Edge Sobolev Spaces, Weakly Hyperbolic Equations, and Branching of Singularities

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Summary

Edge Sobolev spaces are proposed as a main new tool for the investigation of weakly hyperbolic equations. The well–posedness of the linear and the semilinear Cauchy problem in the class of such edge Sobolev spaces is proved. Applications to the propagation of singularities for solutions to semilinear problems are considered.

1 Introduction

We consider the two semilinear Cauchy problems

$$Lu = f(u), \qquad (\partial_t^j u)(0,x) = u_j(x), \quad j = 0,1, \tag{1.1}$$

$$Lu = f(u, \partial_t u, t^{l_*}\nabla_x u), \qquad (\partial_t^j u)(0,x) = u_j(x), \quad j = 0,1, \tag{1.2}$$

where L is the weakly hyperbolic operator

$$L = \partial_t^2 + 2\sum_{j=1}^n \lambda(t)c_j(t)\partial_t\partial_{x_j} - \sum_{i,j=1}^n \lambda(t)^2 a_{ij}(t)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^n \lambda'(t)b_j(t)\partial_{x_j} + c_0(t)\partial_t \tag{1.3}$$

with coefficients a_{ij}, b_j, c_j belonging to $C^\infty([-T_0,T_0],\mathbb{R})$ and $\lambda(t) = t^{l_*}$ with some $l_* \in \mathbb{N}_+ = \{1,2,3,\dots\}$.

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The variables t and x satisfy $(t,x) \in [0,T_0] \times \mathbb{R}^n$; in the end of this paper we will also consider the case $(t,x) \in [-T_0,T_0] \times \mathbb{R}^n$. The operator L is supposed to be weakly hyperbolic with degeneracy for $t=0$ only, i.e.,

$$\left(\sum_{j=1}^{n} c_j(t)\xi_j\right)^2 + \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j \geq \alpha_0|\xi|^2, \quad \alpha_0 > 0, \quad \forall (t,\xi).$$

The choice of the exponents of t in (1.3) reflects so–called Levi conditions which are necessary and sufficient conditions for the C^∞ well–posedness of the *linear* Cauchy problem, see [11], [13]. If, for instance, the t–exponent of the coefficient of ∂_{x_j} were less than l_*-1, the linear Cauchy problem for that L would be well–posed only in certain Gevrey spaces, see [18].
We list some known results. The Cauchy problems (1.1), (1.2) are locally well–posed in $C^k([0,T],H^s(\mathbb{R}^n))$ for s large enough ([7], [12], [13], [15]) and $C^k([0,T],C^\infty(\mathbb{R}^n))$ ([3], [4], [5]).
Furthermore, singularities of the initial data may propagate in an astonishing way: in [14], it has been shown that the solution $v=v(t,x)$ of

$$Lv = v_{tt} - t^2 v_{xx} - (4m+1)v_x = 0, \quad m \in \mathbb{N}, \tag{1.4}$$

with initial data $v(0,x)=u_0(x)$, $v_t(0,x)=0$ is given by

$$v(t,x) = \sum_{j=0}^{m} C_{jm} t^{2j} (\partial_x^j u_0)(x+t^2/2), \quad C_{jm} \neq 0. \tag{1.5}$$

This representation shows that singularities of the initial datum u_0 propagate only to the left.
Taniguchi and Tozaki discovered branching phenomena for similar operators in [19]. They have studied the Cauchy problem

$$v_{tt} - t^{2l_*} v_{xx} - b l_* t^{l_*-1} v_x = 0, \quad (\partial_t^j v)(-1,x) = u_j(x), \quad j=0,1,$$

and assumed that the initial data have a singularity at some point x_0. Since the equation is strictly hyperbolic for $t<0$, this singularity propagates, in general, along each of the two characteristic curves starting at $(-1,x_0)$. When these characteristic curves cross the line $t=0$, they split, and the singularities then propagate along four characteristics for $t>0$. However, in certain cases, determined by a discrete set of values for b, one or two of these four characteristic curves do not carry any singularities. A relation of this phenomenon to Stokes phenomena was described in [1].

The function spaces $C^k([0,T],H^s(\mathbb{R}^n))$ and $C^k([0,T],C^\infty(\mathbb{R}^n))$, for which local well–posedness could be proved, have the disadvantage that their elements have different smoothness with respect to t and x. We do not know any previous result concerning the weakly hyperbolic Cauchy problem stating that solutions belong to a function space that embeds into the Sobolev spaces $H^s_{\rm loc}((0,T)\times\mathbb{R}^n)$, for some $s\in\mathbb{R}$, under the assumption that the initial data and the right–hand side belong to appropriate function spaces of the same kind.

In this paper, solutions to (1.1) and (1.2) are sought in edge Sobolev spaces, a concept which has been initially invented in the analysis of elliptic pseudodifferential equations near edges, see [10], [17].

The operator L can be written as

$$L = t^{-\mu} P\left(t, t\partial_t, \Lambda(t)\partial_x\right),$$

where $\Lambda(t)=\int_0^t \lambda(t')\,dt'$ and $P(t,\tau,\xi)$ is a polynomial in τ, ξ of degree $\mu=2$ with coefficients depending on t smoothly up to $t=0$. Operators with such a structure arise in the investigation of edge pseudodifferential problems on manifolds with cuspidal edges, where cusps are described by means of the function $\lambda(t)$. The singularity of the manifold requires the use of adapted classes of Sobolev spaces, so–called *edge Sobolev spaces*.

We shall define edge Sobolev spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$, where $s\geq 0$ denotes the Sobolev smoothness with respect to (t,x) for $t>0$ and $\delta\in\mathbb{R}$ is an additional parameter. More precisely,

$$H^s_{\rm comp}(\mathbb{R}_+\times\mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n} \subset H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n) \subset H^s_{\rm loc}(\mathbb{R}_+\times\mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n}$$

with continuous embeddings.

The elements of the spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ have different Sobolev smoothness at $t=0$ in the following sense: There are traces τ_j, $\tau_j u(x)=(\partial_t^j u)(0,x)$, with continuous mappings

$$\tau_j : H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)\to H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n), \quad \beta=\frac{1}{l_*+1}$$

for all $j\in\mathbb{N}$, $j<s-1/2$. This reflects the loss of Sobolev regularity observed when passing from the Cauchy data at $t=0$ to the solution. Namely, (1.5) shows that $u_0\in H^{s+m}(\mathbb{R})$ implies $v(t,.)\in H^s(\mathbb{R})$ only, since $C_{mm}\neq 0$.

This phenomenon has consequences for the investigation of the nonlinear problems (1.1), (1.2). The usual iteration procedure giving the existence

of solutions for small times cannot be applied in the case of the standard function space $C([0,T],H^s(\mathbb{R}^n))$, since we have no longer a mapping which maps this Banach space into itself.
However, it turns out, that the iteration approach is applicable if we employ the specially chosen edge Sobolev spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$. Roughly speaking, the iteration algorithm does not feel the loss of regularity, because it has been absorbed in the function spaces. The idea to choose a special function space adapted to the weakly hyperbolic operator has also been used in [6], [8], and [16].
The paper is organized as follows. We construct the edge Sobolev spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ and list their properties in Section 2. Then we show in Section 3 how the $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ well–posedness of (1.1), (1.2) can be proved. The proofs of these results can be found in [9]. In Section 4, we consider the hyperbolic equation from (1.1), but with data prescribed at $t=-T_0$, and show that the strongest singularities of the solution u propagate in the same way as the singularities of the solution v solving $Lv=0$ and having the same initial data as u for $t=-T$. The propagation of the singularities of v was discussed in [19].

2 Edge Sobolev Spaces

In this section, we shall define Sobolev spaces on the cone $\mathbb{R}_+$ first. In a second step, edge Sobolev spaces on the manifold $\mathbb{R}_+\times\mathbb{R}^n$ will be constructed. Then, the restriction of these Sobolev spaces to $(0,T)\times\mathbb{R}^n$ will give us the desired spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$. Details on the abstract approach to edge Sobolev spaces can be found, e.g., in [10], [17]. Proofs of the results listed here are given in [9].

2.1 Weighted Sobolev Spaces on $\mathbb{R}_+$

We say that $u=u(t)$ belongs to the Mellin Sobolev space $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$, $s\in\mathbb{N}$, $\delta\in\mathbb{R}$, if

$$\|u\|^2_{\mathcal{H}^{s,\delta}(\mathbb{R}_+)}=\sum_{k=0}^{s}\int_0^\infty \left|t^{-\delta}(t\partial_t)^k u(t)\right|^2\,dt<\infty.$$

For arbitrary $s,\delta\in\mathbb{R}$, the Mellin Sobolev space $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$ can be defined by means of interpolation and duality, or by the requirement that

$$\|u\|^2_{\mathcal{H}^{s,\delta}(\mathbb{R}_+)}=\frac{1}{2\pi i}\int_{\operatorname{Re} z=1/2-\delta}\langle z\rangle^{2s}|Mu(z)|^2\,dz<\infty,$$

where $Mu(z) = \int_0^\infty t^{z-1}u(t)\,dt$ denotes the Mellin transform. (Both norms coincide if $s \in \mathbb{N}$.) Recall that $M\colon L^2(\mathbb{R}_+) \to L^2\big(\{z \in \mathbb{C}\colon \operatorname{Re} z = 1/2\}; (2\pi i)^{-1}dz\big)$ is an isometry and

$$M\{(-t\partial_t)u\}(z) = zMu(z), \quad M\{t^{-\delta}u\}(z) = Mu(z-\delta).$$

Furthermore, the space $C^\infty_{\mathrm{comp}}(\mathbb{R}_+)$ is dense in $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$.
We introduce the notations

$$\begin{aligned} H^s(\mathbb{R}_+ \times \mathbb{R}^n) &= \big\{v\big|_{\mathbb{R}_+\times\mathbb{R}^n}\colon\ v \in H^s(\mathbb{R}^{1+n})\big\}, \quad n \geq 0, \\ H^s_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) &= \big\{v \in H^s(\mathbb{R}^{1+n})\colon\ \operatorname{supp} v \subseteq \overline{\mathbb{R}}_+ \times \mathbb{R}^n\big\}, \quad n \geq 0, \\ \mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) &= \big\{v\big|_{\mathbb{R}_+\times\mathbb{R}^n}\colon\ v \in \mathcal{S}(\mathbb{R}^{1+n}\big\}, \quad n \geq 0. \end{aligned}$$

Example 2.1. For $s \geq 0$, $H^s_0(\overline{\mathbb{R}}_+) = \mathcal{H}^{0,0}(\mathbb{R}_+) \cap \mathcal{H}^{s,s}(\mathbb{R}_+)$.

Definition 2.2. Let $s \geq 0$, $\delta \in \mathbb{R}$ and $\omega \in C^\infty(\overline{\mathbb{R}}_+)$ be a cut–off function close to $t = 0$, i.e., $\operatorname{supp}\omega$ is bounded and $\omega(t) = 1$ for t close to 0. Then the *cone Sobolev spaces* $H^{s,\delta;\lambda}(\mathbb{R}_+)$, $H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+)$ are defined by

$$\begin{aligned} H^{s,\delta;\lambda}(\mathbb{R}_+) &= \{\omega u_0 + (1-\omega)u_1\colon\ u_0 \in H^s(\mathbb{R}_+),\, u_1 \in \mathcal{H}^{s,\delta;\lambda}_\sharp(\mathbb{R}_+)\}, \\ H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+) &= \{\omega u_0 + (1-\omega)u_1\colon\ u_0 \in H^s_0(\overline{\mathbb{R}}_+),\, u_1 \in \mathcal{H}^{s,\delta;\lambda}_\sharp(\mathbb{R}_+)\}, \end{aligned}$$

where $\mathcal{H}^{s,\delta;\lambda}_\sharp(\mathbb{R}_+) = \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+) \cap \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)$. The space $H^{s,\delta;\lambda}(\mathbb{R}_+)$ is equipped with the norm

$$\|u\|^2_{H^{s,\delta;\lambda}(\mathbb{R}_+)} = \|\omega u_0\|^2_{H^s(\mathbb{R}_+)} + \|(1-\omega)u_1\|^2_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)}\,.$$

Let us list some properties of these spaces.

Proposition 2.3. **(a)** *The spaces* $H^{s,\delta;\lambda}(\mathbb{R}_+)$*,* $H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+)$ *do not depend on the choice of the cut–off function* ω*, up to the equivalence of norms.*

(b) $\mathcal{S}(\overline{\mathbb{R}}_+)$ *is dense in* $H^{s,\delta;\lambda}(\mathbb{R}_+)$*. If* $s \notin 1/2 + \mathbb{N}$*, then* $H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+)$ *is the closure of* $C^\infty_{\mathrm{comp}}(\mathbb{R}_+)$ *in* $H^{s,\delta;\lambda}(\mathbb{R}_+)$*.*

(c) *For fixed* $\delta \in \mathbb{R}$*,* $\{H^{s,\delta;\lambda}(\mathbb{R}_+)\colon\ s \geq 0\}$ *forms an interpolation scale with respect to the complex interpolation method.*

(d) *If* $l_* = 0$*, then* $H^{s,\delta;\lambda}(\mathbb{R}_+) = H^s(\mathbb{R}_+)$ *and* $H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+) = H^s_0(\overline{\mathbb{R}}_+)$*.*

(e) *Let* $\overline{\lambda}(t)$ *be a smooth and strictly increasing function with* $\overline{\lambda}(t) = 1$ *for* $0 \leq t \leq 1$ *and* $\overline{\lambda}(t) = \lambda(t)$ *for* $t \geq 2$*, and set* $\overline{\Lambda}(t) = \int_0^t \overline{\lambda}(t')dt'$*. Then, for all* $s \geq 0$*,* $\delta \in \mathbb{R}$*,*

$$\begin{aligned} H^{s,\delta;\lambda}(\mathbb{R}_+) &= \{\overline{\lambda}(t)^{\delta+1/2}w(\overline{\Lambda}(t))\colon\ w \in H^s(\mathbb{R}_+)\}, \\ H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+) &= \{\overline{\lambda}(t)^{\delta+1/2}w(\overline{\Lambda}(t))\colon\ w \in H^s_0(\overline{\mathbb{R}}_+)\}. \end{aligned}$$

2.2 The Spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$

Recall that the norm of the usual Sobolev space $H^s(\mathbb{R}^{1+n})$ satisfies

$$\begin{aligned}\|u(t,x)\|^2_{H^s(\mathbb{R}\times\mathbb{R}^n)} &= \int_{(\tau,\xi)} \langle(\tau,\xi)\rangle^{2s} \left|F_{(t,x)\to(\tau,\xi)}u(\tau,\xi)\right|^2 \, d\tau\, d\xi \\ &= \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s} \left\|\kappa^{-1}_{\langle\xi\rangle}\hat{u}(.,\xi)\right\|^2_{H^s(\mathbb{R}_t)} \, d\xi,\end{aligned}$$

where $\hat{u}(t,\xi) = F_{x\to\xi}u(t,\xi)$ denotes the partial Fourier transform with respect to x, and $\kappa_\nu : H^s(\mathbb{R}_t) \to H^s(\mathbb{R}_t)$ is the isomorphism defined by

$$\kappa_\nu w(t) = \nu^{1/2} w(\nu t), \quad t \in \mathbb{R}, \quad \nu > 0. \tag{2.1}$$

This leads us to the following definition.

Definition 2.4. Let E be a Hilbert space and $\{\kappa_\nu\}_{\nu>0}$ be a strongly continuous group of isomorphisms acting on E with $\kappa_\nu\kappa_{\nu'} = \kappa_{\nu\nu'}$ for ν, $\nu' > 0$ and $\kappa_1 = \mathrm{id}_E$.
For $s \in \mathbb{R}$, the abstract edge Sobolev space $\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu\}_{\nu>0}\big)\big)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n; E)$ such that $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^n; E)$ and

$$\|u\|^2_{\mathcal{W}^s\left(\mathbb{R}^n;\left(E,\{\kappa_\nu\}_{\nu>0}\right)\right)} = \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \big\|\kappa^{-1}_{\langle\xi\rangle}\hat{u}(\xi)\big\|^2_E \, d\xi < \infty.$$

Example 2.5. For $s \geq 0$, and with κ_ν from (2.1),

$$\begin{aligned}H^s(\mathbb{R}_+ \times \mathbb{R}^n) &= \mathcal{W}^s\big(\mathbb{R}^n; \big(H^s(\mathbb{R}_+), \{\kappa_\nu\}_{\nu>0}\big)\big), \\ H^s_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) &= \mathcal{W}^s\big(\mathbb{R}^n; \big(H^s_0(\overline{\mathbb{R}}_+), \{\kappa_\nu\}_{\nu>0}\big)\big).\end{aligned}$$

Now we are in a position to define the edge Sobolev spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$.

Definition 2.6. Let $s \geq 0$, $\delta \in \mathbb{R}$. Then we define the group $\{\kappa^{(\delta)}_\nu\}_{\nu>0}$ by

$$\kappa^{(\delta)}_\nu w(t) = \nu^{\beta/2-\beta\delta l_*} w(\nu^\beta t), \;\; \nu > 0,$$

where $\beta = 1/(l_* + 1)$, and set

$$\begin{aligned}H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) &= \mathcal{W}^s\big(\mathbb{R}^n; \big(H^{s,\delta;\lambda}(\mathbb{R}_+), \{\kappa^{(\delta)}_\nu\}_{\nu>0}\big)\big), \\ H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) &= \mathcal{W}^s\big(\mathbb{R}^n; \big(H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+), \{\kappa^{(\delta)}_\nu\}_{\nu>0}\big)\big).\end{aligned}$$

The following results are, in part, direct consequences of this definition and Proposition 2.3.

Proposition 2.7. **(a)** *$\mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ is dense in $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$. If $s \notin 1/2 + \mathbb{N}$, then $H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{R}^n)$ in $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$.*

(b) *For every fixed $\delta \in \mathbb{R}$, $\{H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \colon s \geq 0\}$ forms an interpolation scale with respect to the complex interpolation method.*

(c) *If $l_* = 0$, then $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) = H^s(\mathbb{R}_+ \times \mathbb{R}^n)$ and $H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) = H_0^s(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$.*

(d) *For each $a > 0$,*

$$\begin{aligned} &H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(a,\infty)\times\mathbb{R}^n} \\ &\quad = \left\{\lambda(t)^{1/2+\delta} v(\Lambda(t), x) \colon v \in H^s(\mathbb{R}_+ \times \mathbb{R}^n)\right\}\big|_{(a,\infty)\times\mathbb{R}^n}, \end{aligned}$$

where $\big|_{(a,\infty)\times\mathbb{R}^n}$ means restriction of functions $u = u(t,x)$ from the corresponding function space to $(a,\infty) \times \mathbb{R}^n$. In particular,

$$H^s_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{R}^n) \subset H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)$$

with continuous embeddings.

The spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ admit traces at $t = 0$ in the following sense.

Proposition 2.8. *Let $s \geq 0$, $\delta \in \mathbb{R}$. Then, for each $j \in \mathbb{N}$, $j < s - 1/2$, the map $\mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $u \mapsto (\partial_t^j u)(0,x)$, extends by continuity to a map*

$$\tau_j \colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n).$$

Furthermore, the map

$$H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to \prod_{j<s-1/2} H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n), \quad u \mapsto \{\tau_j u\}_{j<s-1/2}$$

is surjective.

Sketch of proof. By interpolation, we may assume that $s \notin 1/2 + \mathbb{N}$. Then

$$H^{s,\delta;\lambda}(\mathbb{R}_+) = \left\{ \sum_{j<s-1/2} \omega(t)\, \frac{t^j}{j!}\, d_j \colon d_j \in \mathbb{C} \ \forall j \right\} \oplus H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+),$$

where $\omega \in C^\infty(\overline{\mathbb{R}}_+)$ is a cut–off function close to $t = 0$. Applying the functor $\mathcal{W}^s(\mathbb{R}^n; (\cdot, \{\kappa_\nu^{(\delta)}\}_{\nu>0}))$ completes the proof. ∎

Similar arguments lead to the following results.

Proposition 2.9. *For $s \geq 0$, $\delta \in \mathbb{R}$, we have continuity of the following maps:*
(a) $\partial_t \colon H^{s+1,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta+1;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$;
(b) $t^l \colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta+l/l_*;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ *for* $l = 0, 1, \ldots, l_*$;
(c) $\partial_{x_j} \colon H^{s+1,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ *for* $1 \leq j \leq n$;
(d) $\varphi \colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ *for each* $\varphi = \varphi(t) \in \mathcal{S}(\overline{\mathbb{R}}_+)$.
Here t^l means the operator of multiplication by t^l. Similarly for φ. In particular, the differential operator L from (1.3) *is continuous from $H^{s+2,\delta;\lambda}((0,T_0)\times \mathbb{R}^n)$ to $H^{s,\delta+2;\lambda}((0,T_0) \times \mathbb{R}^n)$, where the space $H^{s,\delta;\lambda}((0,T_0) \times \mathbb{R}^n)$ for $s \geq 0$, $\delta \in \mathbb{R}$ is defined in* (2.2) *below.*

2.3 The Spaces $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$

For $T > 0$, we set

$$H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n) = H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n} \tag{2.2}$$

and equip this space with its infimum norm. There is an alternative description of this space provided that $s \in \mathbb{N}$.

Lemma 2.10. *Let $s \in \mathbb{N}$, $\delta \in \mathbb{R}$ and $T > 0$. Then the infimum norm of the space $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$ is equivalent to the norm $\|.\|_{s,\delta;T}$, where*

$$\|u\|^2_{s,\delta;T} = \sum_{l=0}^{s} T^{2l-1} \int_0^T \int_{\mathbb{R}^n_\xi} \vartheta_l(t,\xi)^2 \left|\partial_t^l \hat{u}(t,\xi)\right|^2 \, d\xi \, dt,$$

$$\vartheta_l(t,\xi) = \begin{cases} \langle\xi\rangle^{s-l}\lambda(t_\xi)^{-\delta-l} & : 0 \leq t \leq t_\xi, \\ \langle\xi\rangle^{s-l}\lambda(t)^{-\delta-l} & : t_\xi \leq t \leq T. \end{cases}$$

Here we have introduced the notation $t_\xi = \langle\xi\rangle^{-\beta}$, $\beta = 1/(l_* + 1)$.

For a proof of this and the following results, see [9].

Lemma 2.11. *For s, $s' \geq 0$, δ, $\delta' \in \mathbb{R}$, and $T > 0$,*

$$H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n) \subseteq H^{s',\delta';\lambda}((0,T) \times \mathbb{R}^n)$$

if and only if

$$s \geq s', \quad s + \beta\delta l_* \geq s' + \beta\delta' l_*. \tag{2.3}$$

The two conditions in (2.3) are related to the fact that the elements of the edge Sobolev spaces have different smoothness for $t > 0$ and $t = 0$, respectively.
The following two results provide a criterion when the superposition operators defined by the right–hand sides of the hyperbolic equations in (1.1) and (1.2) map an edge Sobolev space into itself.

Proposition 2.12. *Assume that* $s + \delta \geq 0$. *We suppose that* $s \in \mathbb{N}$ *and* $\min\{s, s + \beta\delta l_*\} > (n+2)/2$. *Then* $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$ *is an algebra under pointwise multiplication for any* $0 < T \leq T_0$. *In other words, we have*

$$\|uv\|_{s,\delta;T} \leq C \|u\|_{s,\delta;T} \|v\|_{s,\delta;T}$$

for $u, v \in H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$. *Moreover, the constant* C *is independent of* $0 < T \leq T_0$.

The proof consists in a direct, but quite long calculation using the representation of the norm from Lemma 2.10. Then, an interpolation argument (see [2]) gives us the following result:

Corollary 2.13. *Let* $f = f(u)$ *be an entire function with* $f(0) = 0$, *i.e.,* $f(u) = \sum_{j=1}^{\infty} f_j u^j$ *for all* $u \in \mathbb{R}$. *Assume that* $\lfloor s \rfloor + \delta \geq 0$ *and* $\min\{\lfloor s \rfloor, \lfloor s \rfloor + \beta\delta l_*\} > (n+2)/2$. *Then there is, for each* $R > 0$, *a constant* $C_1(R)$ *with the property that*

$$\|f(u)\|_{s,\delta;T} \leq C_1(R) \|u\|_{s,\delta;T},$$
$$\|f(u) - f(v)\|_{s,\delta;T} \leq C_1(R) \|u - v\|_{s,\delta;T}$$

provided that $u, v \in H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$ *and* $\|u\|_{s,\delta;T} \leq R$, $\|v\|_{s,\delta;T} \leq R$.

3 Linear and Semilinear Cauchy Problems

Our considerations start with the linear Cauchy problem

$$Lw(t,x) = f(t,x), \qquad (\partial_t^j w)(0,x) = w_j(x), \quad j = 0, 1. \tag{3.1}$$

We define

$$a(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(t) \frac{\xi_i \xi_j}{|\xi|^2}, \quad b(t,\xi) = -\sum_{j=1}^{n} b_j(t) \frac{\xi_j}{|\xi|},$$
$$c(t,\xi) = \sum_{j=1}^{n} c_j(t) \frac{\xi_j}{|\xi|}.$$

Further we introduce the number

$$Q_0 = -\frac{1}{2} + \sup_\xi \frac{|b(0,\xi) + c(0,\xi)|}{2\sqrt{c(0,\xi)^2 + a(0,\xi)}}, \tag{3.2}$$

and fix $A_0 = Q_0 l_*/(l_* + 1) = \beta Q_0 l_*$.

Theorem 3.1. *Let s, $Q \in \mathbb{R}$, $s \geq 1$, $Q \geq Q_0$. Further let $w_0 \in H^{s+A}(\mathbb{R}^n)$, $w_1 \in H^{s+A-\beta}(\mathbb{R}^n)$, and $f \in H^{s-1,Q+1;\lambda}((0,T) \times \mathbb{R}^n)$, where $A = \beta Q l_*$. Then there is a solution $w \in H^{s,Q;\lambda}((0,T) \times \mathbb{R}^n)$ to (3.1). Moreover, the solution w is unique in the space $H^{s,Q_0;\lambda}((0,T) \times \mathbb{R}^n)$.*

Remark 3.2. The parameter A_0 describes the loss of regularity. The explicit representations of the solutions for special model operators in [14] and [19] show that the statement of the Theorem becomes false if $A < A_0$.

By interpolation, it suffices to prove Theorem 3.1 when $s \in \mathbb{N}_+$. In this case, Theorem 3.1 will follow by standard functional–analytic arguments if the following *a priori* estimate is established.

Proposition 3.3. *For each $s \in \mathbb{N}_+$, $Q \geq Q_0$, there is a constant $C_0 = C_0(s,Q)$ with the property that*

$$\|w\|_{s,Q;T} \leq C_0 \left(\|w_0\|_{H^{s+A}(\mathbb{R}^n)} + \|w_1\|_{H^{s+A-\beta}(\mathbb{R}^n)} + T\,\|f\|_{s-1,Q+1;T} \right)$$

for all $0 < T \leq T_0$. The constant C_0 does not depend on T.

We only sketch the proof and refer the reader to [9] for the details.
Let $g = g(t,\xi)$ be the temperate weight function

$$g(t,\xi) = \omega(\Lambda(t)\langle\xi\rangle)t_\xi^{-1} + (1 - \omega(\Lambda(t)\langle\xi\rangle))\lambda(t)|\xi|, \quad (t,\xi) \in [0,T_0] \times \mathbb{R}^n,$$

where $\omega \in C^\infty(\overline{\mathbb{R}}_+)$ has support in $[0,2]$, satisfies $\omega(t) = 1$ for $0 \leq t \leq 1$, and takes values in $[0,1]$. Then we introduce the vector $W(t,\xi) = {}^t\left(g(t,\xi)\hat{w}(t,\xi), D_t\hat{w}(t,\xi)\right)$ and obtain the first–order system

$$D_t W(t,\xi) = A(t,\xi)W(t,\xi) + F(t,\xi), \tag{3.3}$$

$$A(t,\xi) = \begin{pmatrix} \frac{D_t g(t,\xi)}{g(t,\xi)} & g(t,\xi) \\ \frac{\lambda(t)^2|\xi|^2 a(t,\xi) - i\lambda'(t)|\xi| b(t,\xi)}{g(t,\xi)} & -2c(t,\xi)\lambda(t)|\xi| + ic_0(t) \end{pmatrix},$$

$$F(t,\xi) = {}^t(0, -\hat{f}(t,\xi)).$$

It is clear that this first–order O.D.E system has a unique solution W for each fixed $\xi \in \mathbb{R}^n$. Point–wise estimates of $W(t,\xi)$ and its time derivatives will allow to establish the *a priori* estimate from Proposition 3.3.
If $X(t,t',\xi)$ denotes the fundamental matrix, i.e.,

$$D_t X(t,t',\xi) = A(t,\xi)X(t,t',\xi), \quad X(t,t,\xi) = I, \quad 0 \le t', t \le T_0, \quad (3.4)$$

then $W(t,\xi) = X(t,t',\xi)W(t',\xi) + i\int_{t'}^{t} X(t,t'',\xi)F(t'',\xi)dt''$. This immediately gives estimates of $|W(t,\xi)|$ if estimates of $X(t,t',\xi)$ have been found. The following lemma is the crucial tool.

Lemma 3.4. *There is a constant $C > 0$ such that*

$$\|X(t,t',\xi)\| \le C\left(\frac{g(t,\xi)}{g(t',\xi)}\right)^{Q_0+1}, \quad 0 \le t' \le t \le T_0,$$

holds for all $\xi \in \mathbb{R}^n$, where Q_0 is given by (3.2).

Proof. This has been proved in [9] for the case of $t_\xi \le t' \le t \le T_0$. From this, $\|A(t,\xi)\| \le Cg(t,\xi)$, and $\|X(t,t',\xi)\| \le \exp(\int_{t'}^{t} \|A(t'',\xi)\|\, dt'')$ we derive the desired inequality for arbitrary $0 \le t' \le t \le T_0$. □

The derivatives $D_t^l W(t,\xi)$ can be estimated by similar arguments and induction on l. Multiplying the resulting point–wise estimates of $|\partial_t^l \hat{w}(t,\xi)|$ with appropriate factors depending on ξ, and integrating the resulting expressions with respect to (t,ξ) completes the proof of Proposition 3.3.
This *a priori* estimate, Corollary 2.13, and the usual iteration procedure give us the following two theorems.

Theorem 3.5. *Let $s \in \mathbb{N}$ and assume that $\min\{s, s+\beta Q_0 l_*\} > (n+2)/2$, where Q_0 be the number from* (3.2)*. Suppose that $f = f(u)$ is an entire function with $f(0) = 0$. Let $Q \ge Q_0$ and $A = \beta Q l_*$. Then, for $u_0 \in H^{s+A}(\mathbb{R}^n)$, $u_1 \in H^{s+A-\beta}(\mathbb{R}^n)$, there is a number $T > 0$ with the property that a solution $u \in H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)$ to the Cauchy problem* (1.1) *exists. This solution u is unique in the space $H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$.*

Theorem 3.6. *Let $s \in \mathbb{N}$ and assume that $s - 1 > (n+2)/2$. Suppose that $f = f(u, v, v_1, \ldots, v_n)$ is entire with $f(0,\ldots,0) = 0$. Let $Q \ge Q_0$ and $A = \beta Q l_*$. Then, for $u_0 \in H^{s+A}(\mathbb{R}^n)$, $u_1 \in H^{s+A-\beta}(\mathbb{R}^n)$, there is a number $T > 0$ with the property that a solution $u \in H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)$ to the Cauchy problem* (1.2) *exists. This solution u is unique in the space $H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$.*

Eventually, we state a result concerning the propagation of mild singularities.

Theorem 3.7. *Let s satisfy the assumptions of* Theorem 3.5. *Assume $u_0 \in H^{s+\beta Q_0 l_*}(\mathbb{R}^n)$, $u_1 \in H^{s+\beta Q_0 l_* - \beta}(\mathbb{R}^n)$, where Q_0 is given by* (3.2). *Let v be the solution to*

$$Lv = 0, \quad (\partial_t^j v)(0,x) = u_j(x), \quad j = 0,1. \tag{3.5}$$

Then the solutions $u, v \in H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$ to (1.1) *and* (3.5) *satisfy*

$$u - v \in H^{s+\beta,Q_0;\lambda}((0,T)\times\mathbb{R}^n).$$

Proof. Corollary 2.13 implies $f(u) \in H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$. From Lemma 2.11 we deduce that $f(u) \in H^{s-1+\beta,Q_0+1;\lambda}((0,T)\times\mathbb{R}^n)$. The function $w(t,x) = (u-v)(t,x)$ solves $Lw = f(u)$ and has vanishing initial data. An application of Theorem 3.1 concludes the proof. ☐

Example 3.8. Consider Qi Min–You's operator L from (1.4). Then $l_* = 1$, $\beta = 1/2$, and $Q_0 = 2m$. Theorems 3.1, 3.5, and 3.7 state that the solutions u, v to (1.1), (3.5) satisfy

$$u,\ v \in H^{s,2m;\lambda}((0,T)\times\mathbb{R}), \qquad u - v \in H^{s+1/2,2m;\lambda}((0,T)\times\mathbb{R})$$

if $u_0 \in H^{s+m}(\mathbb{R})$, $u_1 \in H^{s+m-1/2}(\mathbb{R})$. Proposition 2.7 then implies

$$u,\ v \in H^s_{\text{loc}}((0,T)\times\mathbb{R}), \qquad u - v \in H^{s+1/2}_{\text{loc}}((0,T)\times\mathbb{R}).$$

We find that the strongest singularities of u coincide with the singularities of v. The latter can be looked up in (1.5) in case $u_1 \equiv 0$.

4 Branching Phenomena for Solutions to Semilinear Equations

In this section, we consider the Cauchy problems

$$Lu = f(u), \qquad (\partial_t^j u)(-T_0,x) = \varepsilon w_j(x), \quad j = 0,1, \tag{4.1}$$

$$Lv = 0, \qquad (\partial_t^j v)(-T_0,x) = \varepsilon w_j(x), \quad j = 0,1, \tag{4.2}$$

with L from (1.3), and we are interested in branching phenomena for singularities of the solution u. Our main result is Theorem 4.5.

We know, e.g., from the example of Qi Min–You that we have to expect a loss of regularity when we pass from the Cauchy data at $\{t=0\}$ to the solution at $\{t \neq 0\}$. However, we *also* will observe a loss of smoothness if we prescribe Cauchy data at, say, $t = -T_0$ and look at the solution for $t = 0$.

This can be seen as follows. For simplicity, we only consider the operator of Taniguchi–Tozaki,

$$L = \partial_t^2 - t^{2l_*}\partial_x^2 - bl_* t^{l_*-1}\partial_x, \quad x \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{4.3}$$

In [19], it was shown that the Fourier transform of the solution $v(t,x)$ to

$$Lv = 0, \quad (\partial_t^j v)(0,x) = u_j(x), \quad j = 0,1,$$

can be written in the form

$$\begin{aligned}\hat{v}(t,\xi) &= \exp(-i\Lambda(t)\xi)\,{}_1\mathrm{F}_1\left(\beta(1+b)l_*/2, \beta l_*, 2i\Lambda(t)\xi\right)\hat{u}_0(\xi) \\ &\quad + t\exp(-i\Lambda(t)\xi)\,{}_1\mathrm{F}_1\left(\beta(1+b)l_*/2+\beta, \beta(l_*+2), 2i\Lambda(t)\xi\right)\hat{u}_1(\xi),\end{aligned}$$

where $\Lambda(t) = \int_0^t \lambda(t')\,dt'$, and ${}_1\mathrm{F}_1(\alpha,\gamma,z)$ is the confluent hypergeometric function satisfying

$$\begin{aligned}{}_1\mathrm{F}_1(\alpha,\gamma,z) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} e^{\pm i\pi\alpha} z^{-\alpha}(1+\mathcal{O}(|z|^{-1})) \\ &\quad + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha-\gamma}(1+\mathcal{O}(|z|^{-1}))\end{aligned}$$

for $|z| \to \infty$. This leads us to

$$\begin{aligned}\hat{v}(t,\xi) &= (\exp(-i\Lambda(t)\xi)\mathcal{O}(|\xi|^{\alpha_0^-}) + \exp(i\Lambda(t)\xi)\mathcal{O}(|\xi|^{\alpha_0^+}))\hat{u}_0(\xi) \\ &\quad + (\exp(-i\Lambda(t)\xi)\mathcal{O}(|\xi|^{\alpha_1^-}) + \exp(i\Lambda(t)\xi)\mathcal{O}(|\xi|^{\alpha_1^+}))\hat{u}_1(\xi),\end{aligned} \tag{4.4}$$

for $|\xi| \to \infty$ provided that $t \neq 0$ has been fixed. Here we have set

$$\begin{aligned}\alpha_0^- &= -\beta(1+b)l_*/2, & \alpha_0^+ &= -\beta(1-b)l_*/2, \\ \alpha_1^- &= \alpha_0^- - \beta, & \alpha_1^+ &= \alpha_0^+ - \beta.\end{aligned}$$

In general, one of the exponents α_0^-, α_0^+ and one of the exponents α_1^-, α_1^+ is positive, the other one is negative. We observe a loss of $\max\{\alpha_0^-, \alpha_0^+\}$ derivatives when we start from $t = 0$, and a loss of $\max\{-\alpha_0^-, -\alpha_0^+\}$ derivatives when we arrive at $t = 0$. Therefore, we have to expect both these phenomena when we prescribe Cauchy data at $t = -T_0$ and cross the line $t = 0$. This leads us to the following definition.

Definition 4.1. Let $s \geq 0$, $\delta \in \mathbb{R}$. We say that $u \in H^{s,\delta;\lambda}((-T,T)\times\mathbb{R}^n)$ if $u(t,x) \in H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$, $u(-t,x) \in H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$, and $u(t,x) - u(-t,x) \in H_0^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$.

Let $s_-, s_+ \geq 0$, $\delta_-, \delta_+ \in \mathbb{R}$ and suppose that

$$s_- + \beta\delta_- l_* = s_+ + \beta\delta_+ l_*, \quad s_+ \leq s_-.$$

We say that $u \in H^{s_-,s_+,\delta_-,\delta_+;\lambda}((-T,T)\times\mathbb{R}^n)$ if $u \in H^{s_+,\delta_+;\lambda}((-T,T)\times\mathbb{R}^n)$ and $u(-t,x) \in H^{s_-,\delta_-;\lambda}((0,T)\times\mathbb{R}^n)$. We define the norm by

$$\begin{aligned}&\|u(t,x)\|_{H^{s_-,s_+,\delta_-,\delta_+;\lambda}((-T,T)\times\mathbb{R}^n)}\\&\quad= \|u(t,x)\|_{H^{s_+,\delta_+;\lambda}((0,T)\times\mathbb{R}^n)} + \|u(-t,x)\|_{H^{s_-,\delta_-;\lambda}((0,T)\times\mathbb{R}^n)}.\end{aligned}$$

This choice of the norm is possible, since $H^{s_-,\delta_-;\lambda}((0,T)\times\mathbb{R}^n) \subset H^{s_+,\delta_+;\lambda}((0,T)\times\mathbb{R}^n)$, compare Lemma 2.11.
Let us consider the equation $Lv = 0$, again, where the operator L is from (4.3), and suppose that the initial data at $t = -T_0$ satisfy $w_0 \in H^{s_-}(\mathbb{R})$, $w_1 \in H^{s_--1}(\mathbb{R})$. We set $\delta_+ = Q_0 = (|b|-1)/2$, $\delta_- = -1-Q_0 = (-|b|+1)/2$, $s_+ = s_- + \beta\delta_- l_* - \beta\delta_+ l_*$. The identity (4.4) leads, after some calculation, to a representation of $\hat{v}(t,\xi)$ in terms of $\hat{w}_0(\xi)$ and $\hat{w}_1(\xi)$ which shows that the solution v belongs to $H^{s_-,s_+,\delta_-,\delta_+;\lambda}((-T_0,T_0)\times\mathbb{R})$. Moreover,

$$v(0,x) \in H^{s_-+\beta\delta_- l_*}(\mathbb{R}), \quad (\partial_t v)(0,x) \in H^{s_-+\beta\delta_- l_*-\beta}(\mathbb{R}),$$

and these statements about the smoothness are best possible.
For the general operator L from (1.3), we can prove the following well–posedness result:

Proposition 4.2. *Let L be the operator from* (1.3)*, Q_0 be the number defined in* (3.2)*, and set*

$$Q_+ = Q_0, \quad Q_- = -1 - Q_0, \quad s_+ = s_- + \beta Q_- l_* - \beta Q_+ l_*,$$

where $s_-, s_+ \geq 1$. Suppose that

$$\begin{aligned}&w_0(x) \in H^{s_-}(\mathbb{R}^n), \quad w_1(x) \in H^{s_--1}(\mathbb{R}^n),\\&f(t,x) \in H^{s_--1,s_+-1,Q_-+1,Q_++1;\lambda}((-T_0,T_0)\times\mathbb{R}^n).\end{aligned}$$

Then there is a unique solution $w \in H^{s_-,s_+,Q_-,Q_+;\lambda}((-T_0,T_0)\times\mathbb{R}^n)$ to

$$Lw = f(t,x), \quad (\partial_t^j w)(-T_0,x) = w_j(x), \quad j = 0,1.$$

Moreover, $(\partial_t^j w)(0,x) \in H^{s_-+\beta Q_- l_*-\beta j}(\mathbb{R}^n)$ for $j = 0,1$, and

$$\begin{aligned}&\|w(0,x)\|_{H^{s_-+\beta Q_- l_*}(\mathbb{R}^n)} + \|(\partial_t w)(0,x)\|_{H^{s_-+\beta Q_- l_*-\beta}(\mathbb{R}^n)} \qquad (4.5)\\&\leq C(\|w_0\|_{H^{s_-}(\mathbb{R}^n)} + \|w_1\|_{H^{s_--1}(\mathbb{R}^n)} + \|f(-t,x)\|_{H^{s_--1,Q_-+1;\lambda}((0,T_0)\times\mathbb{R}^n)}),\\&\|w(t,x)\|_{H^{s_-,s_+,Q_-,Q_+;\lambda}((-T_0,T_0)\times\mathbb{R}^n)} \leq C(\|w_0\|_{H^{s_-}(\mathbb{R}^n)} + \|w_1\|_{H^{s_--1}(\mathbb{R}^n)})\\&\quad + C\,\|f(t,x)\|_{H^{s_--1,s_+-1,Q_-+1,Q_++1;\lambda}((-T_0,T_0)\times\mathbb{R}^n)}.\end{aligned}$$

The key of the proof is an estimate of the fundamental matrix.

Lemma 4.3. *Let $X(t,t',\xi)$ be the solution to* (3.4)*. Then it holds*

$$\|X(t,t',\xi)\| \leq C\left(\frac{g(t,\xi)}{g(t',\xi)}\right)^{-Q_0}, \quad 0\leq t\leq t'\leq T_0.$$

Proof. From $X(t,t',\xi)=X(t',t,\xi)^{-1}$ and Lemma 3.4 we deduce that

$$\|X(t,t',\xi)\| \leq C\left(\frac{g(t',\xi)}{g(t,\xi)}\right)^{Q_0+1} |\det X(t',t,\xi)|^{-1}, \quad 0\leq t\leq t'\leq T_0.$$

Since $\partial_t X = iAX$, we obtain

$$\begin{aligned}
&\partial_t \det X(t,t',\xi) = \operatorname{trace}(iA(t,\xi))\det X(t,t',\xi),\\
&\operatorname{trace}(iA(t,\xi)) = \frac{\partial_t g(t,\xi)}{g(t,\xi)} - 2ic(t,\xi)\lambda(t)|\xi| - c_0(t),\\
&|\det X(t,t',\xi)| = \exp\left(\int_{t'}^{t}\frac{\partial_\tau g(\tau,\xi)}{g(\tau,\xi)} - c_0(\tau)\,d\tau\right), \quad 0\leq t,t'\leq T_0.
\end{aligned}$$

This completes the proof. □

Inverting the time direction, we will obtain an estimate of the fundamental matrix for negative times from this lemma.

Proof of Proposition 4.2. The existence and uniqueness of w is clear since its Fourier transform solves an O.D.E. with parameter ξ. It remains to discuss the smoothness of w. By interpolation, we may assume $s_-\in\mathbb{N}$. Then we can make use of Lemma 2.10 to represent the norms of $w\big|_{(-T_0,0)\times\mathbb{R}^n}$ and $f\big|_{(-T_0,0)\times\mathbb{R}^n}$. We define $g(t,\xi)=g(-t,\xi)$ for $(t,\xi)\in(-T_0,0)\times\mathbb{R}^n$ and are going to show (4.5) and

$$\begin{aligned}
&\sum_{l=0}^{s_-}\int_{-T_0}^{0}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2(s_-+Q_-)}g(t,\xi)^{-2(l+Q_-)}|\partial_t^l\hat{w}(t,\xi)|^2\,d\xi\,dt \\
&\quad\leq C(\|w_0\|^2_{H^{s_-}(\mathbb{R}^n)}+\|w_1\|^2_{H^{s_--1}(\mathbb{R}^n)})\\
&\qquad + C\sum_{l=0}^{s_--1}\int_{-T_0}^{0}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2(s_-+Q_-)}g(t,\xi)^{-2(l+Q_-+1)}|\partial_t^l\hat{f}(t,\xi)|^2\,d\xi\,dt,
\end{aligned} \tag{4.6}$$

i.e., the *a priori* estimate for $t\leq 0$. Proposition 3.3 then gives the *a priori* estimate for $t\geq 0$, which completes the proof.

For $(t,\xi) \in (-T_0, 0) \times \mathbb{R}^n$, we set $W(t,\xi) = {}^t(g(t,\xi)\hat{w}(t,\xi), D_t\hat{w}(t,\xi))$ as above, and conclude that

$$W(t,\xi) = X(t,-T_0,\xi)W(-T_0,\xi) + i\int_{-T_0}^{t} X(t,t',\xi)F(t',\xi)\,dt',$$

for $-T_0 \le t \le 0$, where $X(t,t',\xi)$ solves (3.4). Inverting the time direction in (3.3), we conclude from Lemma 4.3 that

$$\|X(t,t',\xi)\| \le C\left(\frac{g(t,\xi)}{g(t',\xi)}\right)^{-Q_0}, \quad -T_0 \le t' \le t \le 0.$$

Consequently,

$$\begin{aligned} &g(t,\xi)^{Q_0}|W(t,\xi)| \\ &\quad \le Cg(-T_0,\xi)^{Q_0}|W(-T_0,\xi)| + C\int_{-T_0}^{t} g(t',\xi)^{Q_0}|F(t',\xi)|\,dt'. \end{aligned} \tag{4.7}$$

Taking squares, setting $t = 0$, and using $g(-T_0,\xi) \sim \langle\xi\rangle$ yields

$$\begin{aligned} &\langle\xi\rangle^{-2\beta Q_0 l_*}|W(0,\xi)|^2 \\ &\quad \le C|W(-T_0,\xi)|^2 + C\int_{-T_0}^{0} \langle\xi\rangle^{-2Q_0} g(t',\xi)^{2Q_0}|F(t',\xi)|^2\,dt'. \end{aligned}$$

Multiplying with $\langle\xi\rangle^{2(s_- -1)}$ and integrating over $\mathbb{R}^n_\xi$ gives (4.5).
Clearly,

$$D_t^l W(t,\xi) = \sum_{m=0}^{l-1} \binom{l-1}{m} (D_t^m A(t,\xi))(D_t^{l-1-m}W(t,\xi)) + D_t^{l-1}F(t,\xi),$$

for every $l \ge 1$. From $\|D_t^m A(t,\xi)\| \le Cg(t,\xi)^{m+1}$ and (4.7) we then get

$$\sum_{l=0}^{s_- -1} g(t,\xi)^{-l}|D_t^l W(t,\xi)| \le C|W(t,\xi)| + \sum_{l=0}^{s_- -2} g(t,\xi)^{-l-1}|D_t^l F(t,\xi)|,$$

$$\begin{aligned} &\sum_{l=0}^{s_- -1} \langle\xi\rangle^{s_- -1-Q_0} g(t,\xi)^{Q_0-l}|D_t^l W(t,\xi)| \\ &\quad \le C\langle\xi\rangle^{s_- -1-Q_0} g(-T_0,\xi)^{Q_0}|W(-T_0,\xi)| \\ &\qquad + C\int_{-T_0}^{t} \langle\xi\rangle^{s_- -1-Q_0} g(t',\xi)^{Q_0}|F(t',\xi)|\,dt' \\ &\qquad + \sum_{l=0}^{s_- -2} \langle\xi\rangle^{s_- -1-Q_0} g(t,\xi)^{Q_0-l-1}|D_t^l F(t,\xi)|. \end{aligned}$$

Squaring this relation and integration over $(-T_0, 0) \times \mathbb{R}^n$ gives

$$\begin{aligned}
&\left\| F^{-1}_{\xi \to x} W(-t,\xi) \right\|^2_{H^{s_- -1, Q_- +1;\lambda}((0,T_0)\times\mathbb{R}^n)} \\
&\quad \le C(\|w_0\|^2_{H^{s_-}(\mathbb{R}^n)} + \|w_1\|^2_{H^{s_- -1}(\mathbb{R}^n)}) \\
&\qquad + C \left\| f(-t,x) \right\|^2_{H^{s_- -1, Q_- +1;\lambda}((0,T_0)\times\mathbb{R}^n)},
\end{aligned}$$

which implies (4.6). The proof is complete. □

Remark 4.4. Due to (3.2), $Q_0 \ge -1/2$, which is equivalent to $s_+ \le s_-$. If $s_+ = s_-$, no loss of regularity occurs when we cross the line of degeneracy. The case of an hyperbolic operator with this property and countably many points of degeneracy (or singularity) accumulating at $t = 0$ has been discussed in [20].

The next theorem relates branching phenomena for the semilinear problem (4.1) with branching phenomena for the linear reference problem (4.2). This relation between a semilinear Cauchy problem and an associated linear reference problem has already been discussed in Example 3.8.

Theorem 4.5. *Let L be the operator from* (1.3)*, Q_0 be the number from* (3.2)*, and suppose that*

$$\min\{\lfloor s_\pm \rfloor, \lfloor s_\pm \rfloor + \beta Q_\pm l_*\} > \frac{n+2}{2}, \quad \lfloor s_\pm \rfloor + Q_\pm \ge 0, \quad s_\pm \ge 1,$$

where $Q_+ = Q_0$, $Q_- = -1 - Q_0$, and $s_+ = s_- + \beta Q_- l_* - \beta Q_+ l_*$.
Assume that $w_0 \in H^{s_-}(\mathbb{R}^n)$, $w_1 \in H^{s_- -1}(\mathbb{R}^n)$, and that $f = f(u)$ is an entire function with $f(0) = f'(0) = 0$.
Then there is an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ there are unique solutions $u, v \in H^{s_-,s_+,Q_-,Q_+;\lambda}((-T_0,T_0)\times\mathbb{R}^n)$ to (4.1) *and* (4.2)*, respectively, which, in addition, satisfy*

$$u - v \in H^{s_- +\beta, s_+ +\beta, Q_-, Q_+;\lambda}((-T_0,T_0)\times\mathbb{R}^n). \tag{4.8}$$

Proof. For the sake of simplicity, let us denote the Banach space $H^{s_-,s_+,Q_-,Q_+;\lambda}((-T_0,T_0)\times\mathbb{R}^n)$ by B. The conditions on $s_\pm$ and $Q_\pm$ imply that B is an algebra, see Corollary 2.13. Since $f(0) = f'(0) = 0$, we can conclude that

$$\begin{aligned}
&\|f(u)\|_{H^{s_- -1, s_+ -1, Q_- +1, Q_+ +1;\lambda}((-T_0,T_0)\times\mathbb{R}^n)} \le C\,\|f(u)\|_B \le C(R)\,\|u\|_B^2, \\
&\|f(u) - f(v)\|_B \le C(R) R\,\|u - v\|_B, \\
&\quad \text{for all } u, v \text{ with } \|u\|_B \le R, \quad \|v\|_B \le R.
\end{aligned}$$

If we choose R small enough, and then choose ε small enough, the usual iteration approach works, leading to a solution u to (4.1) with $\|u\|_B \le R$. From Lemma 2.11 we then deduce that

$$f(u) \in B \subset H^{s_- -1+\beta, s_+ -1+\beta, Q_- +1, Q_+ +1;\lambda}((-T_0, T_0) \times \mathbb{R}^n).$$

The difference $u - v$ solves $L(u - v) = f(u)$ and has vanishing Cauchy data for $t = -T_0$. Then Proposition 4.2 yields (4.8). □

References

[1] K. Amano and G. Nakamura. Branching of singularities for degenerate hyperbolic operators. *Publ. Res. Inst. Math. Sci.*, 20(2):225–275, 1984.

[2] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction*, *Grundlehren Math. Wiss.*, Vol. 223, Springer, Berlin, 1976.

[3] F. Colombini, E. Jannelli, and S. Spagnolo. Well–posedness in the Gevrey classes of the Cauchy problem for a non–strictly hyperbolic equation with coefficients depending on time. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10:291–312, 1983.

[4] F. Colombini and S. Spagnolo. An example of a weakly hyperbolic Cauchy problem not well posed in C^∞. *Acta Math.*, 148:243–253, 1982.

[5] P. D'Ancona and M. Di Flaviano. On quasilinear hyperbolic equations with degenerate principal part. *Tsukuba J. Math.*, 22(3):559–574, 1998.

[6] M. Dreher. Weakly hyperbolic equations, Sobolev spaces of variable order, and propagation of singularities. *submitted.*

[7] M. Dreher. *Local solutions to quasilinear weakly hyperbolic differential equations.* Ph.D. thesis, Technische Universität Bergakademie Freiberg, Freiberg, 1999.

[8] M. Dreher and M. Reissig. Propagation of mild singularities for semilinear weakly hyperbolic differential equations. *J. Analyse Math.*, 82, 2000.

[9] M. Dreher and I. Witt. Edge Sobolev spaces and weakly hyperbolic equations. *submitted to Ann. Mat. Pura Appl. (4).*

[10] Y. Egorov and B.-W. Schulze. *Pseudo-Differential Operators, Singularities, Applications*, volume 93 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1997.

[11] V. Ivrii and V. Petkov. Necessary conditions for the Cauchy problem for non–strictly hyperbolic equations to be well–posed. *Russian Math. Surveys*, 29(5):1–70, 1974.

[12] K. Kajitani and K. Yagdjian. Quasilinear hyperbolic operators with the characteristics of variable multiplicity. *Tsukuba J. Math.*, 22(1):49–85, 1998.

[13] O. Oleinik. On the Cauchy problem for weakly hyperbolic equations. *Comm. Pure Appl. Math.*, 23:569–586, 1970.

[14] M.-Y. Qi. On the Cauchy problem for a class of hyperbolic equations with initial data on the parabolic degenerating line. *Acta Math. Sinica*, 8:521–529, 1958.

[15] M. Reissig. Weakly hyperbolic equations with time degeneracy in Sobolev spaces. *Abstract Appl. Anal.*, 2(3,4):239–256, 1997.

[16] M. Reissig and K. Yagdjian. Weakly hyperbolic equations with fast oscillating coefficients. *Osaka J. Math.*, 36(2):437–464, 1999.

[17] B.-W. Schulze. *Boundary Value Problems and Singular Pseudo-differential Operators.* Wiley Ser. Pure Appl. Math. J. Wiley, Chichester, 1998.

[18] K. Shinkai. Stokes multipliers and a weakly hyperbolic operator. *Comm. Partial Differential Equations*, 16(4,5):667–682, 1991.

[19] K. Taniguchi and Y. Tozaki. A hyperbolic equation with double characteristics which has a solution with branching singularities. *Math. Japon.*, 25(3):279–300, 1980.

[20] T. Yamazaki. Unique existence of evolution equations of hyperbolic type with countably many singular or degenerate points. *J. Differential Equations*, 77:38–72, 1989.