Decay Estimates for a Thermoelastic System in Waveguides

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Abstract

We investigate the linear system of thermoelasticity, consisting of an elasticity equation and a heat conduction equation, in a waveguide $\Omega = (0,1) \times \mathbb{R}^{n-1}$, with certain boundary conditions. We consider the cases of homogeneous and inhomogeneous systems and prove decay estimates of the solutions, which are a key ingredient to showing the global existence of solutions to nonlinear thermoelasticity, after having decomposed the solutions into various parts. We also give a simplified proof to the representation of the solutions to the Cauchy problem of thermoelasticity.

Key words: Thermoelastic system, decay estimates, initial boundary value problems, waveguides

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1. Introduction

Thermoelastic equations describe the elastic and thermal behavior of elastic heat conductive media. The classical equations in thermoelasticity, based on the Fourier law for heat conduction, are of a hyperbolic–parabolic coupled type ([1, 2]).

In this work, we are going to study the long time behavior of solutions to the following initial boundary value problem of linear thermoelastic equations in $\{(t,x) \colon t > 0,\ x \in \Omega\}$ with $\Omega = (0,1) \times \mathbb{R}^{n-1}$ being a so-called waveguide:

$$\begin{cases} u_{tt} - \mu \triangle u - (\mu+\lambda) \operatorname{grad} \operatorname{div} u + \gamma_1 \operatorname{grad} \theta = f(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \theta_t - \kappa \triangle \theta + \gamma_2 \operatorname{div} u_t = g(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega, \end{cases} \tag{1.1}$$

with the initial condition

$$(u, u_t, \theta)(0,x) = (u^0, u^1, \theta^0)(x), \quad x \in \Omega, \tag{1.2}$$

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and the boundary conditions

$$\begin{cases} u_1(t,x) = \partial_\nu u_2(t,x) = \ldots = \partial_\nu u_n(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ \partial_\nu \theta(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.3}$$

where u and θ denote the displacement and temperature deviation to a reference value respectively, all coefficients in (1.1) are constants with all of $\mu, 2\mu+\lambda, \kappa$ and $\gamma_1\gamma_2$ being positive, and ∂_ν is the normal derivative on the boundary of the waveguide Ω. With $B = \gamma_0 \operatorname{diag}(1, \partial_\nu, \ldots, \partial_\nu)$ we can write the boundary conditions for u as $Bu = 0$. Here γ_0 is the standard trace operator.

To study the long time behavior of solutions to linear problems is not only important for understanding the underlying physical phenomena, it is also the crucial step for establishing the global existence of solutions to the corresponding nonlinear problems (cf. [3]). There already have been proved many results on the long time behavior of solutions to the Cauchy problem for equations of thermoelasticity, e.g. see [2, 4, 5, 6] and references therein. It is well–known that the long time behavior of solutions is closely related to the spectral properties of the linear operators. For the Cauchy problems, it is studied usually by taking the Fourier transform in space variables, and investigating the asymptotic behavior of eigenvalues for large/small frequencies. These eigenvalues depend in a non-homogeneous way on the frequencies (see [7] for precise asymptotic expansions of the eigenvalues) which expresses the mixed hyperbolic–parabolic nature of the system. Typically, a power type decay of solutions to the Cauchy problem can then be shown.

In general, this idea does not work for problems in domains with boundaries. First, the Fourier transform is not available. And second, in case of Dirichlet boundary conditions, there are even counter-examples which show that the decay of the solutions to the linear problem (1.1) with vanishing right hand sides can be arbitrarily slow if the domain Ω admits a periodic orbit of billard, see [8]. On the other hand, the energy of rotationally symmetric solutions even decays exponentially, as demonstrated in [9].

An interesting and viable problem is to ask for decay properties in a waveguide. In [10], Lesky and Racke first described the long time behavior of solutions to the initial boundary problems for wave equations in a waveguide. Recently, they have also studied the elasticity problems (without temperature equations) with boundary conditions like (1.3) in [11]. In this paper, we shall develop their idea towards a study of the long time behavior of solutions to the problem (1.1), (1.2) and (1.3). We mainly discuss this problem with homogeneous equations, and the inhomogeneous case can then be studied by using the Duhamel principle, after a special reduction of the problem which makes the right-hand sides satisfy a large number of boundary conditions.

First, by carefully studying the Helmholtz projection associated with the boundary conditions (1.3), we deduce that the solenoidal part of the displacement solves a wave equation, which was considered already by Lesky and Racke in [10], and the potential field and the temperature deviation satisfy the equations of hyperbolic–parabolic coupled type from thermoelasticity. Then, we decompose the unknown functions into zero–modes (which do not depend on the bounded variable) and higher order modes. It is observed that the zero–modes of the potential field and the temperature deviation solve the Cauchy problem of thermoelastic equations in infinite direction variables

($x' \in \mathbb{R}^{n-1}$), so from the classical theory it follows that these zero–modes decay of the order $\mathcal{O}(t^{-(n-1)/2})$, while for the equations of higher order modes, we derive that they decay exponentially when t goes to infinity by using a partial eigenfunction expansion in the bounded direction, and taking the Fourier transform in infinite direction variables. Therefore, the potential field and the temperature difference decay as $\mathcal{O}(t^{-(n-1)/2})$ with respect to the time variable. Finally, by combining the result of the wave equations in a waveguide from [10] for the solenoidal fields, we conclude that the zero–modes of the displacement and temperature deviation decay of the order $\mathcal{O}(t^{-(n-2)/2})$, and their higher order modes decay of the order $\mathcal{O}(t^{-(n-1)/2})$ when t goes to infinity.

The remainder of this paper is arranged as follows: In Section 2, we state two main theorems on decay of solutions to homogeneous and inhomogeneous equations. In Section 3, we present some general facts of the equations (1.1) in a waveguide, including the Helmholtz projection, the zero–mode projection and existence of solutions to the (homogeneous) problem (1.1), (1.2) and (1.3). The homogeneous and inhomogeneous equations will be studied in Section 4 and Section 5 respectively.

2. Main Results

First, we note that we can assume the constants γ_1 and γ_2 in (1.1) to be equal and positive, via the scaling $u = \sqrt{\gamma_1/\gamma_2}\tilde{u}$. Considering first the homogeneous case, we are led to the system

$$\begin{cases} u_{tt} - \mu \triangle u - (\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\, u + \beta\,\mathrm{grad}\,\theta = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \theta_t - \kappa \triangle \theta + \beta\,\mathrm{div}\, u_t = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \end{cases} \tag{2.1}$$

together with the initial condition (1.2) and the boundary conditions (1.3).

We are looking for solutions (u,θ) to this problem in a space X which incorporates the boundary conditions (1.3):

$$\begin{aligned} X &= X_u \times X_\theta, \\ X_u &= \left\{ u \in \bigcap_{j=0}^{2} C^j([0,\infty), H^{2-j}(\Omega))\colon\ Bu = 0 \right\}, \\ X_\theta &= \left\{ \theta \in C([0,\infty), H^2(\Omega)) \bigcap C^1([0,\infty), L^2(\Omega))\colon\ \gamma_0 \partial_\nu \theta = 0 \right\}, \end{aligned}$$

where we do not notationally distinguish $L^2(\Omega)$ and $(L^2(\Omega))^n$. We introduce the notations $\triangle_D$ and $\triangle_N$ for the Dirichlet Laplacian and Neumann Laplacian on Ω, with the respective domains $D(\triangle_D) = H^2(\Omega) \cap H^1_0(\Omega) =: H^2_D(\Omega)$ and $D(\triangle_N) = \{\varphi \in H^2(\Omega)\colon \gamma_0\partial_\nu\varphi = 0\} =: H^2_N(\Omega)$.

Our main results are the following two theorems.

Theorem 2.1. *Assume $\mu > 0$, $\mu + \lambda \geq 0$, $\kappa > 0$ and $\beta \in \mathbb{R}$. If the initial data satisfy*

$$\begin{aligned} &(u^0, u^1, \theta^0) \in H^2(\Omega) \times H^1(\Omega) \times H^2(\Omega), \\ &Bu^0 = 0, \qquad \gamma_0 u^1_1 = 0, \qquad \gamma_0 \partial_\nu \theta^0 = 0, \end{aligned}$$

then the system (2.1) *with the initial condition* (1.2) *and the boundary conditions* (1.3) *has a unique solution* $(u, \theta) \in X$.

Suppose the initial data additionally have the regularity

$$
\begin{aligned}
&u_1^0 \in D(\triangle_D^{K_2/2}) \cap W_1^{K+1}(\Omega), \quad u_1^1 \in D(\triangle_D^{(K_2-1)/2}) \cap W_1^{K-1}(\Omega),\\
&u_k^0 \in D(\triangle_N^{K_2/2}) \cap W_1^{K+1}(\Omega), \quad u_k^1 \in D(\triangle_N^{(K_2-1)/2}) \cap W_1^{K-1}(\Omega), \quad 2 \le k \le n,\\
&\theta^0 \in D(\triangle_N^{K_2/2}) \cap W_1^{K+1}(\Omega),
\end{aligned}
$$

with $K_2 = \lfloor \frac{n}{2} \rfloor + 3$ *and* $K = 2\lfloor \frac{n}{2} \rfloor + 5$, *and the solenoidal parts of* u^0 *and* u^1 *also belong to the Sobolev spaces* $W_1^{K+1}(\Omega)$ *and* $W_1^{K-1}(\Omega)$. *Then the solution* (u, θ) *decays as follows:*

$$
\begin{aligned}
&\|(\nabla u, \partial_t u, \theta)(t, \cdot)\|_{L^\infty(\Omega)}\\
&\quad \le \frac{C}{(1+t)^{(n-2)/2}} \left\|(u^0, u^1, \theta^0)\right\|_{W_1^{K+1}(\Omega) \times W_1^{K-1}(\Omega) \times W_1^{K+1}(\Omega)}, \qquad 0 \le t < \infty.
\end{aligned}
$$

Define a further set of initial data $(u^0_{[0]}, u^1_{[0]}, \theta^0_{[0]})$ *as an average over the cross section of the waveguide like this:*

$$
\begin{aligned}
&u^j_{[0],1} = 0, \quad u^j_{[0],k}(x) = \int_{z_1=0}^1 u_k^j(z_1, x_2, \ldots, x_n)\, \mathrm{d}z_1, \qquad 2 \le k \le n, \quad j = 0, 1,\\
&\theta^0_{[0]}(x) = \int_{z_1=0}^1 \theta^0(z_1, x_2, \ldots, x_n)\, \mathrm{d}z_1,
\end{aligned}
$$

and write $(u_{[0]}, \theta_{[0]})$ *for the solution to* (2.1) *with boundary conditions* (1.3) *and with the initial data* $(u^0_{[0]}, u^1_{[0]}, \theta^0_{[0]})$. *Then the difference* $(u, \theta) - (u_{[0]}, \theta_{[0]})$ *has stronger decay than* (u, θ) *alone:*

$$
\begin{aligned}
&\left\|(\nabla(u - u_{[0]}), \partial_t(u - u_{[0]}), \theta - \theta_{[0]})(t, \cdot)\right\|_{L^\infty(\Omega)}\\
&\quad \le \frac{C}{(1+t)^{(n-1)/2}} \left\|(u^0, u^1, \theta^0)\right\|_{W_1^{K+1}(\Omega) \times W_1^{K-1}(\Omega) \times W_1^{K+1}(\Omega)}, \qquad 0 \le t < \infty.
\end{aligned}
$$

We also consider an inhomogeneous version of the thermoelasticity system:

$$
\begin{cases}
u_{tt} - \mu \triangle u - (\mu + \lambda) \operatorname{grad} \operatorname{div} u + \beta \operatorname{grad} \theta = f(t, x),\\
\qquad\qquad\qquad\qquad \theta_t - \kappa \triangle \theta + \beta \operatorname{div} u_t = g(t, x).
\end{cases} \tag{2.2}
$$

The uniqueness and existence of a solution (u, θ) in the space X follows directly from Duhamel's formula under standard assumptions on the regularity of f and g; therefore we now concentrate on the decay of the solution.

Theorem 2.2. *There are natural numbers* L, K, N *with the following property: If the right–hand sides* f *and* g *are of the regularity*

$$
\begin{aligned}
&f \in \bigcap_{j=0}^{2K} C^j\left([0, \infty), H^{2K-j}(\Omega) \cap W_p^{2K-j}(\Omega)\right) \bigcap C^{2K+1}([0, \infty), L^2(\Omega)),\\
&f, g \in C^{L+1}([0, \infty), L^2(\Omega)),
\end{aligned}
$$

with $1 < p \leq 2$*, and if the Compatibility conditions 5.1 and 5.2 are valid, then the solution* $(u, \theta) \in X$ *to* (2.2) *has an asymptotic behaviour described by the following inequality:*

$$\|(\nabla u, \partial_t u, \theta)(t,\cdot)\|_{L^q(\Omega)} \leq \frac{C}{(1+t)^{(n-2)(1/p-1/q)/2}} \left\|(\nabla u^0, u^0, u^1, \theta^0)\right\|_{W_p^N(\Omega)} \tag{2.3}$$

$$+ \frac{C}{(1+t)^{(n-1)(1/p-1/q)/2}} \sum_{j=0}^{\max(2K-1,L-1)} \left\|(\partial_t^j f, \partial_t^j g)(0,\cdot)\right\|_{W_p^{N-j}(\Omega)}$$

$$+ C \sum_{j=0}^{\max(2K-1,L-1)} \left\|(\partial_t^j f)(t,\cdot)\right\|_{W_p^{N-j}(\Omega)}$$

$$+ C \int_{s=0}^{t} \frac{1}{(1+t-s)^{(n-2)(1/p-1/q)/2}} \|(f,g)(s,\cdot)\|_{W_p^N(\Omega)} \,\mathrm{d}s$$

$$+ C \int_{s=0}^{t} \frac{1}{(1+t-s)^{(n-1)(1/p-1/q)/2}} \left\|\partial_s^{\max(2K,L)}(f,g)(s,\cdot)\right\|_{W_p^N(\Omega)} \,\mathrm{d}s,$$

where $1/p + 1/q = 1$.

3. General Properties of Thermoelastic Systems in Waveguides

The following result is almost immediate:

Lemma 3.1 (Uniqueness). *For a solution* $(u, \theta) \in X$ *to* (2.1)–(1.3)*, we define an energy as*

$$E(t) = \frac{1}{2}\left(\|u_t\|_{L^2(\Omega)}^2 + \mu \|\nabla u\|_{L^2(\Omega)}^2 + (\mu+\lambda)\|\operatorname{div} u\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2\right),$$

where $\|\nabla u\|_{L^2(\Omega)}^2 = \sum_{j,k} \|\partial_j u_k\|_{L^2(\Omega)}^2$*. Then we have the identity*

$$\partial_t E(t) = -\kappa \|\nabla\theta(t,\cdot)\|_{L^2(\Omega)}^2, \qquad t > 0.$$

Under the assumptions $\mu \geq 0$*,* $\mu + \lambda \geq 0$*,* $\kappa \geq 0$*,* $\beta \in \mathbb{R}$ *the solution* (u, θ) *is unique in the space* X*.*

3.1. The Helmholtz projection

Next we recall from [12] the Helmholtz decomposition in an arbitrary bounded or unbounded domain $\omega \subset \mathbb{R}^n$. The space of $L^2(\omega)$ potential vector fields is defined as

$$G_2(\omega) = \left\{\nabla\varphi \in L^2(\omega) \colon \ \varphi \in L^2_{\mathrm{loc}}(\omega)\right\},$$

and it is a closed subspace of $L^2(\omega)$, equipped with the usual $L^2(\omega)$ scalar product.

For a given arbitrary vector field $v \in L^2(\omega)$, the Lax–Milgram theorem guarantees the unique existence of a vector potential field $\nabla\varphi \in G_2(\omega)$ satisfying

$$\langle \nabla\varphi, \nabla\psi \rangle_{L^2(\omega)} = \langle v, \nabla\psi \rangle_{L^2(\omega)}, \qquad \forall\, \nabla\psi \in G_2(\omega).$$

Consequently, the function $\varphi \in L^2_{\text{loc}}(\omega)$ is a weak solution to the boundary value problem

$$\begin{cases} \triangle\, \varphi(x) = \operatorname{div} v(x), & x \in \omega, \\ \partial_\nu \varphi(x) = v(x) \cdot \vec{\nu}(x), & x \in \partial\omega. \end{cases}$$

The Helmholtz projector P then is defined as $Pv := v - \nabla\varphi$, and Pv is orthogonal to any field from $G_2(\omega)$, by construction. The image of P is called $L^2_\sigma(\omega)$, it is the completion of the space of divergence free vector fields from $C_0^\infty(\omega)$ under the $L^2(\omega)$ norm, and we have the orthogonal decomposition $L^2(\omega) = L^2_\sigma(\omega) \oplus G_2(\omega)$.

The Helmholtz projection in general Lebesgue spaces $L^r(\omega)$ with $1 < r < \infty$ is more delicate. First we define some needed function spaces:

$$C^\infty_{(0)}(\overline{\omega}) = \left\{ v\colon\ \exists\, u \in C_0^\infty(\mathbb{R}^n) \text{ with } v = u_{|\omega} \right\},$$
$$\hat{W}^1_r(\omega) = \left\{ \varphi \in L^r_{\text{loc}}(\omega)\colon\ \nabla\varphi \in L^r(\omega) \right\}.$$

The norm in $\hat{W}^1_r(\omega)$ is given by $\|\nabla\varphi\|_{L^r(\omega)}$, and functions which differ only by a constant are considered equal. Then in [13] it has been shown that the Helmholtz decomposition exists in $L^r(\omega)$ if and only if for all vector fields $f \in L^r(\omega)$ there is a unique $\varphi \in \hat{W}^1_r(\omega)$ such that

$$\int_\omega (\nabla\varphi - f) \cdot \nabla\psi \,\mathrm{d}x = 0, \qquad \forall\, \psi \in \hat{W}^1_{r'}(\omega), \qquad \frac{1}{r} + \frac{1}{r'} = 1.$$

There are certain unbounded domains ω with smooth boundary for which the decomposition $L^r(\omega) = L^r_\sigma(\omega) \oplus G_r(\omega)$ is not valid, see [14], [15], and also [16]. Here $L^r_\sigma(\omega)$ is the completion of the space of divergence free vector fields from $C_0^\infty(\omega)$ under the $L^r(\omega)$ norm, and $G_r(\omega)$ is defined similarly to $G_2(\omega)$ above.

However, in case of a waveguide $\omega = \Omega = (0,1) \times \mathbb{R}^{n-1}$, the existence and continuity of the Helmholtz projection has been established in [17], [18], [19] and [20], for instance.

Now we stick to this waveguide Ω.

Lemma 3.2. *For $1 < r < \infty$ and $m \in \mathbb{N}_+$, P is a continuous map from $W^m_r(\Omega)$ into itself.*

PROOF. By [21], the space $C^\infty_{(0)}(\overline{\Omega})$ is dense in $\hat{W}^1_{r'}(\Omega)$. For $f \in L^r(\Omega)$ with $f = Pf + \nabla\varphi$, we have

$$\int_\Omega (\nabla\varphi - f) \cdot \nabla\psi \,\mathrm{d}x = 0, \qquad \forall \psi \in C^\infty_{(0)}(\overline{\Omega}), \tag{3.1}$$

and $\triangle\, \varphi = \operatorname{div} f$ in the sense of distributions. If additionally $f \in W^1_r(\Omega)$, then $\varphi \in W^2_{r,\text{loc}}(\Omega)$, by elliptic regularity. Choose $k \in \{2, \ldots, n\}$. Then $\partial_k f \in L^r(\Omega)$ and we have the Helmholtz decomposition $\partial_k f = P\partial_k f + \nabla\varphi_k$, for some $\varphi_k \in \hat{W}^1_r(\Omega)$, and consequently

$$\int_\Omega (\nabla\varphi_k - \partial_k f)\cdot\nabla\psi\,\mathrm{d}x = 0, \qquad \forall\psi\in C^\infty_{(0)}(\overline{\Omega}).$$

By (3.1) we also have

$$\int_\Omega (\nabla\varphi - f)\cdot\nabla\partial_k\psi\,\mathrm{d}x = 0, \qquad \forall\psi\in C^\infty_{(0)}(\overline{\Omega}),$$

and partial integration then gives

$$\int_\Omega (\nabla\partial_k\varphi - \partial_k f)\cdot\nabla\psi\,\mathrm{d}x = 0,$$

for all ψ from $C^\infty_{(0)}(\overline{\Omega})$ which is dense in $\hat{W}^1_{r'}(\Omega)$. Therefore $\varphi_k = \partial_k\varphi$ modulo constants, and we have shown the improved regularity $\partial_k\varphi\in W^1_r(\Omega)$ for $2\le k\le n$ instead of $\varphi\in\hat{W}^1_r(\Omega)$. Then also $\partial_1\partial_k\varphi\in L^r(\Omega)$ for $k\ge 2$, and it remains to discuss $\partial_1^2\varphi$. But

$$\partial_1^2\varphi = \triangle\varphi - \sum_{k=2}^n \partial_k^2\varphi = \operatorname{div} f - \sum_{k=2}^n \partial_k^2\varphi \in L^r(\Omega),$$

and the consequence is the continuity of the mapping $P\colon W^1_r(\Omega)\to W^1_r(\Omega)$. Higher order derivatives are treated similarly, which finishes the proof.

Lemma 3.3. *If $v\in H^2(\Omega)$, then $Pv\in H^2(\Omega)$ and $B(\mathrm{id}-P)v=0$. If $v\in H^2(\Omega)$ and $Bv=0$, then $BPv=0$.*

PROOF. The third claim follows directly from the first two. The first claim was proved in Lemma 3.2. Concerning the second claim, we can write $(\mathrm{id}-)Pv=\nabla\varphi$ with some $\varphi\in H^3_{\mathrm{loc}}(\Omega)$ and $\gamma_0\partial_1\varphi=0$. For $k\ge 2$ we then have $\gamma_0\partial_1\partial_k\varphi=\gamma_0\partial_k\partial_1\varphi=0$, because second order traces of φ at $\partial\Omega$ exist.

Lemma 3.4. *For a function $v\in H^2(\Omega)$ with $Bv=0$, it holds $P\triangle v=\triangle Pv$. Moreover, if v is a vector field with $v_1\in D(\triangle_D^{K/2})$ and $v_k\in D(\triangle_N^{K/2})$ for $k=2,\dots,n$ and some $K\in\mathbb{N}_+$, then also $(Pv)_1\in D(\triangle_D^{K/2})$ and $(Pv)_k\in D(\triangle_N^{K/2})$.*

PROOF. The map $v\mapsto P\triangle v-\triangle Pv$ is continuous from $H^2(\Omega)$ to $L^2(\Omega)$, and by density it suffices to prove the first assertion for $v\in C^\infty_{(0)}(\overline{\Omega})$. Now we only have to show that $\langle\triangle Pv,\nabla\psi\rangle_{L^2(\Omega)}=0$ for all $v,\psi\in C^\infty_{(0)}(\overline{\Omega})$, which can be done by repeated partial integration and $\operatorname{div}Pv\equiv 0$.

This proves the second claim in case of $K=2$ (and $K=1$ runs similarly). Now consider an even $K\ge 4$. Then we know $v\in H^K(\Omega)$ with $B\triangle^l v=0$ for $l=0,\dots,K/2-1$, and induction gives $P\triangle^l v=\triangle^l Pv$ for such l. By Lemma 3.3, then also $B\triangle^l Pv=0$. And in case of an odd $K\ge 3$, we know $v\in H^K(\Omega)$ with $B\triangle^l v=0$ for $l=0,\dots,(K-1)/2-1$, and also $\gamma_0(\triangle^l v)_1=0$. The rest of the proof goes in a similar way as before.

Coming back to a solution (u,θ) of (2.1)–(1.3), we define the solenoidal part and the potential part of u in the usual way:

$$u^{\mathrm{so}} := Pu, \qquad u^{\mathrm{po}} := (\mathrm{id} - P)u.$$

Similarly, we write $u^{0,\mathrm{so}}$, $u^{1,\mathrm{so}}$, $u^{0,\mathrm{po}}$, $u^{1,\mathrm{po}}$ for the solenoidal and potential parts of the initial data.

Lemma 3.5. *If $(u,\theta) \in X$ is a solution to* (2.1)–(1.3)*, then $u^{\mathrm{so}} \in X_u$ and $(u^{\mathrm{po}},\theta) \in X$ are solutions to the systems*

$$u^{\mathrm{so}}_{tt} - \mu \triangle u^{\mathrm{so}} = 0, \tag{3.2}$$

and

$$\begin{cases} u^{\mathrm{po}}_{tt} - (2\mu+\lambda) \triangle u^{\mathrm{po}} + \beta \operatorname{grad} \theta = 0, \\ \theta_t - \kappa \triangle \theta + \beta \operatorname{div} u^{\mathrm{po}}_t = 0, \end{cases} \tag{3.3}$$

together with the initial conditions

$$(u^{\mathrm{so}}, u^{\mathrm{so}}_t)(0,x) = (u^{0,\mathrm{so}}, u^{1,\mathrm{so}})(x), \qquad (u^{\mathrm{po}}, u^{\mathrm{po}}_t, \theta)(0,x) = (u^{0,\mathrm{po}}, u^{1,\mathrm{po}}, \theta)(x).$$

Conversely, if u^{so} and u^{po} are solenoidal and potential vector fields with $u^{\mathrm{so}} \in X_u$ and $(u^{\mathrm{po}},\theta) \in X$, which solve (3.2) *and* (3.3)*, then $(u,\theta) \in X$ solves* (2.1), (1.3) *where $u := u^{\mathrm{so}} + u^{\mathrm{po}}$.*

PROOF. By Lemma 3.3, the Helmholtz decomposition preserves the $H^2(\Omega)$ regularity and the boundary conditions encoded in the operator B. Now let (u,θ) be a solution to (2.1), apply P to the first equation of that system, and then make use of Lemma 3.4 to obtain (3.2). Subtracting (3.2) from (2.1) then gives (3.3). The converse direction is immediate.

Lemma 3.6. *If a function u from X_u solves the wave equation* (3.2) *and has solenoidal initial data, then the solution u remains solenoidal for all times.*

PROOF. Write $u = u^{\mathrm{so}} + u^{\mathrm{po}}$ and observe that u and u^{so} have the same regularity, solve the same differential equation and the same boundary conditions. It remains to apply Lemma 3.1 with the parameters $\lambda := -\mu$ and $\beta := 0$ to show $u \equiv u^{\mathrm{so}}$.

Lemma 3.7. *If a pair of functions (u,θ) from X solves the thermoelasticity system*

$$\begin{cases} u_{tt} - \alpha \triangle u + \beta \operatorname{grad} \theta = 0, & \alpha > 0, \quad \beta \in \mathbb{R}, \\ \theta_t - \kappa \triangle \theta + \beta \operatorname{div} u_t = 0, & \kappa > 0, \end{cases} \tag{3.4}$$

and the initial data for u are potential fields, then the solution u remains a potential field for all positive times.

PROOF. Write $u = u^{\mathrm{so}} + u^{\mathrm{po}}$, apply Lemma 3.5 with $\mu := \alpha$ and $\lambda := -\mu$, and then make use of Lemma 3.1 to deduce that $u \equiv u^{\mathrm{po}}$.

3.2. The zero mode projection

Solutions of (2.1) that do not depend on x_1 deserve a special treatment. To this end, we define two more orthogonal projectors:

Definition 3.1. For a scalar function $\varphi\colon \Omega \to \mathbb{R}$, we set $P_0\varphi\colon \Omega \to \mathbb{R}$ as

$$(P_0\varphi)(x) := \int_{z_1=0}^{1} \varphi(z_1, x_2, \dots, x_n)\, \mathrm{d}z_1 = \langle \varphi(\cdot, x_2, \dots, x_n), \psi_0(\cdot)\rangle_{L^2(0,1)}\, \psi_0(x_1).$$

For a vector valued function $u\colon \Omega \to \mathbb{R}^n$, we fix $P_{0,B}u\colon \Omega \to \mathbb{R}^n$ as

$$(P_{0,B}u)(x_1, \dots, x_n) := \begin{pmatrix} 0 \\ P_0 u_2 \\ \vdots \\ P_0 u_n \end{pmatrix} (x_1, \dots, x_n).$$

We introduce the notations $\varphi_{[0]} := P_0\varphi$, $\varphi_{[+]} := \varphi - P_0\varphi$, $u_{[0]} := P_{0,B}u$ and $u_{[+]} := u - P_{0,B}u$.

These projectors map $L^2(\Omega)$ continuously into itself.

Here the function $\psi_0(x_1) = 1$ is the normalized eigenfunction to the zero eigenvalue of the Neumann Laplacian on $(0, 1)$; and since $\varphi_{[0]}$ does not depend on x_1, we call $\varphi_{[0]}$ the *zero mode* of φ, and $\varphi_{[+]}$ is said to contain the *higher modes* of φ. And the Dirichlet Laplacian has no zero eigenvalue, which is the reason why the first component of $P_{0,B}u$ has been defined as zero, cf. the definition of the boundary operator B.

Lemma 3.8. *These projectors commute with the usual differential operators and the Helmholtz projector provided appropriate boundary conditions are satisfied:*

- *if* $u \in H^1(\Omega)$ *with* $\gamma_0 u_1 = 0$, *then* $\operatorname{div} P_{0,B}u = P_0 \operatorname{div} u$,
- *if* $\varphi \in H^1(\Omega)$, *then* $\operatorname{grad} P_0\varphi = P_{0,B} \operatorname{grad} \varphi$,
- *if* $\varphi \in H^2(\Omega)$ *with* $\gamma_0 \partial_\nu \varphi = 0$, *then* $P_0 \triangle \varphi = \triangle P_0\varphi$,
- *if* $u \in H^2(\Omega)$ *with* $\gamma_0 \partial_\nu u_k = 0$ *for* $k = 2, \dots, n$, *then* $P_{0,B} \triangle u = \triangle P_{0,B}u$,
- *if* $u \in H^2(\Omega)$ *with* $\gamma_0 u_1 = 0$, *then* $P_{0,B} \operatorname{grad}\operatorname{div} u = \operatorname{grad}\operatorname{div} P_{0,B}u$,
- *if* $u \in H^1(\Omega)$ *with* $\gamma_0 u_1 = 0$, *then* $P_{0,B}Pu = PP_{0,B}u$, *where* P *is the Helmholtz projector.*

Then the following result is proved by very similar methods as the Lemmas 3.5, 3.6 and 3.7, exploiting the commutation relations from Lemma 3.8.

Lemma 3.9. *If* $(u, \theta) \in X$ *is a solution to* (2.1), (1.2), (1.3), *then* $(u_{[0]}, \theta_{[0]}) \in X$ *is a solution to* (2.1), (1.3) *with initial data* $(u^0_{[0]}, u^1_{[0]}, \theta^0_{[0]})$.

If $(u, \theta) \in X$ *is a solution to* (2.1), (1.2), (1.3), *and the initial data* (u^0, u^1, θ^0) *are pure zero modes in the sense of* $P_{0,B}u^0 = u^0$, $P_{0,B}u^1 = u^1$, $P_0\theta^0 = \theta^0$, *then the solution* (u, θ) *remains a pure zero mode for all positive times.*

If $(u, \theta) \in X$ *is a solution to* (2.1), (1.2), (1.3), *and the initial data* (u^0, u^1, θ^0) *contain no zero modes in the sense of* $P_{0,B}u^0 = 0$, $P_{0,B}u^1 = 0$, $P_0\theta^0 = 0$, *then the solution* (u, θ) *remains a zero mode free solution for all positive times.*

The advantage of considering the zero mode part of the solution separately is manifold: first, this part solves a thermoelastic system in the whole space $\mathbb{R}^{n-1}$, and then we can quote well–known results for such situations. Second, the Poincare's inequality becomes available. And third, the Neumann Laplacian becomes an invertible operator if we consider only the subspace of zero mode free functions.

More precisely, this means the following.

Lemma 3.10 (Poincare's inequalities). *We have, with a constant* C *independent of* φ,

$$\|\varphi\|_{L^2(\Omega)} \leq C \|\partial_1\varphi\|_{L^2(\Omega)}, \qquad \varphi \in H_0^1(\Omega), \tag{3.5}$$

$$\|\varphi\|_{L^2(\Omega)} \leq C \|\partial_1\varphi\|_{L^2(\Omega)}, \qquad \varphi \in H^1(\Omega), \quad P_0\varphi \equiv 0. \tag{3.6}$$

PROOF. The estimate (3.5) is standard. Concerning (3.6), we choose an $\varepsilon > 0$ and a $\psi \in H^1(\Omega) \cap C^\infty(\overline{\Omega})$ with $\|\varphi - \psi\|_{H^1(\Omega)} < \varepsilon$. Then

$$\|P_0\psi\|_{L^2(\Omega)} = \|P_0(\psi - \varphi)\|_{L^2(\Omega)} \leq \|\psi - \varphi\|_{L^2(\Omega)} < \varepsilon,$$

and we have

$$\|\varphi\|_{L^2(\Omega)} \leq \|\varphi - \psi\|_{L^2(\Omega)} + \|P_0\psi\|_{L^2(\Omega)} + \|(\mathrm{id} - P_0)\psi\|_{L^2(\Omega)} < 2\varepsilon + \|\chi\|_{L^2(\Omega)},$$

with $\chi := (\mathrm{id} - P_0)\psi$. Note that $\chi \in H^1(\Omega) \cap C^\infty(\overline{\Omega})$ and $\partial_1\chi \equiv \partial_1\psi$ as well as

$$\int_{z_1=0}^{1} \chi(z_1, x_2, \dots, x_n)\,\mathrm{d}z_1 = 0, \qquad (x_2, \dots, x_n) \in \mathbb{R}^{n-1}.$$

By Poincare's inequality on $(0, 1)$, we have

$$\begin{aligned} \|\chi(\cdot, x_2, \dots, x_n)\|_{L^2(0,1)} &\leq C \|\partial_1\chi(\cdot, x_2, \dots, x_n)\|_{L^2(0,1)}, \\ \|\chi\|_{L^2(\Omega)} &\leq C \|\partial_1\psi\|_{L^2(\Omega)} \leq C\varepsilon + C \|\partial_1\varphi\|_{L^2(\Omega)}, \end{aligned}$$

which finally gives us

$$\|\varphi\|_{L^2(\Omega)} < (2 + C)\varepsilon + C \|\partial_1\varphi\|_{L^2(\Omega)}, \qquad \forall \varepsilon > 0.$$

Sending ε to zero completes the proof.

By either (3.5) or (3.6), then the Lax–Milgram lemma gives us the existence, uniqueness and *a priori* estimates of solutions $w \in H^1(\Omega)$ to the scalar weak Dirichlet and Neumann problems

$$\begin{aligned} &\triangle w = f, \qquad \gamma_0 w = 0, \qquad f \in L^2(\Omega), \\ &\triangle w = f, \qquad \gamma_0\partial_1 w = 0, \qquad P_0 w \equiv 0, \qquad f \in L^2(\Omega), \qquad P_0 f \equiv 0. \end{aligned} \tag{3.7}$$

The solutions w can then be found by spectral decomposition on $L^2(0,1)$, which means, in case of (3.7) and ψ_j being the eigenfunctions of the Neumann Laplacian,

$$f(x) = \sum_{j=1}^{\infty} \psi_j(x_1) f_j(x_2, \dots, x_n), \qquad w(x) = \sum_{j=1}^{\infty} \psi_j(x_1) w_j(x_2, \dots, x_n),$$
$$\left(-\pi^2 j^2 + \partial_2^2 + \dots + \partial_n^2\right) w_j = f_j \qquad \text{on } \mathbb{R}^{n-1}.$$

For each of the boundary conditions, we have $\|\triangle w\|_{L^2(\Omega)}^2 = \sum_{j,k=1}^n \|\partial_j \partial_k w\|_{L^2(\Omega)}^2$, and consequently

$$\|w\|_{H^2(\Omega)} \le C \|f\|_{L^2(\Omega)}^2 . \tag{3.8}$$

3.3. Existence of solutions

Now the splitting of u into solenoidal and potential part is justified, and we discuss the existence of solutions.

The existence of the solenoidal part is obvious:

Proposition 3.1 (Existence of the solenoidal part). *Necessary and sufficient for the global existence of a solution $u \in X_u$ of the wave equation* (3.2) *with initial data u^0 and u^1 is*

$$(u^0, u^1) \in H^2(\Omega) \times H^1(\Omega), \qquad Bu^0 = 0, \qquad \gamma_0 u_1^1 = 0.$$

The existence of the potential part u^{po} will be established in Proposition 3.3, after several preparatory lemmas.

Now let (u, θ) from X be a solution to (3.4) and introduce a vector W with $n(n+1)+1$ components,

$$W = (W^{(1)}, \dots, W^{(n+2)})^\top, \tag{3.9}$$

where $W^{(1)}$, …, $W^{(n+1)}$ are n–vectors,

$$W^{(k)} = \operatorname{grad} u_k, \qquad (1 \le k \le n), \qquad W^{(n+1)} = \frac{1}{\sqrt{\alpha}} \partial_t u, \tag{3.10}$$

and $W^{(n+2)}$ is a scalar,

$$W^{(n+2)} = \frac{1}{\sqrt{\alpha}} \theta. \tag{3.11}$$

Then we obtain the system

$$\partial_t W + \mathcal{A} W = 0, \tag{3.12}$$

$$\mathcal{A} = \begin{pmatrix} 0 & \dots & 0 & a_1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & a_n & 0 \\ a_1^\top & \dots & a_n^\top & 0 & \beta \operatorname{grad} \\ 0 & \dots & 0 & \beta \operatorname{div} & -\kappa \triangle \end{pmatrix}, \tag{3.13}$$

where a_k is an $n \times n$ matrix of the form

$$a_k = -\sqrt{\alpha} \begin{pmatrix} 0 & \dots & \partial_1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \partial_n & \dots & 0 \end{pmatrix}$$

with non-vanishing entries in the k–th column.

Lemma 3.11. *If $(u, \theta) \in X$ is a solution to* (3.4)*, then the function W constructed above is a solution to* (3.12) *of the regularity*

$$\begin{cases} W^{(k)} \in C^1([0,\infty), \nabla H^1(\Omega)) \bigcap C([0,\infty), \nabla H^2(\Omega)), \qquad 1 \le k \le n, \\ W^{(n+1)} \in C^1([0,\infty), L^2(\Omega)) \bigcap C([0,\infty), H^1(\Omega)), \\ W^{(n+2)} \in C([0,\infty), H^2(\Omega)) \bigcap C^1([0,\infty), L^2(\Omega)) \end{cases} \tag{3.14}$$

and boundary conditions

$$\gamma_0 W_k^{(1)} = \gamma_0 W_1^{(k)} = \gamma_0 W_1^{(n+1)} = \gamma_0 \partial_\nu W^{(n+2)} = 0, \qquad 2 \le k \le n. \tag{3.15}$$

Conversely, let W be given with the properties (3.14) *and* (3.15)*. Then there are $(u^0, u^1, \theta^0) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ such that*

$$\theta(t,x) := \sqrt{\alpha} W^{(n+2)}(t,x),$$
$$u(t,x) := u^0(x) + \sqrt{\alpha} \int_{\tau=0}^t W^{(n+1)}(\tau, x)\,\mathrm{d}\tau$$

satisfy $(u, \theta) \in X$ and solve (3.3) *with initial conditions* (1.2)*.*

PROOF. We start with constructing (u^0, u^1, θ^0). By (3.14), there is a scalar function $\psi_1 \in H^2(\Omega)$ with $W^{(1)}(0,x) = \operatorname{grad} \psi_1(x)$. From the boundary condition $\gamma_0 W_k^{(1)} = 0$ we then find that ψ_1 is constant on $\{0\} \times \mathbb{R}^{n-1}$ and on $\{1\} \times \mathbb{R}^{n-1}$. Since the trace $\gamma_0 \psi_1$ at the boundary belongs to $H^{3/2}(\mathbb{R}^{n-1})$, both constants must be zero, and then $u_1^0 := \psi_1$ has homogeneous Dirichlet boundary conditions. For $k \ge 2$, there are functions $\psi_k \in H^2(\Omega)$ with $W^{(k)}(0,x) = \operatorname{grad} \psi_k(x)$. Then $u_k^0 := \psi_k$ has vanishing Neumann boundary values. Next, we put $u^1(x) = \sqrt{\alpha} W^{(n+1)}(0,x)$ and $\theta^0(x) := \sqrt{\alpha} W^{(n+2)}(0,x)$.

For $k \ge 1$, we remark

$$\operatorname{grad} u_k(t,x) = \operatorname{grad} u_k^0(x) + \sqrt{\alpha} \int_{\tau=0}^t \operatorname{grad} W_k^{(n+1)}(\tau,x)\,\mathrm{d}\tau$$
$$= \operatorname{grad} u_k^0(x) + \int_{\tau=0}^t \partial_\tau W^{(k)}(\tau,x)\,\mathrm{d}\tau = W^{(k)}(t,x).$$

Then (3.3) follows easily, and the proof is finished.

To define the domain of the operator $\mathcal{A}$, we bring into play that u shall be a vector potential field. But first we define the ground space $\mathcal{H}$.

Definition 3.2. A vector $(W^{(1)},\dots,W^{(n+2)})$ belongs to $\mathcal{H}$ if and only if

- there is a scalar function φ such that $\nabla\varphi \in H^1(\Omega)\cap G_2(\Omega)$ and $B\nabla\varphi = 0$, with $W^{(k)} = \partial_k\nabla\varphi$ for $k = 1,\dots,n$,
- $W^{(n+1)} \in G_2(\Omega)$,
- $W^{(n+2)} \in L^2(\Omega)$.

Lemma 3.12. *The space $\mathcal{H}$, equipped with the $L^2(\Omega)$ norm, is a Banach space.*

PROOF. The space $H^1(\Omega)\cap G_2(\Omega)$, endowed with the $H^1(\Omega)$ norm, is the intersection space of two Banach spaces. Its subspace Z_{tmp} consisting of all those elements $\nabla\varphi$ with $B\nabla\varphi = 0$ is a closed subspace, since Z_{tmp} is the null space of a bounded trace operator. Applying the closed graph theorem to the operators $\partial_k\colon Z_{tmp} \to L^2(\Omega)$ for $k = 1,\dots,n$ concludes the proof.

Of course, the condition $B\nabla\varphi = 0$ reduces to $\gamma_0\partial_1\varphi = 0$, due to the limited smoothness of φ.

We also define a closed subspace of $\mathcal{H}$:

$$\begin{aligned}\mathcal{H}_+ := \Big\{&(W^{(1)},\dots,W^{(n+2)}) = (\partial_1\nabla\varphi,\dots,\partial_n\nabla\varphi,\nabla\psi,\vartheta) \in \mathcal{H}\colon \\ &P_0\varphi \equiv P_0\psi \equiv P_0\vartheta \equiv 0\Big\}.\end{aligned} \tag{3.16}$$

By repeated use of Poincare's inequalities, we deduce the estimates

$$\begin{aligned}\|\varphi\|_{L^2(\Omega)} &\le C\,\|\partial_1\varphi\|_{L^2(\Omega)} \le C'\,\|\partial_1\nabla\varphi\|_{L^2(\Omega)}\,,\\ \|\psi\|_{L^2(\Omega)} &\le C\,\|\partial_1\psi\|_{L^2(\Omega)}\,,\end{aligned}$$

valid for $W \in \mathcal{H}_+$.

We also note that the vector $W = (\partial_1\nabla\varphi,\dots,\partial_n\nabla\varphi,\nabla\psi,\vartheta)$ is mapped by $\mathcal{A}$ to the vector $\mathcal{A}W = (-\sqrt{\alpha}\partial_1\nabla\psi,\dots,-\sqrt{\alpha}\partial_n\nabla\psi,\nabla(-\sqrt{\alpha}\triangle\varphi+\beta\vartheta),\beta\triangle\psi-\kappa\triangle\vartheta)$.

Definition 3.3. The domain of the operator $\mathcal{A}$ is fixed as

$$D(\mathcal{A}) = \Big\{W = (W^{(1)},\dots,W^{(n+2)}) \in \mathcal{H}\colon\ \mathcal{A}W \in \mathcal{H},\quad W^{(n+2)} \in D(\triangle_N)\Big\}.$$

Lemma 3.13. *A vector W belongs to $D(\mathcal{A})$ if and only if $W \in \mathcal{H}$ and*

- $W^{(k)} \in H^1(\Omega)$ *for* $1 \le k \le n+1$ *and* $W^{(n+2)} \in H^2(\Omega)$,
- *the boundary conditions* (3.15) *hold.*

Furthermore, if $W \in C([0,\infty),D(\mathcal{A}))$ and $\partial_t W \in C([0,\infty),\mathcal{H})$, then W has the regularity (3.14)*.*

PROOF. From $(\mathcal{A}W)^{(k)} \in L^2(\Omega)$ for $1 \le k \le n$ we find $W^{(n+1)} \in H^1(\Omega) \cap G_2(\Omega)$. And $W^{(n+2)} \in D(\triangle_N) \subset H^2(\Omega)$ together with $(\mathcal{A}W)_k^{(n+1)} \in L^2(\Omega)$ imply $\operatorname{div} W^{(k)} \in L^2(\Omega)$ for $1 \le k \le n$. We know that $W^{(k)} = \partial_k \nabla\varphi$ for some scalar function φ and all k, hence we obtain $\triangle\, \partial_k\varphi \in L^2(\Omega)$. From

$$\triangle(\partial_1\varphi) \in L^2(\Omega), \qquad \gamma_0(\partial_1\varphi) = 0, \qquad \partial_1\varphi \in H^1(\Omega),$$

we conclude that $\partial_1\varphi \in H^2_D(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$. Then the boundary values $\gamma_0\partial_1\varphi$ belong to $H^{3/2}(\partial\Omega)$, and for $k \ge 2$ we find $\gamma_0(\partial_1\partial_k\varphi) = 0$, and therefore we know for such k that

$$\triangle(\partial_k\varphi) \in L^2(\Omega), \qquad \gamma_0\partial_1(\partial_k\varphi) = 0, \qquad \partial_k\varphi \in H^1(\Omega),$$

hence $\partial_k\varphi \in H^2_N(\Omega)$ and $B\nabla\varphi = 0$, from which $W^{(k)} \in H^1(\Omega)$ and the conditions (3.15) follow. The proof is complete.

Lemma 3.14. *The embedding $D(\mathcal{A}) \subset \mathcal{H}$ is dense.*

PROOF. We concentrate our discussion to the components $W^{(1)}, \dots, W^{(n)}$. Let $\nabla\varphi \in H^1(\Omega) \cap G_2(\Omega)$ be given with $\gamma_0\partial_1\varphi = 0$ and $W^{(k)} = \partial_k\nabla\varphi$. Choose a smooth vector field v (which will in general not be a potential field) with $\gamma_0 v_1 = 0$ and $\|\nabla\varphi - v\|_{H^1(\Omega)}$ small and set $\nabla\psi := (\operatorname{id} - P)v$. Then also $\|\nabla\varphi - \nabla\psi\|_{H^1(\Omega)}$ will be small, by the continuity of $\operatorname{id} - P$ as mapping between Sobolev spaces of the same order. Lemma 3.3 gives $B\nabla\psi = 0$. It remains to set $\tilde{W}^{(k)} := \partial_k\nabla\psi$ for $1 \le k \le n$.

Further, we define an operator $\mathcal{A}_+$ with domain $D(\mathcal{A}_+) := D(\mathcal{A}) \cap \mathcal{H}_+$, with $\mathcal{A}_+W := \mathcal{A}W$ for $W \in D(\mathcal{A}_+)$. This operator maps $D(\mathcal{A}_+)$ into $\mathcal{H}_+$.

Lemma 3.15. *Suppose $\mathcal{A}W = F$ with $F \in \mathcal{H}$, $W \in D(\mathcal{A})$, and $W^{(k)} = \nabla\partial_k\varphi$ for $1 \le k \le n$. Then we have the estimates*

$$\left\|\nabla W^{(n+1)}\right\|_{L^2(\Omega)} \le C\,\|F\|_{L^2(\Omega)}\,, \tag{3.17}$$

$$\left\|\triangle\, W^{(n+2)}\right\|_{L^2(\Omega)} \le C\,\|F\|_{L^2(\Omega)}\,, \tag{3.18}$$

$$\|\partial_1\varphi\|_{H^2(\Omega)} \le C\left(\left\|F_1^{(n+1)}\right\|_{L^2(\Omega)} + \left\|\partial_1 W^{(n+2)}\right\|_{L^2(\Omega)}\right). \tag{3.19}$$

To each $F \in \mathcal{H}_+$, there is exactly one $W \in D(\mathcal{A}_+)$ with $\mathcal{A}_+W = F$, and we have the following estimate, with some constant C independent of F:

$$\left\|(W^{(1)}, \dots, W^{(n+1)})\right\|_{H^1(\Omega)} + \left\|W^{(n+2)}\right\|_{H^2(\Omega)} \le C\,\|F\|_{\mathcal{H}}\,. \tag{3.20}$$

PROOF. Estimate (3.17) follows from $F^{(k)} = a_k W^{(n+1)}$ for $1 \le k \le n$, having (3.18) as a direct consequence. Finally, the equation for the first component $F_1^{(n+1)}$ of $F^{(n+1)}$ implies

$$\triangle(\partial_1\varphi) = c_1 F_1^{(n+1)} + c_2\partial_1 W^{(n+2)}, \qquad \gamma_0\partial_1\varphi = 0,$$

with certain constants c_1, c_2, and then (3.8) completes the proof of (3.19).

Concerning the last claim, we know $F^{(k)} = \partial_k \nabla \varphi_F$ for $k = 1, \dots, n$, with $\varphi_F \in H^2(\Omega)$, $\gamma_0 \partial_1 \varphi_F = 0$ and $P_0 \varphi_F \equiv 0$; $F^{(n+1)} = \nabla \psi_F$ with $\psi_F \in H^1(\Omega)$ and $P_0 \psi_F \equiv 0$; as well as $F^{(n+2)} = \vartheta_F \in L^2(\Omega)$ with $P_0 \vartheta_F \equiv 0$.

We wish to find φ, ψ, ϑ with $-\sqrt{\alpha}\partial_k \nabla \psi = \partial_k \nabla \varphi_F$, $\nabla(-\sqrt{\alpha} \triangle \varphi + \beta\vartheta) = \nabla \psi_F$, $\beta \triangle \psi - \kappa \triangle \vartheta = \vartheta_F$, and additionally $\nabla \varphi \in H^2(\Omega)$, $\nabla \psi \in H^1(\Omega)$, $\vartheta \in H^2(\Omega)$, $\gamma_0 \partial_\nu (\varphi, \psi, \theta) = 0$, and $P_0 \varphi \equiv P_0 \psi \equiv P_0 \vartheta \equiv 0$.

Then necessarily $\psi = -\varphi_F/\sqrt{\alpha}$. Next ϑ is uniquely determined via $\triangle(-\kappa\vartheta + \beta\psi) = \vartheta_F$, which is solvable since $P_0 \vartheta_F \equiv 0$. After having determined ϑ, we finally consider $-\sqrt{\alpha} \triangle \varphi = \psi_F - \beta\vartheta + \mathrm{const}$. It turns out that this integration constant must be zero, since the other three items of this equation are members of $L^2(\Omega)$. Then φ is uniquely determined with the desired properties. Here we have made repeated use of (3.8), which also gives (3.20), finishing the proof.

Lemma 3.16. *The operator $\mathcal{A}$ is a closed operator, $\mathcal{A}\colon D(\mathcal{A}) \to \mathcal{H}$.*

PROOF. Take a sequence $(W_s)_{s\in\mathbb{N}} \subset D(\mathcal{A})$ which converges in the norm of $\mathcal{H}$ to an element $W_* \in \mathcal{H}$. Additionally, suppose $\lim_{s\to\infty} \|\mathcal{A}W_s - Y_*\|_{\mathcal{H}} = 0$ for some $Y_* \in \mathcal{H}$. We intend to show $W_* \in D(\mathcal{A})$ and $\mathcal{A}W_* = Y_*$.

From our assumption, we have already the convergence $\lim_{s\to\infty} W_s^{(k)} = W_*^{(k)}$ in the topology of $L^2(\Omega)$, for $k = 1, \dots, n+2$. From (3.17) we directly obtain $W_*^{(n+1)} \in H^1(\Omega)$, with the $H^1(\Omega)$ convergence $\lim_{s\to\infty} W_s^{(n+1)} = W_*^{(n+1)}$, and $\gamma_0 W_{*,1}^{(n+1)} = 0$. The Neumann Laplacian $\triangle_N\colon D(\triangle_N) \to L^2(\Omega)$ is closed, then (3.18) implies $W_*^{(n+2)} \in D(\triangle_N) = H_N^2(\Omega)$, and we have the convergence $\lim_{s\to\infty} W_s^{(n+2)} = W_*^{(n+2)}$ in $H^2(\Omega)$.

Next, we know that there are functions φ_s and φ_* with the properties

$$\begin{aligned} W_s^{(k)} &= \partial_k \nabla \varphi_s \qquad (1 \le k \le n), \qquad \nabla \varphi_s \in H^2(\Omega), \qquad B\nabla\varphi_s = 0, \\ W_*^{(k)} &= \partial_k \nabla \varphi_* \qquad (1 \le k \le n), \qquad \nabla \varphi_* \in H^1(\Omega), \qquad \gamma_0 \partial_1 \varphi_* = 0. \end{aligned}$$

Since $(W_s)_{s\in\mathbb{N}}$ converges to W_* in $\mathcal{H}$, we get $\lim_{s\to\infty} \|\partial_1(\partial_1\varphi_s - \partial_1\varphi_*)\|_{L^2(\Omega)} = 0$, from which (3.5) yields $\lim_{s\to\infty} \|\partial_1\varphi_s - \partial_1\varphi_*\|_{L^2(\Omega)} = 0$. Now consider the case $k = 1$ first. From (3.19) and the closedness of the Dirichlet Laplacian $\triangle_D\colon D(\triangle_D) \to L^2(\Omega)$, we find $\partial_1\varphi_* \in D(\triangle_D) = H_D^2(\Omega)$ with $\lim_{s\to\infty} \|\partial_1\varphi_s - \partial_1\varphi_*\|_{H^2(\Omega)} = 0$.

Next take $2 \le k \le n$. The k-th component of $((\mathcal{A}W_s)^{(n+1)})_{s\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, hence also the sequence $(\triangle \partial_k \varphi_s)_{s\in\mathbb{N}}$. We know already that the sequence $(\partial_1^2 \partial_k \varphi_s)_{s\in\mathbb{N}}$ converges in $L^2(\Omega)$ to the limit $\partial_1^2\partial_k\varphi_*$, and therefore also the sequence $(\sum_{l=2}^n \partial_l^2 \partial_k \varphi_s)_{s\in\mathbb{N}}$ has a limit in $L^2(\Omega)$. Via a partial Fourier transform which replaces the variable $x' := (x_2, \dots, x_n)$ with $\xi' := (\xi_2, \dots, \xi_n)$, we conclude that also $(\partial_{x'}^{\alpha'} \partial_k \varphi_s)_{s\in\mathbb{N}}$ is a Cauchy sequence, for all multi–indices $\alpha' = (\alpha_2, \dots, \alpha_n)$ with $|\alpha'| = 2$. Then it follows that

$$\lim_{s\to\infty} \|\partial_x^\alpha (\varphi_s - \varphi_*)\|_{L^2(\Omega)} = 0, \qquad \forall\, \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = 3.$$

This means that $(W_s^{(k)})_{s\in\mathbb{N}}$ converges to $W_*^{(k)}$ not only in the norm of $L^2(\Omega)$, but also in the norm of $H^1(\Omega)$. Now the closedness of $\mathcal{A}$ quickly follows.

The next result is proved by a similar technique as the previous one.

Lemma 3.17. *The adjoint operator $\mathcal{A}^*$ has the same domain, $D(\mathcal{A}) = D(\mathcal{A}^*)$, and it is given by*

$$\mathcal{A}^* = \begin{pmatrix} 0 & \dots & 0 & -a_1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -a_n & 0 \\ -a_1^\top & \dots & -a_n^\top & 0 & -\beta \operatorname{grad} \\ 0 & \dots & 0 & -\beta \operatorname{div} & -\kappa \triangle \end{pmatrix}.$$

For $W \in D(\mathcal{A})$, we have

$$\Re \langle \mathcal{A}W, W \rangle_{L^2(\Omega)} = \kappa \left\| \nabla W^{(n+2)} \right\|_{L^2(\Omega)}^2 .$$

Proposition 3.2. *The operator $-\mathcal{A}$ generates a C_0 semigroup of contractions on the space $\mathcal{H}$. For each $W_0 \in D(\mathcal{A})$, the system* (3.12) *has a unique solution $W \in C([0,\infty), D(\mathcal{A}))$ with $W(0) = W_0$ and $\|W(t)\|_{\mathcal{H}} \le \|W_0\|_{\mathcal{H}}$ for all $t \in (0,\infty)$.*

PROOF. This follows from the above results on $\mathcal{A}$ as well as $\mathcal{A}^*$, and the Lumer–Phillips theorem.

For later use, we study higher regularity of the solution:

Lemma 3.18. *Let $W \in D(\mathcal{A}^m) := \{W \in D(\mathcal{A}) \colon \mathcal{A}W \in D(\mathcal{A}^{m-1})\}$ for an odd integer $m \ge 1$. Then*

- *$W^{(k)} \in H^m(\Omega)$ for $1 \le k \le n+1$ and $W^{(n+2)} \in H^{m+1}(\Omega)$,*
- *if $W = (\partial_1 \nabla\varphi, \dots, \partial_n \nabla\varphi, \nabla\psi, \theta)^\top$, then $\nabla\varphi \in H^{m+1}(\Omega)$ as well as $\nabla\psi \in H^m(\Omega)$, and the following boundary conditions are valid:*

$$\gamma_0 \partial_1^m \varphi = \gamma_0 \partial_1^m \psi = \gamma_0 \partial_1^m \theta = 0.$$

PROOF. By Lemma 3.13, the claim is valid for $m = 1$. Now let the assertion be shown for m, and suppose $W \in D(\mathcal{A}^{m+2})$, where $W = (\partial_1 \nabla\varphi, \dots, \partial_n \nabla\varphi, \nabla\psi, \theta)$. Then we have $\mathcal{A}W \in D(\mathcal{A}^m)$, to which we apply the induction assumption and find $(\mathcal{A}W)^{(k)} \in H^m(\Omega)$ and $(\mathcal{A}W)^{(n+2)} \in H^{m+1}(\Omega)$. In the manner of the proof of Lemma 3.16 we show step by step: $W^{(n+1)} \in H^{m+1}(\Omega)$, $W^{(n+2)} \in H^{m+2}(\Omega)$, $\nabla\varphi \in H^{m+2}(\Omega)$ and $W^{(k)} \in H^{m+1}(\Omega)$ for $1 \le k \le n$.

Moreover, we have

$$\begin{aligned} \mathcal{A}W &= \Big(a_1 W^{(n+1)}, \dots, a_n W^{(n+1)}, \sum_{k=1}^n a_k^\top W^{(k)} + \beta \nabla W^{(n+2)}, \\ &\qquad - \kappa \triangle W^{(n+2)} + \beta \operatorname{div} W^{(n+1)} \Big)^\top \\ &= \left(-\sqrt{\alpha}\partial_1 \nabla\psi, \dots, -\sqrt{\alpha}\partial_n \nabla\psi, \nabla(-\sqrt{\alpha} \triangle \varphi + \beta\theta), -\kappa \triangle \theta + \beta \triangle \psi \right)^\top , \end{aligned}$$

and the induction assumption concerning the boundary values gives

$$\gamma_0\partial_1^m(-\sqrt{\alpha}\triangle\varphi+\beta\theta)=\gamma_0\partial_1^m(-\kappa\triangle\theta+\beta\triangle\psi)=0. \tag{3.21}$$

Because we also have $\gamma_0\partial_1^m\theta=0$, the first desired identity $\gamma_0\partial_1^{m+2}\varphi=0$ is obtained.

In a second step, we consider $\mathcal{A}^2W\in D(\mathcal{A}^m)$. By the same procedure as in the first step, we show $W^{(k)}\in H^{m+2}(\Omega)$ for $1\le k\le n+1$ and $W^{(n+2)}\in H^{m+3}(\Omega)$. And we also have

$$\mathcal{A}^2W=\left(\dots,\dots,\dots,\sum_{k=1}^n a_k^\top(\mathcal{A}W)^{(k)}+\beta\nabla(\mathcal{A}W)^{(n+2)},\dots\right)^\top,$$

$$\sum_{k=1}^n a_k^\top(\mathcal{A}W)^{(k)}+\beta\nabla(\mathcal{A}W)^{(n+2)}=\alpha\nabla\triangle\psi+\beta\nabla(-\kappa\triangle\theta+\beta\triangle\psi),$$

and the induction assumption is

$$\gamma_0\partial_1^m\Big(\alpha\triangle\psi+\beta(-\kappa\triangle\theta+\beta\triangle\psi)\Big)=0,$$

from which we find $\gamma_0\partial_1^m\triangle\psi=0$, by (3.21), hence $\gamma_0\partial_1^{m+2}\psi=0$, and then also $\gamma_0\partial_1^{m+2}\theta=0$. The proof is finished.

To obtain a better feeling of the elements of $D(\mathcal{A}^m)$, we describe a subset:

Lemma 3.19. *For m being an odd integer ≥ 1, we define*

$$Y_m=\Big\{W=(\partial_1\nabla\varphi,\dots,\partial_n\nabla\varphi,\nabla\psi,\theta)^\top:\ (\nabla\varphi,\nabla\psi,\theta)\in H^{2m}(\Omega),$$
$$\gamma_0\partial_1^k(\varphi,\psi,\theta)=0,\quad k=1,3,\dots,2m-1\Big\}.$$

Then $Y_m\subset D(\mathcal{A}^m)$.

For the proof, we only note that $\mathcal{A}^2$ maps Y_m into Y_{m-2}, and $Y_1\subset D(\mathcal{A})$.

Proposition 3.3 (Existence of the potential part). *Suppose that we are given initial data (u^0,u^1,θ^0) with the regularity*

$$u^0\in H^2(\Omega)\cap G_2(\Omega),\qquad Bu^0=0,$$
$$u^1\in H^1(\Omega)\cap G_2(\Omega),\qquad \gamma_0u_1^1=0,$$
$$\theta^0\in H^2(\Omega),\qquad \gamma_0\partial_1\theta^0=0.$$

Then the thermoelastic system (3.4) *with the initial conditions* (1.2) *possesses a unique solution $(u,\theta)\in X$.*

Moreover, for each $m\in\mathbb{N}$ there is a number M such that: if the above introduced initial data (u^0,u^1,θ^0) additionally satisfy $(u^0,u^1,\theta^0)\in H^M(\Omega)$ and

$$B\triangle^k u^0=0,\qquad \gamma_0\triangle^k u_1^1=0,\qquad \gamma_0\partial_1\triangle^k\theta^0=0,\qquad 1\le k\le M/2-1,$$

then the solution (u,θ) has higher regularity in the sense of $(\triangle^k u,\triangle^k\theta)\in X$ for $0\le k\le m$.

PROOF. It suffices to combine Lemma 3.7, Lemma 3.11, Lemma 3.13, Proposition 3.2 and Lemma 3.19.

4. Decay Estimates

In this section, we prove decay properties of the solution u to (2.1) which will be the main part of the proof of Theorem 2.1. First we consider the solenoidal part u^{so}:

Proposition 4.1. *If u^{so} solves* (3.2) *with $u^{\mathrm{so}} \in X_u$, then $E^{\mathrm{so}}(t) = \mathrm{const}$ with*

$$E^{\mathrm{so}}(t) = \frac{1}{2}\left(\|u_t^{\mathrm{so}}\|^2_{L^2(\Omega)} + \mu\,\|\nabla u^{\mathrm{so}}\|^2_{L^2(\Omega)}\right).$$

Moreover, we have the decay estimates

$$\begin{aligned} &\left\|(\nabla u^{\mathrm{so}}_{[0]}, \partial_t u^{\mathrm{so}}_{[0]})(t,\cdot)\right\|_{L^\infty(\Omega)} \\ &\qquad \leq \frac{C}{(1+t)^{(n-2)/2}}\left(\left\|u^{0,\mathrm{so}}_{[0]}\right\|_{W_1^{K+1}(\Omega)} + \left\|u^{1,\mathrm{so}}_{[0]}\right\|_{W_1^{K-1}(\Omega)}\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\left\|(\nabla u^{\mathrm{so}}_{[+]}, \partial_t u^{\mathrm{so}}_{[+]})(t,\cdot)\right\|_{L^\infty(\Omega)} \\ &\qquad \leq \frac{C}{(1+t)^{(n-1)/2}}\left(\left\|u^{0,\mathrm{so}}_{[+]}\right\|_{W_1^{K+1}(\Omega)} + \left\|u^{1,\mathrm{so}}_{[+]}\right\|_{W_1^{K-1}(\Omega)}\right), \end{aligned} \tag{4.2}$$

under the following assumptions on the inital data:

$$\begin{aligned} u_1^{0,\mathrm{so}} &\in D(\triangle_D^{K_{\mathrm{so}}/2}) \cap W_1^{K+1}(\Omega), & u_1^{1,\mathrm{so}} &\in D(\triangle_D^{(K_{\mathrm{so}}-1)/2}) \cap W_1^{K-1}(\Omega), \\ u_k^{0,\mathrm{so}} &\in D(\triangle_N^{K_{\mathrm{so}}/2}) \cap W_1^{K+1}(\Omega), & u_k^{1,\mathrm{so}} &\in D(\triangle_N^{(K_{\mathrm{so}}-1)/2}) \cap W_1^{K-1}(\Omega), \end{aligned}$$

where $2 \leq k \leq n$, $K_{\mathrm{so}} = \lfloor \frac{n}{2} \rfloor + 3$ and $K = 2\lfloor \frac{n}{2} \rfloor + 5$. Here $u_1^{0,\mathrm{so}}(x) = u_1^{\mathrm{so}}(0,x)$ and $u_1^{1,\mathrm{so}}(x) = \partial_t u^{\mathrm{so}}(0,x)$ denote the initial data of the first component of u^{so}, and the initial data $u^{0,\mathrm{so}}$ and $u^{1,\mathrm{so}}$ are split into zero mode parts and higher mode parts as in Definition 3.1.

PROOF. The energy estimate follows from Lemma 3.1, and the estimates (4.1), (4.2) can be found in [10].

Now we handle the potential part: let $(v,\theta) \in X$ with $v := u^{\mathrm{po}}$ be a solution to (3.4) and write

$$v = \begin{pmatrix} v_1 \\ v' \end{pmatrix}, \qquad v' = (v_2, \dots, v_n)^\top,$$

$x' = (x_2, \dots, x_n)$, $\nabla' = (\partial_2, \dots, \partial_n)$, and $\triangle' = \sum_{k=2}^n \partial_k^2$. Accordingly, we also introduce div'.

Zero mode solutions to (2.1) solve thermoelasticity problems in domains of one dimension less, because zero mode functions do not depend on x_1:

Proposition 4.2. *If $(u_{[0]}, \theta_{[0]}) \in X$ solves* (2.1) *and* (1.3) *with $u_{[0]}$ being a potential field, then $E^{\mathrm{po}}_{[0]}(t) = \mathrm{const}$ with*

$$E^{\mathrm{po}}_{[0]}(t) = \frac{1}{2}\left(\left\| \partial_t u^{\mathrm{po}}_{[0]} \right\|^2_{L^2(\Omega)} + \mu \left\| \nabla u^{\mathrm{po}}_{[0]} \right\|^2_{L^2(\Omega)} + (\mu+\lambda) \left\| \operatorname{div} u^{\mathrm{po}}_{[0]} \right\|^2_{L^2(\Omega)} + \left\| \theta_{[0]} \right\|^2_{L^2(\Omega)} \right).$$

Moreover, we have the decay estimates

$$\left\| \partial_x^\alpha (\nabla u_{[0]}, \partial_t u_{[0]}, \theta_{[0]})(t,\cdot) \right\|_{L^\infty(\Omega)} \leq \frac{C}{(1+t)^{(n-1+|\alpha|)/2}} \left\| (\nabla u^0_{[0]}, u^1_{[0]}, \theta^0_{[0]}) \right\|_{W_1^{N+|\alpha|}(\Omega)}, \tag{4.3}$$

where $\alpha \in \mathbb{N}^n$, and $N \geq n$ is an integer, under the following assumption on the initial data:

$$(u^0_{[0]}, u^1_{[0]}, \theta^0_{[0]}) \in W_1^{N+|\alpha|}(\Omega).$$

PROOF. We only have to remark that the inequality (4.3) is just the L^∞–L^1 decay estimate for the potential part of a solution to a thermoelasticity problem in the spatial domain $\mathbb{R}^{n-1}$, compare [2, Lemma 4.15].

Note that the vector potential part of a zero mode vector field remains a zero mode vector field, by Lemma 3.8.

The Propositions 4.1 and 4.2 describe the decay of the solenoidal part and the zero mode of the potential part, and therefore we can now restrict our attention to $(v, \theta) \in X$ as solution to (3.4) with $v = u^{\mathrm{po}}_{[+]}$ and $\theta = \theta_{[+]}$. The decay estimate will be given in Proposition 4.3.

4.1. Fourier decomposition and fundamental solution

Let $\phi_1, \phi_2, \ldots$ denote the eigenfunctions to $-\partial_1^2$ with Dirichlet boundary conditions on $(0,1)$, with eigenvalues $\lambda_1, \lambda_2, \ldots$, and write ψ_0, $\psi_1, \ldots$ for the eigenfunctions to $-\partial_1^2$ with Neumann boundary conditions on $(0,1)$, with the associated eigenvalues $\lambda_0, \lambda_1, \ldots$. We assume the usual normalization:

$$\int_0^1 \phi_j^2(x_1)\,\mathrm{d}x_1 = \int_0^1 \psi_j^2(x_1)\,\mathrm{d}x_1 = 1,$$

and we have the explicit representations $\phi_j(x_1) = \mathrm{const}\sin(j\pi x_1)$ and $\psi_j(x_1) = \mathrm{const}\cos(j\pi x_1)$, as well as $\lambda_j = \pi^2 j^2$.

Recalling the boundary conditions $Bv = 0$, the following Fourier decomposition is natural:

$$v_1(t,x) = \sum_{j=1}^\infty v_{1,j}(t,x')\phi_j(x_1), \qquad v'(t,x) = \sum_{j=1}^\infty v'_j(t,x')\psi_j(x_1),$$

$$\theta(t,x) = \sum_{j=1}^\infty \theta_j(t,x')\psi_j(x_1),$$

and we regard the Fourier coefficients $v_{1,j}\colon [0,\infty)\times\mathbb{R}^{n-1}\to\mathbb{R}$, $v_j'\colon [0,\infty)\times\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$, $\theta_j\colon [0,\infty)\times\mathbb{R}^{n-1}\to\mathbb{R}$ as new unknown functions for which we seek estimates. Note that terms with the function ψ_0 do not appear because of the absence of zero modes.

Making use of $\partial_1\phi_k = \lambda_k^{1/2}\psi_k$, we then derive the systems for $j\ge 1$:

$$\begin{aligned}\partial_t^2 v_{1,j} - \alpha(\triangle' - \lambda_j)v_{1,j} - \beta\lambda_j^{1/2}\theta_j &= 0,\\ \partial_t^2 v_j' - \alpha(\triangle' - \lambda_j)v_j' + \beta\nabla'\theta_j &= 0,\\ \partial_t\theta_j - \kappa(\triangle' - \lambda_j)\theta_j + \beta\partial_t\lambda_j^{1/2}v_{1,j} + \beta\partial_t \operatorname{div}' v_j' &= 0.\end{aligned}$$

Next we perform a Fourier transform, exchanging the variable $x'\in\mathbb{R}^{n-1}$ against $\xi' = (\xi_2,\dots,\xi_n)\in\mathbb{R}^{n-1}$:

$$\hat v_{1,j}(t,\xi') = \int_{\mathbb{R}^{n-1}_{x'}} \exp(-\mathrm{i}x'\cdot\xi')v_{1,j}(t,x')\,\mathrm{d}x',$$

and accordingly for $\hat v_j'$ and $\hat\theta_j$. Setting $\varrho_j = \varrho_j(\xi') := \sqrt{\alpha(|\xi'|^2+\lambda_j)}$, we then find for all $j\ge 1$:

$$\begin{aligned}\partial_t^2\hat v_{1,j} + \varrho_j^2\hat v_{1,j} - \beta\lambda_j^{1/2}\hat\theta_j &= 0,\\ \partial_t^2\hat v_j' + \varrho_j^2\hat v_j' + \beta\mathrm{i}\xi'\hat\theta_j &= 0,\\ \partial_t\hat\theta_j + \frac{\kappa}{\alpha}\varrho_j^2\hat\theta_j + \beta\lambda_j^{1/2}\partial_t\hat v_{1,j} + \beta\partial_t\mathrm{i}\xi'\cdot\hat v_j' &= 0,\end{aligned}$$

where $\xi'\cdot\hat v_j'$ stands for the euclidean bilinear product of vectors with $n-1$ components.

To bring this system into first order form, we set for $j\ge 1$:

$$\begin{aligned}\hat w_{1,j}(t,\xi') &:= \partial_t\hat v_{1,j}(t,\xi') + \mathrm{i}\varrho_j(\xi')\hat v_{1,j}(t,\xi'),\\ \hat w_{2,j}(t,\xi') &:= \partial_t\hat v_j'(t,\xi') + \mathrm{i}\varrho_j(\xi')\hat v_j'(t,\xi'),\\ \hat w_{3,j}(t,\xi') &:= \partial_t\hat v_{1,j}(t,\xi') - \mathrm{i}\varrho_j(\xi')\hat v_{1,j}(t,\xi'),\\ \hat w_{4,j}(t,\xi') &:= \partial_t\hat v_j'(t,\xi') - \mathrm{i}\varrho_j(\xi')\hat v_j'(t,\xi'),\\ \hat w_{5,j}(t,\xi') &:= \sqrt{2}\hat\theta_j(t,\xi')\end{aligned}$$

and $\hat W_j := (\hat w_{1,j},\dots,\hat w_{5,j})^\top$. Then we obtain the system

$$\partial_t\hat W_j + \hat A_j\hat W_j = 0,$$

$$\hat A_j = \begin{pmatrix} -\mathrm{i}\varrho_j & 0 & 0 & 0 & -\beta\lambda_j^{1/2}/\sqrt{2}\\ 0 & -\mathrm{i}\varrho_j & 0 & 0 & \beta\mathrm{i}\xi'/\sqrt{2}\\ 0 & 0 & \mathrm{i}\varrho_j & 0 & -\lambda_j^{1/2}/\sqrt{2}\\ 0 & 0 & 0 & \mathrm{i}\varrho_j & \beta\mathrm{i}\xi'/\sqrt{2}\\ \beta\lambda_j^{1/2}/\sqrt{2} & \beta\mathrm{i}(\xi')^\top/\sqrt{2} & \beta\lambda_j^{1/2}/\sqrt{2} & \beta\mathrm{i}(\xi')^\top/\sqrt{2} & (\kappa/\alpha)\varrho_j^2\end{pmatrix}.$$

In the sequel, we will determine approximately the fundamental solution to this ODE system, and therefore we wish to find the eigenvalues and eigenvectors of $\hat A_j$, modulo some remainder terms. To take a more general approach, we fix a cone $\mathcal{K}$ in a "frequency space" $\mathbb{R}^N$, and introduce symbol classes: a function f mapping from $\mathcal{K}$ into $\mathbb{C}$ (or $\mathbb{C}^k$ or $\mathbb{C}^{p\times q}$) is said to belong to the symbol class S^α if we have the estimate $|f(\eta)| \le C_f|\eta|^\alpha$ for all $\eta \in \mathcal{K}$. And it belongs to the homogeneous symbol class S^α_{hom} if additionally $f(\varrho\eta) = \varrho^\alpha f(\eta)$ for all $\varrho > 0$ and $\eta \in \mathcal{K}$.

In our case, $\mathcal{K} = [0,\infty) \times \mathbb{R}^{n-1}$ with $\eta = (\eta_1, \dots, \eta_n) = (\sqrt{\lambda_j}, \xi')$. We may ignore during our computations that λ_j takes discrete values only and is separated from zero.

Then we write $A(\eta) := \hat{A}_j(\xi')$ and $U(t,\eta) := \hat{W}_j(t,\xi')$. We end up with the system

$$\begin{aligned}
&\partial_t U + AU = 0, \qquad\qquad (4.4)\\
&A(\eta) = A_2(\eta) + A_1(\eta),\\
&A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \in S^2_{\mathrm{hom}}, \qquad a_2(\eta) := \frac{\kappa}{\alpha}\varrho_j^2(\xi'),\\
&A_1 = \begin{pmatrix} -D & 0 & b \\ 0 & D & b \\ -b^* & -b^* & 0 \end{pmatrix} \in S^1_{\mathrm{hom}},
\end{aligned}$$

with D being an imaginary multiple of the $n \times n$ identity matrix, b being a column vector with n complex-valued entries, and $b^* = \overline{(b^\top)}$ the hermitian adjoint. Note that there are positive constants c_1 and c_2 with $c_1|\eta|^2 \le a_2(\eta) \le c_2|\eta|^2$ for all $\eta \in \mathcal{K}$.

To the $(2n+1) \times (2n+1)$ matrix A, $2n-2$ eigenvalues and eigenvectors can be found right away:

Lemma 4.1. *Let $c_1, \dots, c_{n-1}$ be vectors from $\mathbb{C}^n$ with the properties*

$$b^* c_m = 0, \qquad c_m^* c_k = \delta_{mk}, \qquad 1 \le m, k \le n-1.$$

Then the vectors

$$\begin{aligned}
f_k &= (c_k, 0, 0)^\top, \qquad 1 \le k \le n-1,\\
f_{n+k} &= (0, c_k, 0)^\top, \qquad 1 \le k \le n-1,
\end{aligned}$$

are eigenvectors to the matrix A, and the associated eigenvalues are $-\mathrm{i}\varrho_j$ and $\mathrm{i}\varrho_j$, respectively.

The characteristic polynomial of A is

$$\det(A - \lambda I) = (-\mathrm{i}\varrho_j - \lambda)^{n-1}(\mathrm{i}\varrho_j - \lambda)^{n-1}\left((a_2 - \lambda)(\varrho_j^2 + \lambda^2) - 2|b|^2\lambda\right),$$

and this polynomial has appeared many times in thermoelasticity, cp. the works [3] or [7]. Naturally, a detailed understanding of the eigenvalues and eigenvectors of A is indispensable for proving decay properties for the system (4.4). Our formulas (4.5) and (A.1) recover results known from [7] and [3], but it seems that the method presented here is considerably simpler, and we also find the eigenvectors with minimal additional effort.

To stay away from trivialities, we assume $\beta \neq 0$.

Lemma 4.2. *If $\eta \neq 0$, then A can not have purely imaginary eigenvalues except $\pm \mathrm{i}\varrho_j$, both with multiplicity $n-1$.*

PROOF. The last factor in the characteristic polynomial of A does not vanish for $\lambda = \pm \mathrm{i}\varrho_j$. Now suppose $\lambda = \mathrm{i}\sigma$ were an eigenvalue of A with $\mathbb{R} \ni \sigma \neq \pm\varrho_j$, then $(a_2 - \mathrm{i}\sigma)(\varrho_j^2 - \sigma^2) - 2\mathrm{i}|b|^2\sigma = 0$, and taking the real part gives a contradiction.

4.1.1. The case of large $|\eta|$; $|\eta| \geq C_0 \gg 1$

Lemma 4.3. *There is a constant C_0 such that for $|\eta| \geq C_0$, the remaining eigenvalues of A are*

$$\begin{cases} \lambda_n = -\mathrm{i}\varrho_j + \dfrac{|b|^2}{a_2} + \mathcal{O}(|\eta|^{-1}), \\ \lambda_{2n} = \mathrm{i}\varrho_j + \dfrac{|b|^2}{a_2} + \mathcal{O}(|\eta|^{-1}), \\ \lambda_{2n+1} = a_2 - \dfrac{2|b|^2}{a_2} + \mathcal{O}(|\eta|^{-2}), \end{cases} \tag{4.5}$$

with $|b|^2/a_2 = \beta^2/(2\kappa)$, and with the normalized eigenvectors

$$\begin{aligned} f_n &= (b/|b|, 0, 0)^\top + \mathcal{O}(|\eta|^{-1}), \\ f_{2n} &= (0, b/|b|, 0)^\top + \mathcal{O}(|\eta|^{-1}), \\ f_{2n+1} &= (0, 0, 1)^\top + \mathcal{O}(|\eta|^{-1}). \end{aligned}$$

PROOF. We clearly have

$$\lambda_1\lambda_2 \dots \lambda_{2n+1} = \det A = a_2\varrho_j^{2n}, \qquad \lambda_1 + \lambda_2 + \dots + \lambda_{2n+1} = \operatorname{tr} A = a_2.$$

By Lemma 4.1, only the eigenvalues λ_n, λ_{2n} and λ_{2n+1} are not yet known. Then it follows that

$$\lambda_n\lambda_{2n}\lambda_{2n+1} = a_2\varrho_j^2, \qquad \lambda_n + \lambda_{2n} + \lambda_{2n+1} = a_2. \tag{4.6}$$

We apply the Gershgorin principle (see [22]) to the last row of the matrix A and find

$$\lambda_{2n+1} = a_2 + \mathcal{O}(|\eta|). \tag{4.7}$$

This eigenvalue must be a solution to $(a_2 - \lambda)(\varrho_j^2 + \lambda^2) - 2|b|^2\lambda = 0$, which we can rewrite as

$$a_2 - \lambda = \frac{2|b|^2\lambda}{\varrho_j^2 + \lambda^2} = \frac{2|b|^2}{\lambda(1 + \varrho_j^2/\lambda^2)}.$$

Plugging (4.7) into the right-hand side gives the desired expression of λ_{2n+1} from (4.5). By (4.6) we then get

$$\lambda_n\lambda_{2n} = \varrho_j^2 + \mathcal{O}(1), \qquad \lambda_n + \lambda_{2n} = \frac{2|b|^2}{a_2} + \mathcal{O}(|\eta|^{-2}),$$

which has the representations of λ_n and λ_{2n} in (4.5) as direct consequences.

For the eigenvector f_n to the eigenvalue λ_n, we make the ansatz $f_n = (z_n, z_n', z_n'')^\top$ with $|f_n| = 1$. Then we obtain

$$\begin{pmatrix} (|b|^2/a_2 + \mathcal{O}(|\eta|^{-1}))I_n & 0 & b \\ 0 & (2\mathrm{i}\varrho_j + \mathcal{O}(1))I_n & b \\ -b^* & -b^* & a_2 - \lambda_n \end{pmatrix} \begin{pmatrix} z_n \\ z_n' \\ z_n'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and reading this system from the bottom line up we show $z_n'' = \mathcal{O}(|\eta|^{-1})$ and $z_n' = \mathcal{O}(|\eta|^{-1})$. Then necessarily $|z_n| = 1 - \mathcal{O}(|\eta|^{-1})$, and the first line gives $z_n \parallel b$.

The representations of f_{2n} and f_{2n+1} are shown in a similar manner. The proof is complete.

Let $g_1, \dots, g_{2n+1} \in \mathbb{C}^n$ be the eigenvectors to $A^* = A_2 - A_1$ with eigenvalues $\overline{\lambda_m}$, hence $A^* g_m = \overline{\lambda_m} g_m$, where we can arrange that $g_m = f_m$ for $m = 1, \dots, n-1$ and $m = n+1, \dots, 2n-1$. A normalization of the g_m is given by the condition $g_m^* f_m = 1$ for $m = 1, \dots, 2n+1$; and then we have even $g_m^* f_k = \delta_{mk}$. Since the vectors f_k form an asymptotically unitary matrix for $|\eta| \to \infty$, the norms $|g_k|$ are uniformly bounded with respect to η if $|\eta| \geq C_0$.

Lemma 4.4. *Let $(v, \theta) \in X$ be a solution to* (3.4)*, with v being a vector potential field. Then the Fourier coefficients $\hat{W}_j(t, \xi')$ satisfy, for $|(\lambda_j, \xi')| \geq C_0$, the decay estimate*

$$\left|\hat{W}_j(t, \xi')\right| \leq Ce^{-ct} \left|\hat{W}_j(0, \xi')\right|, \qquad 0 \leq t < \infty,$$

where the constants C and c are independent of j, ξ', t.

PROOF. We can write $\hat{W}_j(0, \xi) = \sum_{k=1}^{2n+1} \alpha_k f_k$, and the $\alpha_k \in \mathbb{C}$ can be determined via $\alpha_k = g_k^* \hat{W}_j(0, \xi)$. However, if v is the gradient of a scalar function that satisfies homogeneous Neumann boundary conditions, then $\alpha_k = 0$ for $k = 1, \dots, n-1$ and $k = n+1, \dots, 2n-1$.

4.1.2. The case of intermediate $|\eta|$; $\varepsilon \leq |\eta| \leq C_0$

Keep C_0 fixed as in the previous part, and choose an arbitrary ε between 0 and 1.

Lemma 4.5. *Let $(v, \theta) \in X$ be a solution to* (3.4)*, with v being a vector potential field. Then the Fourier coefficients $\hat{W}_j(t, \xi')$ satisfy, for $\varepsilon \leq |(\sqrt{\lambda_j}, \xi')| \leq C_0$, the decay estimate*

$$\left|\hat{W}_j(t, \xi')\right| \leq Ce^{-ct} \left|\hat{W}_j(0, \xi')\right|, \qquad 0 \leq t < \infty,$$

where the constants C and c depend only on ε and C_0.

PROOF. We start with describing the eigenvalues λ_n, λ_{2n}, λ_{2n+1} (the other eigenvalues do not participate in the representation of the solution since v is a vector potential field). If $|\eta| = C_0$, then they have positive real part, and by Lemma 4.2, these eigenvalues are never on the imaginary axis. Then a compactness argument gives us a positive number c such that $\Re\lambda_k(\eta) > c$ for $k = n, 2n, 2n+1$, when $\varepsilon \leq |\eta| \leq C_0$. It may happen that two such eigenvalues coincide and Jordan blocks appear during the construction of the fundamental solution; this situation can be resolved with the same technique as in [6].

4.2. Reconstruction

In this section, we show how the reconstruction of the zero mode free vector v from its Fourier coefficients leads to decay estimates of v, coming from the pointwise estimates of the Fourier coefficients $|\hat{W}_j(t,\xi')|$.

Proposition 4.3. *Let $(u^{\rm po}_{[+]},\theta_{[+]})\in X$ be a solution to* (3.4)*, with $u^{\rm po}$ being a potential field. Then $E^{\rm po}_{[+]}(t)=\mathrm{const}$ with*

$$E^{\rm po}_{[+]}(t)=\frac{1}{2}\left(\left\|\partial_t u^{\rm po}_{[+]}\right\|^2_{L^2(\Omega)}+\mu\left\|\nabla u^{\rm po}_{[+]}\right\|^2_{L^2(\Omega)}\right.$$
$$\left.+(\mu+\lambda)\left\|\operatorname{div} u^{\rm po}_{[+]}\right\|^2_{L^2(\Omega)}+\left\|\theta_{[+]}\right\|^2_{L^2(\Omega)}\right).$$

Set $v:=u^{\rm po}_{[+]}$ and $\theta:=\theta_{[+]}$ for sake of brevity. Choose a non-negative integer m, and set $K_{\rm po}=2m+\lfloor\frac{n-1}{2}\rfloor+3$. Then under the assumptions

$$(\triangle^k v,\triangle^k\theta)\in X,\qquad 0\le k\le m\in\mathbb{N},\tag{4.8}$$

$$\begin{cases} v_1^0\in D(\triangle_D^{K_{\rm po}/2}), & v_1^1\in D(\triangle_D^{(K_{\rm po}-1)/2}),\\ v_k^0\in D(\triangle_N^{K_{\rm po}/2}), & v_k^1\in D(\triangle_N^{(K_{\rm po}-1)/2})\qquad (2\le k\le n),\\ \theta^0\in D(\triangle_N^{K_{\rm po}/2}), & \end{cases}\tag{4.9}$$

the following decay estimates hold:

$$\|\partial_x^\alpha(\nabla v,\partial_t v,\theta)(t,\cdot)\|_{L^\infty(\Omega)}\le Ce^{-ct}\left\|(v^0,v^1,\theta^0)\right\|_{H^{K_{\rm po}}(\Omega)\times H^{K_{\rm po}-1}(\Omega)\times H^{K_{\rm po}}(\Omega)},\tag{4.10}$$

where $\alpha\in\mathbb{N}^n$ with $|\alpha|\le 2m$.

PROOF. The point-wise estimates obtained so far can be combined into

$$\left|\hat{W}_j(t,\xi')\right|\le Ce^{-ct}\left|\hat{W}_j(0,\xi')\right|,\qquad j\ge 1,\quad \xi'\in\mathbb{R}^{n-1}.$$

Then we deduce that

$$\left\|\partial_{x'}^{\alpha'}W_j(t,\cdot)\right\|_{H^s(\mathbb{R}^{n-1})}\le Ce^{-ct}\left\|\partial_{x'}^{\alpha'}W_j(0,\cdot)\right\|_{H^s(\mathbb{R}^{n-1})},\tag{4.11}$$

where $j\ge 1$, $s\in\mathbb{R}$, $W_j=(w_{1,j},\dots,w_{5,j})^\top$ and

$$\begin{aligned}
w_{1,j}(t,x')&=(\partial_t+\mathrm{i}\sqrt{\alpha(\lambda_j-\triangle')})v_{1,j}(t,x'),\\
w_{2,j}(t,x')&=(\partial_t+\mathrm{i}\sqrt{\alpha(\lambda_j-\triangle')})v'_j(t,x'),\\
w_{3,j}(t,x')&=(\partial_t-\mathrm{i}\sqrt{\alpha(\lambda_j-\triangle')})v_{1,j}(t,x'),\\
w_{4,j}(t,x')&=(\partial_t-\mathrm{i}\sqrt{\alpha(\lambda_j-\triangle')})v'_j(t,x'),\\
w_{5,j}(t,x')&=\sqrt{2}\theta_j(t,x').
\end{aligned}$$

We continue with the norm equivalences

$$\left\|\partial_{x'}^{\alpha'} W_j\right\|_{H^s(\mathbb{R}^{n-1})} \sim \left\|\partial_{x'}^{\alpha'}(\partial_t v_{1,j}, \partial_t v_j', \nabla' v_{1,j}, \nabla' v_j', j v_{1,j}', j v_j', \theta_j)\right\|_{H^s(\mathbb{R}^{n-1})}, \tag{4.12}$$

for $s \in \mathbb{R}$, which follow from Fourier transform in $\mathbb{R}^{n-1}$ and Plancherel.

For $\alpha = (\alpha_1, \alpha') \in \mathbb{N}^n$ with $\alpha_1 + |\alpha'| \leq 2m+1$, the components of the vector valued function $x_1 \mapsto \partial_1^{\alpha_1}\partial_{x'}^{\alpha'} v(t, x_1, x')$ either fulfill homogeneous Dirichlet boundary conditions, or we have

$$\int_0^1 \partial_1^{\alpha_1}\partial_{x'}^{\alpha'} v(t, x_1, x')\, \mathrm{d}x_1 = 0, \qquad \text{a.e. } x' \in \mathbb{R}^{n-1}.$$

Here we have used the absence of zero modes and (4.8). In both cases we can bring Poincare's inequality on the bounded domain $(0,1) \subset \mathbb{R}^1$ into play:

$$\begin{aligned}
\left|\partial_1^{\alpha_1}\partial_{x'}^{\alpha'} v(t, x_1, x')\right|^2 &\leq C \left\|\partial_1^{\alpha_1}\partial_{x'}^{\alpha'} v(t, \cdot, x')\right\|^2_{W_2^1((0,1))} \\
&\leq C \left\|\partial_1^{\alpha_1+1}\partial_{x'}^{\alpha'} v(t, \cdot, x')\right\|^2_{L^2((0,1))} \\
&= \sum_{j=1}^{\infty} j^{2(\alpha_1+1)} \left(\left|\partial_{x'}^{\alpha'} v_{1,j}(t, x')\right|^2 + \left|\partial_{x'}^{\alpha'} v_j'(t, x')\right|^2\right).
\end{aligned}$$

In the last step, we have exploited (4.8) once again. Similar estimates can be derived for

$$\left|\partial_1^{\alpha_1}\partial_{x'}^{\alpha'}\partial_t v(t, x_1, x')\right| \quad (|\alpha| \leq 2m), \qquad \left|\partial_1^{\alpha_1}\partial_{x'}^{\alpha'}\theta(t, x_1, x')\right| \quad (|\alpha| \leq 2m).$$

Now choose a real number b with $(n-1)/2 < b \leq n/2$. Then with the notation $\langle\xi'\rangle := (1 + |\xi'|^2)^{1/2}$ and by Sobolev's embedding $H^b(\mathbb{R}^{n-1}) \subset L^\infty(\mathbb{R}^{n-1})$ and (4.11),

$$\begin{aligned}
j^2 \left|\partial_{x'}^{\alpha'} v_{1,j}(t, x')\right|^2 &\leq C j^2 \left\|\partial_{x'}^{\alpha'} v_{1,j}(t, \cdot)\right\|^2_{H^b(\mathbb{R}^{n-1})} \\
&= C j^2 \int_{\mathbb{R}^{n-1}_{\xi'}} \langle\xi'\rangle^{2b} \left|(\xi')^{\alpha'} \hat{v}_{1,j}(t, \xi')\right|^2 \mathrm{d}\xi' \\
&\leq C \int_{\mathbb{R}^{n-1}_{\xi'}} \langle\xi'\rangle^{2b} \left|(\xi')^{\alpha'} \hat{W}_j(t, \xi')\right|^2 \mathrm{d}\xi' \\
&\leq C e^{-2ct} \int_{\mathbb{R}^{n-1}_{\xi'}} \langle\xi'\rangle^{2b} \left|(\xi')^{\alpha'} \hat{W}_j(0, \xi')\right|^2 \mathrm{d}\xi' \\
&= C e^{-2ct} \left\|\partial_{x'}^{\alpha'} W_j(0, \cdot)\right\|^2_{H^b(\mathbb{R}^{n-1})},
\end{aligned}$$

and consequently, by (4.12),

$$\left|\partial_1^{\alpha_1}\partial_{x'}^{\alpha'} v(t,x_1,x')\right|^2 \le Ce^{-2ct}\sum_{j=1}^{\infty} j^{2\alpha_1}\left\|\partial_{x'}^{\alpha'} W_j(0,\cdot)\right\|^2_{H^b(\mathbb{R}^{n-1})} \tag{4.13}$$

$$\le Ce^{-2ct}\sum_{j=1}^{\infty}\left(j^{2\alpha_1}\left\|v^1_{1,j}\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})} + j^{2\alpha_1}\left\|v^{1\,\prime}_j\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})}\right)$$

$$+ Ce^{-2ct}\sum_{j=1}^{\infty}\left(j^{2\alpha_1}\left\|\nabla' v^0_{1,j}\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})} + j^{2\alpha_1}\left\|\nabla' v^{0\,\prime}_j\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})}\right)$$

$$+ Ce^{-2ct}\sum_{j=1}^{\infty}\left(j^{2(\alpha_1+1)}\left\|v^0_{1,j}\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})} + j^{2(\alpha_1+1)}\left\|v^{0\,\prime}_j\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})}\right)$$

$$+ Ce^{-2ct}\sum_{j=1}^{\infty} j^{2\alpha_1}\left\|\theta^0_j\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})},$$

where $v^0_{1,j}$, $v^{0\,\prime}_j$ are the Fourier coefficients of the initial function v^0, and similarly for $v^1_{1,j}$, $v^{1\,\prime}_j$ and θ^0_j.

By the assumption (4.9), we have for $\alpha_1 + |\alpha'| \le 2m+1$

$$\sum_{|\beta'|\le b+|\alpha'|}\left\|\partial_1^{\alpha_1}\partial_{x'}^{\beta'} v^1_1\right\|^2_{L^2(\Omega)} = \sum_{|\beta'|\le b+|\alpha'|}\sum_{j=1}^{\infty} j^{2\alpha_1}\left\|\partial_{x'}^{\beta'} v^1_{1,j}\right\|^2_{L^2(\mathbb{R}^{n-1})}$$

$$= \sum_{j=1}^{\infty} j^{2\alpha_1}\left\|v^1_{1,j}\right\|^2_{H^{b+|\alpha'|}(\mathbb{R}^{n-1})}.$$

The other terms in the right-hand side of (4.13) can be treated similarly, and then (4.10) follows, finishing the proof of Proposition 4.3.

Now we come to the proof of the first main result.

PROOF (PROOF OF THEOREM 2.1). First we split the vector field u into solenoidal part u^{so} and potential part u^{po}, and then we split the potential part u^{po} into zero mode part $u^{\mathrm{po}}_{[0]}$ and higher mode part $u^{\mathrm{po}}_{[+]}$. Similarly we write $\theta = \theta_{[0]} + \theta_{[+]}$. These splittings have been justified in the Lemmas 3.5, 3.6, 3.7, 3.8, 3.9. By Lemma 3.4, the many boundary conditions on the initial data u^0 and u^1, as described in Theorem 2.1, survive the Helmholtz projection. Then Proposition 3.1 and Proposition 3.3 have shown that these three parts exist with the desired regularity. The uniqueness was proved in Lemma 3.1. The decay under the assumption of higher regularity of the initial data is then proved in the Propositions 4.1, 4.2 and 4.3.

5. Proof of Theorem 2.2

PROOF. We begin by considering the zero mode part $(u_{[0]}, \theta_{[0]})$ of the solution $(u,\theta) \in X$ to (2.2) separately. This pair $(u_{[0]}, \theta_{[0]})$ does not depend on x_1, and it solves the system

$$\partial_t^2 u_{[0]} - \mu \triangle u_{[0]} - (\mu+\lambda)\operatorname{grad}\operatorname{div} u_{[0]} + \beta \operatorname{grad} u_{[0]} = f_{[0]},$$
$$\partial_t \theta_{[0]} - \kappa \triangle \theta_{[0]} + \beta \operatorname{div} \partial_t u_{[0]} = g_{[0]},$$

with initial data $(u^0_{[0]}, u^1_{[0]}, \theta^0_{[0]})$. This can be read as a thermoelastic system in the spatial domain $\mathbb{R}^{n-1}$, and we can quote the following decay estimate

$$\begin{aligned}
&\left\|(\nabla u_{[0]}, \partial_t u_{[0]}, \theta_{[0]})(t,\cdot)\right\|_{L^q(\Omega)} \qquad (5.1)\\
&\leq \frac{C}{(1+t)^{(n-2)(1/p-1/q)/2}} \left\|(\nabla u_{[0]}, u_{[0]}, \partial_t u_{[0]}, \theta_{[0]})(0,\cdot)\right\|_{W^N_p(\Omega)}\\
&\quad + C \int_{s=0}^t \frac{1}{(1+t-s)^{(n-2)(1/p-1/q)/2}} \left\|(f_{[0]}, g_{[0]})(s,\cdot)\right\|_{W^N_p(\Omega)} \,\mathrm{d}s,
\end{aligned}$$

from [2]. Here $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that the mappings $f \mapsto f_{[0]}$ and $g \mapsto g_{[0]}$ are continuous in the norms of $W^N_1(\Omega)$.

Next we study the higher mode part $(u_{[+]}, \theta_{[+]})$. We split $u_{[+]}$ further into the solenoidal part $u^{\mathrm{so}}_{[+]}$ and the potential part $u^{\mathrm{po}}_{[+]}$. Then $u^{\mathrm{so}}_{[+]}$ solves the decoupled system of wave equations

$$\partial_t^2 u^{\mathrm{so}}_{[+]} - \mu \triangle u^{\mathrm{so}}_{[+]} = f^{\mathrm{so}}_{[+]}$$

with homogeneous Dirichlet boundary conditions for the first component $u^{\mathrm{so}}_{[+],1}$, and homogeneous Neumann boundary conditions for the other components $u^{\mathrm{so}}_{[+],k}$, $k \geq 2$. However, the Neumann Laplacian on Ω has a spectrum separated from zero, since all functions here are free from zero modes. The initial data are called $u^{0,\mathrm{so}}_{[+]}$ and $u^{1,\mathrm{so}}_{[+]}$. Then we can quote the following decay estimate from [10]:

$$\begin{aligned}
&\left\|(\nabla u^{\mathrm{so}}_{[+]}, \partial_t u^{\mathrm{so}}_{[+]})(t,\cdot)\right\|_{L^q(\Omega)} \qquad (5.2)\\
&\leq \frac{C}{(1+t)^{(n-1)(1/p-1/q)/2}} \Bigg(\left\|(\nabla u^{0,\mathrm{so}}_{[+]}, u^{1,\mathrm{so}}_{[+]}, u^{0,\mathrm{so}}_{[+]})\right\|_{W^{2K+1}_p(\Omega)}\\
&\qquad + \sum_{j=0}^{2K-1} \left\|\partial_t^j f^{\mathrm{so}}_{[+]}(0,\cdot)\right\|_{W^{2K-j}_p(\Omega)} \Bigg)\\
&\quad + \int_{s=0}^t \frac{C}{(1+t-s)^{(n-1)(1/p-1/q)/2}} \left\|\partial_s^{2K} f^{\mathrm{so}}_{[+]}(s,\cdot)\right\|_{L^p(\Omega)} \,\mathrm{d}s\\
&\quad + C \sum_{j=0}^{2K-1} \left\|\partial_t^j f^{\mathrm{so}}_{[+]}(t,\cdot)\right\|_{W^{2K-1-j}_p(\Omega)},
\end{aligned}$$

where now $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. Here we have to suppose that

$$f^{\mathrm{so}}_{[+]} \in \bigcap_{j=0}^{2K} C^j\left([0,\infty), H^{2K-j}(\Omega) \cap W^{2K-j}_p(\Omega)\right) \bigcap C^{2K+1}([0,\infty), L^2(\Omega)), \quad (5.3)$$

with $K \geq \lfloor \frac{n}{2} \rfloor + \frac{1}{2}$. Additionally, we have to assume:

Compatibility Condition 5.1. *The formally computed higher order derivatives of* $u^{\mathrm{so}}_{[+]}$ *at* $t=0$, $(\partial_t^j u^{\mathrm{so}}_{[+]})(0,\cdot) = u^{j,\mathrm{so}}_{[+]}(\cdot)$ *with*

$$u^{j,\mathrm{so}}_{[+]} := \begin{cases} \sum_{k=0}^{\frac{j}{2}-1} (\mu\,\triangle)^k \partial_t^{j-2-2k} f^{\mathrm{so}}_{[+]}(0,\cdot) + (\mu\,\triangle)^{\frac{j}{2}} u^{0,\mathrm{so}}_{[+]} & : j \text{ even}, \\ \sum_{k=0}^{\frac{j-1}{2}-1} (\mu\,\triangle)^k \partial_t^{j-2-2k} f^{\mathrm{so}}_{[+]}(0,\cdot) + (\mu\,\triangle)^{\frac{j-1}{2}} u^{1,\mathrm{so}}_{[+]} & : j \text{ odd}, \end{cases}$$

satisfy the compatibility conditions of order $2K$*:*

$$u^{j,\mathrm{so}}_{[+]} \in \begin{cases} H^{2K+2-j}(\Omega) \cap W_1^{2K+2-j}(\Omega) & : j = 0, \dots, 2K+1, \\ L^2(\Omega) & : j = 2K+2. \end{cases}$$

Here $K \geq \lfloor \frac{n}{2} \rfloor + \frac{1}{2}$*. Additionally, the first component* $u^{j,\mathrm{so}}_{[+],1}$ *satisfies homogeneous Dirichlet boundary conditions, and the other components* $u^{j,\mathrm{so}}_{[+],k}$ *satisfy homogeneous Neumann boundary conditions.*

Note that, by Lemma 3.2, the Helmholtz projector maps Sobolev spaces $W_r^k(\Omega)$ continuously into themselves, but only for $1 < r < \infty$. Therefore we cannot expect (5.2) to hold for $q = \infty$.

It remains to study the potential part $u^{\mathrm{po}}_{[+]}$ of $u_{[+]}$. We construct a vector W from $u^{\mathrm{po}}_{[+]}$ and $\theta_{[+]}$ as in (3.9), (3.10), (3.11), and obtain the system

$$\partial_t W + \mathcal{A}_+ W = F_+ \tag{5.4}$$

with $\mathcal{A}_+$ as in (3.13) with domain $D(\mathcal{A}_+) = D(\mathcal{A}) \cap \mathcal{H}_+$, compare Definition 3.2, (3.16) and Definition 3.3. And we have

$$F_+ = \left(0, \dots, 0, \frac{1}{\sqrt{\alpha}} f^{\mathrm{po}}_{[+]}, \frac{1}{\sqrt{\alpha}} g_{[+]}\right)^\top \in \mathcal{H}_+.$$

Under the assumptions $F_+ \in C^1([0,\infty), \mathcal{H}_+)$ and $W^0 := W(t=0) \in D(A_+)$, we then have a unique solution $W \in C^1([0,\infty), \mathcal{H}_+) \cap C([0,\infty), D(\mathcal{A}_+))$. Then the higher order derivatives of W at $t=0$ are formally given by

$$\begin{aligned} (\partial_t W)(t=0) = W^1 &:= F_+^0(0) - \mathcal{A}_+ W^0 := F_+(0,\cdot) - \mathcal{A}_+ W^0, \\ (\partial_t^2 W)(t=0) = W^2 &:= F_+^1(0) - \mathcal{A}_+ F_+^0(0) - \mathcal{A}_+^2 W^0 \\ &:= (\partial_t F_+)(0,\cdot) - \mathcal{A}_+ F_+(0,\cdot) - \mathcal{A}_+^2 W^0, \\ &\dots \\ (\partial_t^L W)(t=0) = W^L &:= F_+^{L-1}(0) - \sum_{l=1}^{L-1} \mathcal{A}_+^{L-l} F_+^{l-1}(0) - \mathcal{A}_+^L W^0 \\ &:= (\partial_t^{L-1} F_+)(0,\cdot) - \sum_{l=1}^{L-1} \mathcal{A}_+^{L-l} (\partial_t^{l-1} F_+)(0,\cdot) - \mathcal{A}_+^L W^0. \end{aligned}$$

The following condition will become useful soon:

Compatibility Condition 5.2. *For $l = 1, \dots, L$, the above defined term W^l is a member of $D(\mathcal{A}_+)$. Here L is chosen in such a way that $L+1$ is the smallest odd integer greater than or equal to $K_{\mathrm{po}} = \lfloor \frac{n-1}{2} \rfloor + 3$.*

Assuming $F \in C^{L+1}([0,\infty), \mathcal{H}_+)$ we then find $W \in C^{L+1}([0,\infty), \mathcal{H}_+)$ by standard semigroup theory, and we can introduce $\tilde{W} := \mathcal{A}_+^{-L} \partial_t^L W$, where we use that $\mathcal{A}_+$ is continuously invertible on $\mathcal{H}_+$ by (3.20). This function $\tilde{W}$ then solves

$$\partial_t \tilde{W} + \mathcal{A}_+ \tilde{W} = \tilde{F} := \mathcal{A}_+^{-L} \partial_t^L F, \qquad \tilde{W}(0,\cdot) = \mathcal{A}_+^{-L} W^L(\cdot). \tag{5.5}$$

More precisely, we have:

Lemma 5.1. *If the Compatibility condition 5.2 is valid and $F_+ \in C^{L+1}([0,\infty), \mathcal{H}_+)$, and if $\tilde{W} \in C^1([0,\infty), D(\mathcal{A}_+^L)) \cap C([0,\infty), D(\mathcal{A}_+^{L+1}))$ solves* (5.5)*, then*

$$W(t,x) := \mathcal{A}_+^{-L} F_+^{L-1}(t,x) - \sum_{l=1}^{L-1} \mathcal{A}_+^{-l} F_+^{l-1}(t,x) - \tilde{W}(t,x)$$

is a solution to (5.4) *with initial data W^0. Here $F_+^M := \partial_t^M F_+$.*

The proof is straightforward.

The key advantage of this approach is that decay estimates for $\tilde{W}$ can be obtained via Proposition 4.3 and Duhamel's principle because the right–hand side $\tilde{F}$ satisfies a large number of boundary conditions, compare Lemma 3.18.

Then $\hat{W}(0,\cdot)$ from (5.5) belongs to $D(\mathcal{A}_+^{L+1})$, and Proposition 4.3 in connection with Lemma 3.18 give us

$$\left\| \tilde{W}(t,\cdot) \right\|_{L^q(\Omega)} \le C e^{-ct} \left\| \tilde{W}(0,\cdot) \right\|_{H^{K_{\mathrm{po}}}(\Omega)} + C \int_{s=0}^t e^{-c(t-s)} \left\| \tilde{F}(s,\cdot) \right\|_{H^{K_{\mathrm{po}}}(\Omega)} \mathrm{d}s,$$

where $K_{\mathrm{po}} = \lfloor \frac{n-1}{2} \rfloor + 3$ is as in Proposition 4.3.

Going back to the function W, we obtain, by the continuity of the embeddings $W_p^{K_{\mathrm{po}}+n/2}(\Omega) \subset H^{K_{\mathrm{po}}}(\Omega)$ and $W_p^n(\Omega) \subset L^q(\Omega)$,

$$\begin{aligned}
\|W(t,\cdot)\|_{L^q(\Omega)} &\le \sum_{l=0}^{L-1} \left\| \mathcal{A}_+^{l-L} \partial_t^{L-1-l} F_+(t,\cdot) \right\|_{W_p^n(\Omega)} \\
&\quad + C e^{-ct} \left(\|W(0,\cdot)\|_{W_p^{K_{\mathrm{po}}+n/2}(\Omega)} + \sum_{l=0}^{L-1} \left\| \mathcal{A}_+^{l-L} \partial_t^{L-1-l} F_+(0,\cdot) \right\|_{W_p^{K_{\mathrm{po}}+n/2}(\Omega)} \right) \\
&\quad + C \int_{s=0}^t e^{-c(t-s)} \left\| \mathcal{A}_+^{-L} \partial_t^L F_+(s,\cdot) \right\|_{W_p^{K_{\mathrm{po}}+n/2}(\Omega)}.
\end{aligned}$$

By a method very similar to the proof of (3.20), we can show that

$$\left\| \mathcal{A}_+^{-m} G_+ \right\|_{W_r^k(\Omega)} \le C \left\| G_+ \right\|_{W_r^{k-m}(\Omega)}, \qquad 0 \le m \le k,$$

for $G_+ \in \mathcal{H}_+ \cap W_r^{k-m}(\Omega)$ and $1 < r < \infty$. Then it follows that

$$
\begin{aligned}
\|W(t,\cdot)\|_{L^q(\Omega)} &\leq C \sum_{l=1}^{L} \left\| \partial_t^{l-1} F_+(t,\cdot) \right\|_{W_p^{\max(n-l,0)}(\Omega)} \\
&+ Ce^{-ct} \left(\|W(0,\cdot)\|_{W_p^{K_{\mathrm{po}}+n/2}(\Omega)} + \sum_{l=1}^{L} \left\| \partial_t^{l-1} F_+(0,\cdot) \right\|_{W_p^{K_{\mathrm{po}}+n/2-l}(\Omega)} \right) \\
&+ C \int_{s=0}^{t} e^{-c(t-s)} \left\| \partial_t^{L} F_+(s,\cdot) \right\|_{W_p^{K_{\mathrm{po}}+n/2-L}(\Omega)} \, \mathrm{d}s.
\end{aligned} \tag{5.6}
$$

Then the proof of the decay estimate (2.3) can be concluded by addition of (5.1), (5.2) and (5.6). This completes the proof of Theorem 2.2.

A. Appendix

The estimate (4.3) follows from an L^∞–L^1 estimate of vector potential solutions to thermoelastic systems in a whole space, and such an estimate can be derived from pointwise estimates of solutions to (4.4) via Fourier transform. This in turn requires knowledge about the eigenvalues and eigenvectors to the matrix A from (4.4). By the Lemmas 4.1 and 4.3 and the proof of Lemma 4.5, only the case of $|\eta| \leq \varepsilon$ is not yet covered in this paper. For reasons of self-containedness, we close this gap now, and the proof we give here seems to have the advantage of being considerably shorter and less technical than other approaches.

Lemma A.1. *There is a positive constant ε such that the eigenvalues λ_n, λ_{2n} and λ_{2n+1} of the matrix A are*

$$
\begin{cases}
\lambda_n = -\mathrm{i}\sqrt{\alpha+\beta^2}|\eta| + \dfrac{\kappa\beta^2}{2(\alpha+\beta^2)}|\eta|^2 + \mathcal{O}(|\eta|^3), \\
\lambda_{2n} = \mathrm{i}\sqrt{\alpha+\beta^2}|\eta| + \dfrac{\kappa\beta^2}{2(\alpha+\beta^2)}|\eta|^2 + \mathcal{O}(|\eta|^3), \\
\lambda_{2n+1} = \dfrac{\kappa\alpha}{\alpha+\beta^2}|\eta|^2 + \mathcal{O}(|\eta|^4),
\end{cases} \tag{A.1}
$$

for $0 < |\eta| \leq \varepsilon$. The normalized eigenvectors f_n, f_{2n}, f_{2n+1} have the form

$$
(f_n, f_{2n}, f_{2n+1}) = (f_n^{(0)}, f_{2n}^{(0)}, f_{2n+1}^{(0)}) + \mathcal{O}(|\eta|),
$$

the vectors $f_k^{(0)}$ chosen such that $(f_1, \dots, f_{n-1}, f_n^{(0)}, f_{n+1}, \dots, f_{2n-1}, f_{2n}^{(0)}, f_{2n+1}^{(0)})$ is an orthornormal family of eigenvectors of the anti-self-adjoint matrix A_1.

Proof. As in the proof of Lemma 4.3, we start with determining the missing eigenvalues λ_n, λ_{2n} and λ_{2n+1}. To this end, we write

$$
\tilde{A} = |\eta|^{-1} A(\eta) = \begin{pmatrix} -\tilde{D} & 0 & \tilde{b} \\ 0 & \tilde{D} & \tilde{b} \\ -\tilde{b}^* & -\tilde{b}^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_2/|\eta| \end{pmatrix} = \tilde{A}_0 + \tilde{A}_1,
$$

with $\tilde{A}_0 \in S^0_{\rm hom}$ and $\tilde{A}_1 \in S^1_{\rm hom}$. If η approaches 0 radially, then $\tilde{A}_0 \equiv \mathrm{const}$ with the eigenvalues

$$\begin{aligned}
&\tilde{\lambda}_1 = \ldots = \tilde{\lambda}_{n-1} = -\mathrm{i}\varrho_j/|\eta|, \qquad \tilde{\lambda}_{n+1} = \ldots = \tilde{\lambda}_{2n-1} = \mathrm{i}\varrho_j/|\eta|, \\
&\tilde{\lambda}_n = -\mathrm{i}\sqrt{\varrho_j^2 + 2|b|^2}/|\eta|, \qquad \tilde{\lambda}_{2n} = \mathrm{i}\sqrt{\varrho_j^2 + 2|b|^2}/|\eta|, \qquad \tilde{\lambda}_{2n+1} = 0,
\end{aligned}$$

which all belong to $S^0_{\rm hom}$. It is well–known that *simple* eigenvalues depend analytically on the perturbation of the matrix, which is in our case $a_2/|\eta|$. This gives us the asymptotic expansions

$$\begin{aligned}
\lambda_n &= -\mathrm{i}\sqrt{\varrho_j^2 + 2|b|^2} + c_n|\eta|^2 + \mathcal{O}(|\eta|^3), \\
\lambda_{2n} &= \mathrm{i}\sqrt{\varrho_j^2 + 2|b|^2} + c_{2n}|\eta|^2 + \mathcal{O}(|\eta|^3), \\
\lambda_{2n+1} &= c_{2n+1}|\eta|^2 + \mathcal{O}(|\eta|^3),
\end{aligned}$$

with coefficients c_n, c_{2n}, c_{2n+1} not yet known. Due to $\det(A - \lambda I) = 0$, these three λ_k solve

$$(a_2 - \lambda)(\varrho_j^2 + \lambda^2) - 2|b|^2\lambda = 0, \tag{A.2}$$

with $a_2 = \kappa|\eta|^2$, $\varrho_j^2 = \alpha|\eta|^2$, $|b|^2 = \frac{1}{2}\beta^2|\eta|^2$. Bringing (A.2) into the form

$$\lambda = \frac{a_2(\varrho_j^2 + \lambda^2) - \lambda^3}{\varrho_j^2 + 2|b|^2},$$

we find the expression for λ_{2n+1} in (A.1). Then we deduce that

$$\begin{aligned}
\lambda_n\lambda_{2n} &= \varrho_j^2\frac{a_2}{\lambda_{2n+1}} = (\alpha + \beta^2)|\eta|^2 + \mathcal{O}(|\eta|^4), \\
\lambda_n + \lambda_{2n} &= a_2 - \lambda_{2n+1} = \frac{\kappa\beta^2}{\alpha + \beta^2}|\eta|^2 + \mathcal{O}(|\eta|^4),
\end{aligned}$$

which yields the formulas for λ_n and λ_{2n} in (A.1).

For the proof of the statement concerning the eigenvectors, we only note that the normalized eigenvectors to simple eigenvalues depend smoothly on the perturbation of the matrix.

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