \overset{o}{W}{}^1_{2,g}(\Omega)$ be a solution of*

$$\frac{1}{h}(u - u_{j-1}, v) + a_j(u, v) = (f_j, v) \quad \forall v \in \overset{o}{W}{}^1_{2,g}(\Omega).$$

Then $u \in L^\infty(\Omega)$, if $u_{j-1} \in L^\infty(\Omega)$.

P r o o f. Let $g_j = f_j + \frac{1}{h}u_{j-1}$, $K > 1$, $m = \frac{2N'}{N'+1}$ and $l = \frac{m}{m-1}$, hence $m^{-1} + l^{-1} = 1$. We choose $v = \max(u - K, 0)$. This function belongs to $\overset{o}{W}{}^1_{2,g}(\Omega)$, see [Dre96]. Thus, we get

$$\begin{aligned}&\frac{1}{h}(u, u-K)_{A_K} + \sum_{i,k=1}^{N}\int_{A_K} g a_{ik}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\,dx\\ &= (g_j, u-K)_{A_K} - \sum_{i,k=1}^{N}\int_{A_K} a_i\frac{\partial u}{\partial x_i}(u-K)\,dx,\end{aligned}$$

hence

$$\begin{aligned}&\frac{1}{h}\|u-K\|_{2,A_K}^2+c\|\nabla u\|_{2,A_K,g}^2\\ &\le \|g_j\|_{m,A_K}\|u-K\|_{l,A_K}+C\|\nabla u\|_{m,A_K}\|u-K\|_{l,A_K}-\frac{1}{h}(K,u-K)_{A_K}\\ &\le \|g_j\|_{m,A_K}^2+\|u-K\|_{l,A_K}^2+\varepsilon\|\nabla u\|_{m,A_K}^2+C_\varepsilon\|u-K\|_{l,A_K}^2 .\end{aligned}$$

From $u_{j-1}\in B_R(U_0)\subset L^r(\Omega)$, $r>N$ we get $\|g_j\|_{m,A_K}^m\le\|g_j\|_{r,A_K}^m(\mathrm{mes}\,A_K)^{\frac{r-m}{r}}$. It follows that

$$\frac{1}{h}\|u-K\|_{2,A_K}^2+c\|\nabla u\|_{2,A_K,g}^2\le C\|u-K\|_{l,A_K}^2+C(\mathrm{mes}\,A_K)^{\frac{2}{m}\frac{r-m}{r}} .$$

(The constant C depends on h due to the construction of g_j. But this does not matter, since h is fixed.) From this and $L_g^2(A_K)\subset L^m(A_K)$ it may be concluded that

$$\int_{A_K}|\nabla u|^m\,dx\le C\left(\|u-K\|_{l,A_K}^m+(\mathrm{mes}\,A_K)^{\frac{r-m}{r}}\right).$$

Since $r>N$ there is an $\varepsilon>0$ with $\frac{r-m}{r}>1-\frac{m}{N}+\varepsilon$ and $0<\varepsilon\le\frac{m}{N}$. Without loss of generality we may assume that K is sufficiently large, such that $\mathrm{mes}\,A_K<1$. Consequently,

$$(\mathrm{mes}\,A_K)^{\frac{r-m}{r}}\le(\mathrm{mes}\,A_K)^{1-\frac{m}{N}+\varepsilon},$$

and so

$$\int_{A_K}|\nabla u|^m\,dx\le C\left(\|u-K\|_{l,A_K}^m+K^m(\mathrm{mes}\,A_K)^{1-\frac{m}{N}+\varepsilon}\right).$$

Since $l<\frac{Nm}{N-m}$, we can apply the proposition described above and obtain

$$\sup_\Omega \mathrm{ess}\,u(x)\le C.$$

We can apply similar considerations to $\tilde u=-u$ and deduce $\inf\mathrm{ess}_\Omega u(x)\ge -C$. Consequently, $u\in L^\infty(\Omega)$. □

Now we are able to prove the L^∞–estimate of u_j-U_0.

Theorem 3.4. *There exist constants $h_0>0$, $T^*>0$ and a monotonically increasing function $M(t)\in C([0,T^*])$ with $M(0)=0$ such that:*
If $h_0\le h$ and $t_j\le T^*$ then $\|u_j-U_0\|_\infty\le M(t_j)\le R$.

P r o o f. Let $u_0,u_1,\dots,u_{i-1}\in B_R(U_0)$, $0\le j\le i$, $z_j=u_j-U_0$. Then $\delta z_j=\delta u_j$ and

$$(\delta z_j,v)+a_j(z_j,v)=(f_j,v)-a_j(U_0,v)\quad\forall v\in \overset{o}{W}{}^1_{2,g}(\Omega).$$

It holds $z_j\in L^\infty(\Omega)\cap\overset{o}{W}{}^1_{2,g}(\Omega)$. We choose $p>2$ and $v=|z_j|^{p-2}z_j$, $w_j=|z_j|^{\frac{p-2}{2}}z_j$. This gives with $\frac{1}{r}+\frac{1}{r'}=1$

$$\begin{aligned}&\frac{1}{h}\left(z_j,|z_j|^{p-2}z_j\right)-\frac{1}{h}\left(z_{j-1},|z_j|^{p-2}z_j\right)+\frac{c_1}{p}\|w_j\|_{2,1,g}^2\\ &\le\|f_j\|_r\left\||z_j|^{p-1}\right\|_{r'}+|a_j(U_0,|z_j|^{p-2}z_j)|+\frac{c_2}{p}\|w_j\|_2^2 .\end{aligned}$$

We set $\frac{1}{p}+\frac{1}{p'}=1$, and it holds

$$\left(z_j,|z_j|^{p-2}z_j\right)-\left(z_{j-1},|z_j|^{p-2}z_j\right)\geq\frac{1}{p}\left\|w_j\right\|_2^2-\frac{1}{p}\left\|w_{j-1}\right\|_2^2,$$

$$\left\||z_j|^{p-1}\right\|_{r'}=\left\|w_j\right\|_{\frac{2r'(p-1)}{p}}^{\frac{2(p-1)}{p}}\leq C\left\|w_j\right\|_{2r'}^{\frac{2(p-1)}{p}}.$$

Since $g^{-N'}\in L^1(\Omega)$, $N'>N$, Lemma 2.2 and $U_0\in W^1_{2r,g}(\Omega)$, we have $U_0\in W^1_{r\frac{2N'}{N'+1}}(\Omega)$, hence $\nabla U_0\in L^r(\Omega)\cap L^{2r}_g(\Omega)$. It follows that

$$\begin{aligned}
&|a_j(U_0,|z_j|^{p-2}z_j)|\\
&\leq C\frac{p-1}{p}\sum_{i,k=1}^{N}\int_\Omega\underbrace{\sqrt{g}\left|\frac{\partial U_0}{\partial x_k}\right|}_{2r}\underbrace{|w_j|^{\frac{p-2}{p}}}_{\frac{2r}{r-1}}\underbrace{\sqrt{g}\left|\frac{\partial w_j}{\partial x_k}\right|}_{2}\,dx+C\sum_{i=1}^{N}\int_\Omega\underbrace{\left|\frac{\partial U_0}{\partial x_i}\right|}_{r}\underbrace{|z_j|^{p-1}}_{r'}\,dx\\
&\leq C\left\|w_j\right\|_{2r'\frac{p-2}{p}}^{\frac{p-2}{p}}\left\|w_j\right\|_{2,1,g}+C\left\||z_j|^{p-1}\right\|_{r'}\\
&\leq C\frac{p}{\varepsilon}\left\|w_j\right\|_{2r'}^{2\frac{p-2}{p}}+\frac{\varepsilon}{p}\left\|w_j\right\|_{2,1,g}^2+C\left\|w_j\right\|_{2r'}^{2\frac{p-1}{p}}.
\end{aligned}$$

Combining these inequalities and choosing $\varepsilon<c_1$ we get

$$\begin{aligned}
&\frac{1}{h}\left(\left\|w_j\right\|_2^2-\left\|w_{j-1}\right\|_2^2\right)+c\left\|w_j\right\|_{2,1,g}^2\\
&\leq Cp(\left\|f_j\right\|_r+1)\left\|w_j\right\|_{2r'}^{2\frac{p-1}{p}}+C_\varepsilon p^2\left\|w_j\right\|_{2r'}^{2\frac{p-2}{p}}+C\left\|w_j\right\|_2^2\\
&\leq C\left(p\left\|w_j\right\|_{2r'}^{2\frac{p-1}{p}}+p^2\left\|w_j\right\|_{2r'}^{2\frac{p-2}{p}}+\left\|w_j\right\|_2^2\right).
\end{aligned}$$

Here we used $\left\|f_j\right\|_r\leq C$ which follows from $u_{j-1}\in B_R(U_0)$.

We now apply Lemma 2.5 to estimate the norms on the right-hand side. We can apply this lemma to $\left\|w_j\right\|_{2r'}^{2\frac{p-1}{p}}$, since $2\leq 2r'<\frac{r_1N}{N-r_1}$. Let $\alpha=\frac{p-1}{p}$, $1<q<r_1$ (independent of p),

$$\overline{\theta}=\frac{\frac{1}{q}-\frac{1}{2r'}}{\frac{1}{q}-\frac{1}{r_1}+\frac{1}{N}},\quad\overline{\sigma}=\frac{\overline{\theta}}{1-\overline{\theta}},\quad\overline{\beta}=\frac{\alpha}{1+(1-\alpha)\overline{\sigma}}=\frac{p-1}{p+\overline{\sigma}},$$

where $\overline{\sigma}$ does not depend on p. Write $\beta=\beta_1(p):=\overline{\beta}$. There is a σ_1 such that for all p

$$\frac{\alpha-\beta}{1-\alpha}=\frac{(p-1)\overline{\sigma}}{p+\overline{\sigma}}\leq\sigma_1.$$

The result is

$$\left\|w_j\right\|_{2r'}^{2\alpha}\leq\varepsilon\left\|w_j\right\|_{2,1,g}^2+C\varepsilon^{-\sigma_1}\left\|w_j\right\|_q^{2\beta_1},$$

where C does neither depend on p nor on h. Choosing δ independent of p sufficiently small and setting $\varepsilon=\frac{\delta}{p}$, we obtain:

$$p\left\|w_j\right\|_{2r'}^{2\alpha}\leq\delta\left\|w_j\right\|_{2,1,g}^2+C_\delta p^{\sigma_1+1}\left\|w_j\right\|_q^{2\beta_1}.$$

Now we estimate $\|w_j\|_{2r'}^{2\frac{p-2}{p}}$. Let $\alpha = \frac{p-2}{p}$, $1 < q < r_1$ (same value as above),

$$\overline{\theta} = \frac{\frac{1}{q} - \frac{1}{2r'}}{\frac{1}{q} - \frac{1}{r_1} + \frac{1}{N}}, \quad \overline{\sigma} = \frac{\overline{\theta}}{1-\overline{\theta}}, \quad \overline{\beta} = \frac{\alpha}{1+(1-\alpha)\overline{\sigma}} = \frac{p-2}{p+2\overline{\sigma}},$$

where $\overline{\sigma}$ does not depend on p. We set $\beta = \beta_2(p) := \overline{\beta}$. We choose a σ_2, such that for all p

$$\frac{\alpha-\beta}{1-\alpha} = \frac{(p-2)\overline{\sigma}}{2(p+\overline{\sigma})} \leq \sigma_2$$

and deduce that

$$\|w_j\|_2^{2\alpha} \leq \varepsilon \|w_j\|_{2,1,g}^2 + C\varepsilon^{-\sigma_2} \|w_j\|_q^{2\beta_2}.$$

Now let $\varepsilon = \frac{\delta}{p^2}$, δ independent of p sufficiently small. We obtain

$$p^2 \|w_j\|_2^{2\alpha} \leq \delta \|w_j\|_{2,1,g}^2 + C_\delta p^{2\sigma_2+2} \|w_j\|_q^{2\beta_2}.$$

Finally, there is a σ_3 with

$$\|w_j\|_2^2 \leq \delta \|w_j\|_{2,1,g}^2 + C\delta^{-\sigma_3} \|w_j\|_q^2.$$

For small δ we conclude that

$$\frac{1}{h}\left(\|w_j\|_2^2 - \|w_{j-1}\|_2^2\right) + c\|w_j\|_{2,1,g}^2 \leq C_\delta p^{\sigma_M}\left(\|w_j\|_q^{2\beta_1} + \|w_j\|_q^{2\beta_2} + \|w_j\|_q^2\right),$$

$\sigma_M = \max(\sigma_1+1, 2\sigma_2+2, 1)$. We point out that the constant C_δ does neither depend on h nor on p. Summing up ($j = 1, \dots, i$) we see that

$$\|w_i\|_2^2 - \|w_0\|_2^2 \leq Chp^{\sigma_M}\sum_{j=1}^{i}\left(\|w_j\|_q^{2\beta_1} + \|w_j\|_q^{2\beta_2} + \|w_j\|_q^2\right).$$

From $w_0 = |u_0 - U_0|^{\frac{p-2}{2}}(u_0 - U_0) = 0$ and $\|w_j\|_q = \|z_j\|_{pq/2}^{\frac{p}{2}}$ we obtain

$$\|z_i\|_p^p \leq ct_i p^{\sigma_M}\max_{1\leq j\leq i}\left(\|z_j\|_{pq/2}^{p\beta_1} + \|z_j\|_{pq/2}^{p\beta_2} + \|z_j\|_{pq/2}^p\right).$$

We define $p_\nu = 2\left(\frac{2}{q}\right)^\nu$, $\beta_{1,\nu} = \beta_1(p_\nu) = \frac{p_\nu-1}{p_\nu+\overline{\sigma}}$, $\beta_{2,\nu} = \beta_2(p_\nu) = \frac{p_\nu-2}{p_\nu+2\overline{\sigma}}$ and $m_{i,\nu} = \max_{1\leq j\leq i}\|z_j\|_{p_\nu}$ for $\nu = 0, 1, 2\dots$. It follows that

$$m_{i,\nu}^{p_\nu} \leq ct_i p_\nu^{\sigma_M}\left(m_{i,\nu-1}^{p_\nu\beta_{1,\nu}} + m_{i,\nu-1}^{p_\nu\beta_{2,\nu}} + m_{i,\nu-1}^{p_\nu}\right).$$

We have $\prod_{\nu=1}^\infty \beta_{1,\nu} = \prod_{\nu=1}^\infty \frac{p_\nu-1}{p_\nu+\overline{\sigma}} > 0$, since $p_\nu = 2\left(\frac{2}{q}\right)^\nu$ and

$$\sum_{\nu=1}^\infty\left|1 - \frac{p_\nu-1}{p_\nu+\overline{\sigma}}\right| = \sum_{\nu=1}^\infty\frac{\overline{\sigma}+1}{p_\nu+\overline{\sigma}} \leq (\overline{\sigma}+1)\sum_{\nu=1}^\infty\frac{1}{p_\nu} < \infty.$$

Now we are able to apply Lemma 2.6 and deduce that

$$\limsup_{\nu\to\infty} m_{i,\nu} \le c Q_{\gamma_1,\gamma_2}(t_i) m_{i,0}^{\beta_0}.$$

Furthermore,

$$m_{i,0} = \max_{1\le j\le i} \|u_j - U_0\|_2 \le \max_{1\le j\le i} \|u_j\|_2 + \|U_0\|_2 \le C(t_i) \le C.$$

Since $\lim_{p\to\infty} \|v\|_p = \|v\|_\infty$ for every function $v \in L^\infty(\Omega)$, it follows that

$$\|u_i - U_0\|_\infty \le C Q_{\gamma_1,\gamma_2}(t_i) =: M(t_i).$$

This function M is continuous, independent of h and fulfils $M(0) = 0$. We fix $0 < T^* \le T$ such that $M(t) \le R$ for all $t \le T^*$. □

Let us summarise the a-priori-estimates:

Theorem 3.5. *There are constants $N_0 \in \mathbb{N}$, $C > 0$, $0 < T^* \le T$, such that for all $n \ge N_0$, $t \le T^*$ it holds:*

$$\|\overline{u}^n\|_{L^\infty(I^*, \overset{o}{W}{}^1_{2,g}(\Omega))} \le C, \tag{3.4^n}$$

$$\|u_t^n\|_{L^\infty(I^*, L^2(\Omega))} \le C, \tag{3.5^n}$$

$$\|u_t^n\|_{L^2(I^*, \overset{o}{W}{}^1_{2,g}(\Omega))} \le C, \tag{3.6^n}$$

$$\|u^n(t) - U_0\|_\infty \le M(t+h).$$

Corollary 3.6. *If $t_{j-1} < t \le t_j$, then*

$$u^n(t) = \frac{t - t_{j-1}}{t_j - t_{j-1}} u_j + \frac{t_j - t}{t_j - t_{j-1}} u_{j-1},$$

consequently,

$$\|u^n\|_{L^\infty(I^*, \overset{o}{W}{}^1_{2,g}(\Omega))} \le C.$$

These approximations $u^n, \overline{u}^n$ satisfy

$$(u_t^n(t), v) + a_i(\overline{u}^n(t), v) = \left(\overline{f}^n(t), v\right), \tag{3.7^n}$$

where $v \in \overset{o}{W}{}^1_{2,g}(\Omega)$, $t_{i-1} < t \le t_i$ and $\overline{f}^n(t) := f(., t_i, u_{i-1}(.))$.

4. Convergence and Existence

In this section we prove that the sequences (u^n), $(\overline{u}^n)$ of piecewise linear and piecewise constant interpolations converge to a function u. We will show that u is the uniquely determined solution of (2.2).

First we list some auxiliary results. Let $h = \frac{T}{n}$ and $t_{j-1} < t \le t_j$. Then

$$\|\overline{u}^n(t-h) - \overline{u}^n(t)\|_2 = \|\delta u_j\|_2 \, h \le \frac{C}{n},$$
$$\|\overline{u}^n(t) - u^n(t)\|_2 \le \|\delta u_j\|_2 \, h \le \frac{C}{n},$$
$$\|u^n(t) - \overline{u}^n(t)\|_{2,1,g} \le \int_t^{t_j} \|\delta u_j\|_{2,1,g} \, d\tau \le \frac{C}{\sqrt{n}}. \tag{4.1^n}$$

Writing $h_n = \frac{T}{n}$, $h_m = \frac{T}{m}$, $t_{i-1,n} < t \le t_{i,n}$, $t_{j-1,m} < t \le t_{j,m}$ we conclude that

$$\begin{aligned}&\left\|\overline{f}^n(t) - \overline{f}^m(t)\right\|_2 = \|f(.,t_{i,n},\overline{u}^n(t-h_n)) - f(.,t_{j,m},\overline{u}^m(t-h_m))\|_2 \\ &\le C(h_n + h_m + \|\overline{u}^n(t-h_n) - u^n(t)\|_2 + \|u^n(t) - u^m(t)\|_2 + \|u^m(t) - \overline{u}^m(t-h_m)\|_2) \\ &\le C\left(h_n + h_m + \|u^n(t) - u^m(t)\|_2\right).\end{aligned} \tag{4.2}$$

Using these tools we can prove:

Theorem 4.1. *There exists a function $u \in W_2^1(I^*, \overset{o}{W}{}_{2,g}^1(\Omega))$ with $u_t \in L^\infty(I^*, L^2(\Omega))$, such that:*

$$u^n \to u \text{ in } C(I^*, \overset{o}{W}{}_{2,g}^1(\Omega)), \tag{4.3}$$
$$u^n \rightharpoonup u \text{ in } W_2^1(I^*, \overset{o}{W}{}_{2,g}^1(\Omega)), \tag{4.4}$$
$$u_t^n \to u_t \text{ in } L^2(I^*, L^2(\Omega)), \tag{4.5}$$
$$u_t^n \rightharpoonup u_t \text{ in } L^2(I^*, \overset{o}{W}{}_{2,g}^1(\Omega)), \tag{4.6}$$
$$u_t^n \rightharpoonup^* u_t \text{ in } L^\infty(I^*, L^2(\Omega)). \tag{4.7}$$

Proof. Let $t_{i-1,n} < t \le t_{i,n}$ and $t_{j-1,m} < t \le t_{j,m}$. Inserting $v = u^n(t) - u^m(t)$ into (3.7^n) and (3.7^m) and subtracting we obtain

$$\begin{aligned}&((u^n - u^m)_t, u^n - u^m) + a_{i,n}(u^n - u^m, u^n - u^m) = \left(\overline{f}^n - \overline{f}^m, u^n - u^m\right) \\ &- (a_{i,n}(\overline{u}^m, u^n - u^m) - a_{j,m}(\overline{u}^m, u^n - u^m)) \\ &- a_{i,n}(\overline{u}^n - u^n, u^n - u^m) + a_{i,n}(\overline{u}^m - u^m, u^n - u^m).\end{aligned} \tag{4.8}$$

We have

$$a_{i,n}(u^n - u^m, u^n - u^m) \ge C_1 \|u^n - u^m\|_{2,1,g}^2 - C_2 \|u^n - u^m\|_2^2, \tag{4.9}$$
$$\left(\overline{f}^n - \overline{f}^m, u^n - u^m\right) \le C\,(h_n + h_m)^2 + C\,\|u^n - u^m\|_2^2, \tag{4.10}$$
$$\begin{aligned}&|a_{i,n}(\overline{u}^m, u^n - u^m) - a_{j,m}(\overline{u}^m, u^n - u^m)| \\ &\le C(h_n + h_m)\,\|\overline{u}^m\|_{2,1,g}\,\|u^n - u^m\|_{2,1,g} \\ &\le C_\varepsilon\,(h_n + h_m)^2 + \varepsilon\,\|u^n - u^m\|_{2,1,g}^2,\end{aligned} \tag{4.11}$$
$$|a_{i,n}(\overline{u}^n - u^n, u^n - u^m)| \le \frac{C_\varepsilon}{n^2}\,\|\delta u^n\|_{2,1,g}^2 + \varepsilon\,\|u^n - u^m\|_{2,1,g}^2,$$

consequently,

$$\begin{aligned}&((u^n-u^m)_t,u^n-u^m)+c\left\|u^n-u^m\right\|_{2,1,g}^2\\&\le C\left(h_n+h_m\right)^2+C\left\|u^n-u^m\right\|_2^2+C\left(h_n+h_m\right)^2\left(\left\|u_t^n\right\|_{2,1,g}^2+\left\|u_t^m\right\|_{2,1,g}^2\right).\end{aligned}$$

Integrating this inequality over $[0,t]$ we obtain

$$\begin{aligned}&\left\|u^n(t)-u^m(t)\right\|_2^2+c\int_0^t\left\|u^n(\tau)-u^m(\tau)\right\|_{2,1,g}^2\,d\tau\\&\le C_1\left(h_n+h_m\right)^2+C_2\int_0^t\left\|u^n(\tau)-u^m(\tau)\right\|_2^2\,d\tau,\end{aligned}$$

and the application of Gronwall's Lemma yields

$$\left\|u^n(t)-u^m(t)\right\|_2^2\le C_1'\left(h_n+h_m\right)^2e^{C_2't}\le C\left(h_n+h_m\right)^2,$$

consequently,

$$\left\|u^n(t)-u^m(t)\right\|_2^2+c\int_0^t\left\|u^n(\tau)-u^m(\tau)\right\|_{2,1,g}^2\,d\tau\le C_T\left(\frac{1}{n}+\frac{1}{m}\right)^2.$$

It follows that (u^n) is a Cauchy sequence in the Banach spaces $C(I^*,L^2(\Omega))$ and $L^2(I^*,\overset{o}{W}{}^1_{2,g}(\Omega))$ with limit u. This technique of proving convergence order 1 goes back to Slodička, [Slo90].

From (4.8), (4.9), (4.10), (4.11) and

$$\left|a_{i,n}(\overline{u}^n-u^n,u^n-u^m)\right|\le\frac{C_\varepsilon}{n}+\varepsilon\left\|u^n-u^m\right\|_{2,1,g}^2$$

we get (4.3). It holds

$$\begin{aligned}&((u^n-u^m)_t,(u^n-u^m)_t)\\&=\left(\overline{f}^n-\overline{f}^m,(u^n-u^m)_t\right)-a_{i,n}(\overline{u}^n-\overline{u}^m,(u^n-u^m)_t)\\&-(a_{i,n}(\overline{u}^m,(u^n-u^m)_t)-a_{j,m}(\overline{u}^m,(u^n-u^m)_t)).\end{aligned}$$

By (2.7), (2.6), (3.4^m), (3.6^m), (3.6^n) we obtain

$$\begin{aligned}&\left\|(u^n-u^m)_t\right\|_{L^2(I^*,L^2(\Omega))}^2\\&\le C_\varepsilon\left\|\overline{f}^n-\overline{f}^m\right\|_{L^2(I^*,L^2(\Omega))}^2+\varepsilon\left\|(u^n-u^m)_t\right\|_{L^2(I^*,L^2(\Omega))}^2\\&+C\left\|\overline{u}^n-\overline{u}^m\right\|_{L^2(I^*,\overset{o}{W}{}^1_{2,g}(\Omega))}\left\|(u^n-u^m)_t\right\|_{L^2(I^*,\overset{o}{W}{}^1_{2,g}(\Omega))}+C\left(h_n+h_m\right).\end{aligned}$$

From this and (4.2), (4.1^n), (4.1^m), (4.3), (3.6^n), (3.6^m), we deduce that (u_t^n) is a Cauchy sequence in $L^2(I^*,L^2(\Omega))$ with limit v. By standard arguments we get $v=u_t$, hence (4.5). Since the sequence (u_t^n) is bounded in the Hilbert space $L^2(I^*,\overset{o}{W}{}^1_{2,g}(\Omega))$

and the limit is unique, we get (4.6), thus, (4.4). From the uniqueness of the limit and (3.5), we have (4.7). □

The last two lemmata show that u is a uniquely determined solution of (2.2).

Lemma 4.2. *The function* u *satisfies* (2.2).

Proof. We integrate (3.7^n) over I^* and obtain

$$\int_{I^*} (u_t^n, v)\, dt + \int_{I^*} a^n(\overline{u}^n, v)\, dt = \int_{I^*} \left(\overline{f}^n, v\right)\, dt \quad \forall n, \quad \forall v \in L^2(I, \overset{o}{W}{}^1_{2,g}(\Omega)).$$

Taking account of

$$\left\|\overline{f}^n - f(.,.,u(.,.))\right\|^2_{L^2(I^*,L^2(\Omega))} \le C \int_{I^*} h_n^2 + \|\overline{u}^n(t) - u(t)\|_2^2\, dt \le \frac{C}{n^2}$$

we have

$$\int_{I^*} \left(\overline{f}^n, v\right)\, dt \xrightarrow{n\to\infty} \int_{I^*} (f(.,t,u(.,t)), v)\, dt.$$

Writing $a^n(\overline{u}^n, v) = a^n(\overline{u}^n, v) - a^n(u, v) + a^n(u, v) - a_t(u, v) + a_t(u, v)$ and applying the Theorem of Lebesgue and (4.3) we obtain the assertion. □

Lemma 4.3. *There is at most one solution of* (2.2) *in* $L^2(I^*, \overset{o}{W}{}^1_{2,g}(\Omega)) \cap W_2^1(I^*, L^2(\Omega))$.

Proof. Let u_1, u_2 be two solutions of (2.2). Setting $w := u_1 - u_2$ and

$$v(t) = \begin{cases} w(t) & : t \le t_0, \\ 0 & : \text{ otherwise} \end{cases}$$

we conclude that

$$\begin{aligned} &\|w(t_0)\|_2^2 + C_1 \int_0^{t_0} \|w(t)\|_{2,1,g}^2\, dt \\ &\le C \int_0^{t_0} \|f(.,t,u_1(.,t)) - f(.,t,u_2(.,t))\|_2^2\, dt + C \int_0^{t_0} \|w(t)\|_2^2\, dt \\ &\le C_0 \int_0^{t_0} \|w(t)\|_2^2\, dt. \end{aligned}$$

Gronwall's Lemma implies $\|w(t_0)\|_2^2 \le 0e^{C_0 t} = 0$. □

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Fakultät für Mathematik und Informatik
Technische Universität Bergakademie Freiberg
Bernhard–von–Cotta–Str. 2
D–09596 Freiberg
Germany

Fachbereich für Mathematik und Informatik
Universität Halle–Wittenberg
D–06099 Halle
Germany