Theorie und Numerik Spezieller Funktionen

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0. Introduction

0.1 Eigenfunctions of the Laplace Operator

For various reasons, it is desirable to know the eigenfunctions of the Laplace–operator (with homogeneous Dirichlet boundary conditions) on a domain Ω:

$$\begin{cases} \triangle u(x) + \lambda u(x) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a domain (an open and connected set), $u \in C^2(\Omega)$, $\lambda \in \mathbb{R}$, and

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

In many cases, the eigenfunctions will form an orthogonal basis of $L^2(\Omega)$, which is what makes them interesting.

For general domains Ω, it is hopeless to find a formula for the eigenvalues λ and the eigenfunctions u. Only numerical techniques can help, and they are also quite elaborate.

For this reason, let us stick to simple domains Ω first.

0.1.1 An Interval in $\mathbb{R}^1$

Take $n = 1$ and $\Omega = (a, b)$. Then we have

$$\begin{cases} u''(x) + \lambda u(x) = 0, & a < x < b, \\ u(a) = u(b) = 0. \end{cases}$$

The general solution is $u(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$. From the boundary conditions, you get the system

$$\begin{pmatrix} \cos(\sqrt{\lambda}a) & \sin(\sqrt{\lambda}a) \\ \cos(\sqrt{\lambda}b) & \sin(\sqrt{\lambda}b) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which needs a vanishing determinant, in order to have a nontrivial solution. Therefore, we obtain the equation $\sin(\sqrt{\lambda}(b-a)) = 0$, which gives

$$\lambda = \lambda_n = \frac{(n\pi)^2}{(b-a)^2}, \qquad u_n = u_n(x) = \sin \frac{n\pi(b-x)}{b-a}.$$

0.1.2 A Cuboid in $\mathbb{R}^2$

Now to some less trivial domain: $\Omega = (a, b) \times (c, d) \subset \mathbb{R}^2$. We make the ansatz $u(x, y) = X(x)Y(y)$, and obtain the system

$$\begin{cases} X''(x)Y(y) + X(x)Y''(y) + \lambda X(x)Y(y) = 0, & a < x < b,\ c < y < d, \\ X(a) = X(b) = 0, \\ Y(c) = Y(d) = 0. \end{cases}$$

We divide the differential equation by XY (for a moment, we don't care whether this could be zero), and obtain

$$\frac{X''(x)}{X(x)} = -\lambda - \frac{Y''(y)}{Y(y)} = -\mu,$$

since the LHS[1] depends only on x, and the RHS[2] depends only on y. This gives the two BVPs[3] to ODEs[4]

$$X''(x) + \mu X(x) = 0, \qquad X(a) = X(b) = 0,$$
$$Y''(y) + (\lambda - \mu)Y(y) = 0, \qquad Y(c) = Y(d) = 0.$$

We have already learned how to handle these two problems: only certain values for μ and λ will lead to non-trivial solutions X and Y; and the functions X and Y will be certain Sine functions.

The function $u = XY$ will then be an eigenfunction to the Laplace–operator. Surprisingly, every eigenfunction can be constructed like this (we will not prove this result).

0.1.3 A Disk in $\mathbb{R}^2$

Next, we consider a disk $\Omega = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < r_0^2\}$. Introducing polar coordinates (r, φ) and looking up the representation of the Laplace-operator in polar coordinates, we end up with the following problem:

$$\begin{cases} \triangle u + \lambda u = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}u + \lambda u = 0, & 0 < r < r_0,\ 0 \le \varphi \le 2\pi, \\ u(r_0, \varphi) = 0, & 0 \le \varphi \le 2\pi, \\ u(r, 0) = u(r, 2\pi), & 0 < r < r_0, \\ u(r, \varphi) \text{ bounded}, & r \to 0. \end{cases}$$

We copy the idea of splitting the variables from the cuboid case:

$$u(r, \varphi) = R(r)\Phi(\varphi),$$

leading us to

$$\frac{1}{r}\left(\partial_r(rR'(r))\right)\Phi(\varphi) + \frac{R(r)}{r^2}\Phi''(\varphi) + \lambda R(r)\Phi(\varphi) = 0,$$
$$\frac{r(\partial_r(rR'(r)))}{R(r)} + \lambda r^2 = -\frac{\Phi''(\varphi)}{\Phi(\varphi)}.$$

The LHS depends only on r, the RHS only on φ. Call their value μ. Then we obtain

$$\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \qquad \Phi(0) = \Phi(2\pi),$$

hence $\sqrt{\mu}$ must be an integer, $\mu = m^2$, $m = 0, 1, 2, \dots$. This will give us the following equation for the function R:

$$R''(r) + \frac{1}{r}R'(r) + \left(\lambda - \frac{m^2}{r^2}\right)R(r) = 0, \qquad R(r_0) = 0, \qquad R(r) \text{ bounded for } r \to 0.$$

And now we are stuck. The standard theory of ODE does not tell us how to solve such equations. Pay attention to the singularities of the coefficients for $r = 0$.

It is custom to do some cosmetic changes:

$$x = \sqrt{\lambda}r, \qquad R(r) = y(x),$$

from which we can derive the differential equation (the so-called Bessel equation)

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{m^2}{x^2}\right)y(x) = 0.$$

The advantage is that we have one parameter less, but we still have no idea how solutions look like, and if they even exist.

[1] left–hand side
[2] right–hand side
[3] boundary value problem
[4] ordinary differential equation

0.1.4 A Ball in $\mathbb{R}^3$

Another reasonable domain is a ball in $\mathbb{R}^3$, $\Omega = \{(x,y,z) \in \mathbb{R}^2 \colon x^2+y^2+z^2 < r_0^2\}$. Introducing polar coordinates and the Laplace operator in polar coordinates, we end up with the problem

$$\begin{cases} \triangle_{r,\varphi,\theta} u + \lambda u = 0, & 0 < r < r_0 \; 0 \le \varphi \le 2\pi,\ 0 < \theta < \pi, \\ u(r_0,\varphi,\theta) = 0, & 0 \le \varphi \le 2\pi,\ 0 < \theta < \pi, \\ u(r,0,\theta) = u(r,2\pi,\theta), & 0 < r < r_0 \; 0 < \theta < \pi, \\ u(r,\varphi,\theta) \text{ bounded} & r \to 0, \end{cases}$$

where

$$\triangle_{r,\varphi,\theta} = \frac{1}{r^2}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\triangle_{\varphi,\theta},$$
$$\triangle_{\varphi,\theta} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

We make the ansatz $u(r,\varphi,\theta) = R(r)S(\varphi,\theta)$, and obtain, after some calculation,

$$R''(r) + \frac{2}{r}R'(r) + \frac{\lambda r^2 - \mu}{r^2}R(r) = 0, \qquad 0 < r < r_0, \qquad R(r_0) = 0.$$

The substitution

$$x = \sqrt{\lambda} r, \qquad y(x) = \sqrt{x} R(r)$$

then yields again a Bessel equation:

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{\mu + 1/4}{x^2}\right) y(x) = 0.$$

Next, we should consider the other factor $S = S(\varphi,\theta)$. We split the variables again: $S(\varphi,\theta) = \Phi(\varphi)\Theta(\theta)$, and obtain an ODE for Φ which should be known by now:

$$\Phi''(\varphi) + \nu\Phi(\varphi) = 0, \qquad \Phi(0) = \Phi(2\pi),$$

which forces us to choose $\nu = m^2$ with $m \in \mathbb{N}_0$.

We substitute $\cos\theta = x$ and $y(x) = \Theta(\theta)$ with $-1 < x < 1$, and, after some calculation, we find

$$y''(x) - \frac{2x}{1-x^2}y'(x) + \frac{\mu(1-x^2) - \nu}{(1-x^2)^2}y(x) = 0.$$

Note the poles in the coefficients for $x \to \pm 1$. Again, we have no idea whether solutions to this differential equation exist or not.

0.2 Gaussian Integration

Let us be given a function f on the interval $[-1,1]$. We would like to find an approximate value of the integral $\int_{x=-1}^{x=1} f(x)\,\mathrm{d}x$ like this:

$$\int_{x=-1}^{x=1} f(x)\,\mathrm{d}x \approx \sum_{j=1}^{n} w_j f(x_j),$$

where the x_j are certain points in the interval $[-1,1]$ (not necessarily equidistant), and the w_j are so–called weights.

Is it possible to choose the x_j and w_j in such a way that all polynomials up to a certain degree are integrated exactly ? How high can that degree be ?

We have $2n$ parameters available, so we hope to integrate all polynomials of degree less than or equal to $2n-1$ exactly. This leads us to the GAUSS quadrature formulas. The x_j must be chosen as the zeroes of the LEGENDRE[5] polynomial $P_n = P_n(x)$, which is defined as being the (only) polynomial solution to the differential equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0$$

with $P_n(1) = 1$. The functions P_0, P_1, P_2, …, P_n form an $L^2(-1,1)$–*orthogonal* basis of the space of all polynomials of degree at most n. All the n zeroes of P_n are in the interval $[-1,1]$. And the weights w_k must be chosen as

$$w_k = \int_{x=-1}^{x=1} \left(\prod_{j=1, j\neq k}^{n} \frac{x-x_j}{x_k-x_j} \right) \mathrm{d}x = \int_{x=-1}^{x=1} \left(\prod_{j=1, j\neq k}^{n} \frac{x-x_j}{x_k-x_j} \right)^2 \mathrm{d}x.$$

For the convenience of the reader, we list the data for the case $n=7$:

i	x_i	w_i
1	−0.949107912342759	0.129484966168870
2	−0.741531185599384	0.279705391489277
3	−0.405845151377397	0.381830050505119
4	0	0.417959183673469
5	+0.405845151377397	0.381830050505119
6	+0.741531185599384	0.279705391489277
7	+0.949107912342759	0.129484966168870

More parameters for the Gaussian quadrature (up to $n=96$) can be found in [1].

As an example, we use this to evaluate $\ln 5 = \int_{x=1}^{x=5} \frac{1}{x}\,\mathrm{d}x$. The exact value is

$$\ln 5 \approx 1.6094379124341002818\ldots.$$

First, we shift the interval $[1,5]$ to $[-1,1]$:

$$t = \frac{x-3}{2}, \quad x = 2t+3, \qquad \ln 5 = \int_{x=1}^{x=5} \frac{\mathrm{d}x}{x} = \int_{t=-1}^{t=1} \frac{2\,\mathrm{d}t}{2t+3}.$$

Evaluating this last integral with GAUSSIAN quadrature with $n=7$ and with one or two sub-divisions, as well as with the trapezoidal rule ($n=6$) and Kepler's rule ($n=6$), we obtain the following numbers:

sub-division	value	error
0	1.6094346840305430703	$3.228e-06$
1	1.6094378965041162519	$1.592e-08$
2	1.6094379124141617306	$1.993e-11$

Table 1: Gauss quadrature

sub-division	value	error
0	1.6436008436008436009	$3.416e-02$
1	1.6182289932289932289	$8.791e-03$
2	1.611653797587057737	$2.215e-03$

Table 2: Trapezoidal rule

[5] ADRIEN–MARIE LEGENDRE, 1752 – 1833

sub-division	value	error
0	1.6131128131128131129	$3.675e-03$
1	1.6097717097717097716	$3.338e-04$
2	1.6094620657064125734	$2.415e-05$

Table 3: Kepler's rule

0.3 Further Examples

There are several more differential equations which appear quite often, but are very hard to investigate with the standard tools for ODE:

$$
\begin{aligned}
&y''(x) - 2xy'(x) + \lambda y = 0 && \text{Hermite differential equation,}\\
&x(1-x)y''(x) + (\gamma - (\alpha+\beta+1)x)y'(x) - \alpha\beta\gamma y = 0 && \text{hypergeometric differential equation,}\\
&xy''(x) + (\gamma - x)y'(x) - \alpha y = 0 && \text{confluent hypergeometric differential eq.}
\end{aligned}
$$

0.4 Summary

We should not consider each of the above differential equations separately. Instead, we should have a look at the general structure:

All the equations look like

$$y''(x) + \frac{a(x)}{b(x)}y'(x) + \frac{c(x)}{b(x)^2}y(x) = 0,$$

where a, b, and c are polynomials of the form

$$
\begin{aligned}
a(x) &= \alpha_0 + \alpha_1 x,\\
b(x) &= \beta_0 + \beta_1 x + \beta_2 x^2,\\
c(x) &= \gamma_0 + \gamma_1 x + \gamma_2 x^2.
\end{aligned}
$$

In the sequel, a general theory of ODEs of that form will be presented, taking advantage from complex analysis.

Chapter 1

The Gamma Function and Related Functions

1.1 Euler's Gamma function

The material in this section is taken from [12, Vol. 2], [13, Vol. 1] and [6, Vol. 1].

Definition 1.1. *For $\Re z > 0$, we define*

$$\Gamma(z) := \int_{t=0}^{t=\infty} e^{-t} t^{z-1}\,\mathrm{d}t.$$

Obviously, $\Gamma(1) = 1$; and by partial integration, we deduce that $z\Gamma(z) = \Gamma(z+1)$, for $\Re z > 0$. The natural consequence is $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}_+$.

Next, we answer the following questions:

- Is Γ analytic ?
- Can the Γ function be extended to whole $\mathbb{C}$?
- If not, does it have poles ? What are the residues of Γ ?

Suppose $0 < \delta \le \Re z \le X < \infty$. Writing $z = x + \mathrm{i}y$ and $t^{z-1} = \exp((x+1-\mathrm{i}y)\ln t)$, we see that t^{z-1} is an analytic function of z, for $t > 0$, and $|t^{z-1}| = t^{x-1}$. Then it follows that

$$\begin{aligned}
&0 < t \le 1 && : |e^{-t}t^{z-1}\ln t| \le e^{-t}t^{\delta-1}|\ln t|,\\
&1 \le t < \infty && : |e^{-t}t^{z-1}\ln t| \le e^{-t}t^{X-1}\ln t,
\end{aligned}$$

which implies that the integral $\int_0^\infty e^{-t}t^{z-1}\ln t\,dt$ converges uniformly, for the above z. Then the differentiation under the integral is allowed, and we obtain

$$\Gamma^{(n)}(z) = \int_{t=0}^{t=\infty} e^{-t}t^{z-1}(\ln t)^n\,\mathrm{d}t, \qquad \Re z > 0.$$

The analyticity of the Γ function then enables us to extend it to the left half-plane.

first variant: If $\Re z > 0$ and $n \in \mathbb{N}$, then we have the identity

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)\cdot\ldots\cdot z},$$

where the RHS is trivially an analytic function for $\Re z > -n$, provided that $z \notin \{0, -1, \ldots, -n+1\}$. Then this formula gives us the unique analytic extension of the Γ function to the left half-plane, assuming that the number n has been chosen larger enough.

second variant: We split the integral at $t = 1$:

$$\Gamma(z) = I_1(z) + I_2(z) := \int_{t=0}^{t=1} e^{-t} t^{z-1}\,\mathrm{d}t + \int_{t=1}^{t=\infty} e^{-t} t^{z-1}\,\mathrm{d}t, \qquad \Re z > 0.$$

The second integral I_2 converges uniformly for $\Re z \leq X < \infty$ and can be extended to the left half-plane without problems; it defines an analytic function there. The first integral converges I_1 uniformly for $\Re z \geq \delta > 0$. We expand the exponential function into a series:

$$I_1(z) = \int_{t=0}^{t=1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} t^{\nu+z-1}\,\mathrm{d}t, \qquad \Re z > 0.$$

Fix a z with $\Re z > 0$. Then the series converges uniformly for $0 < t \leq 1$, and we can commute integration and summation:

$$I_1(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}, \qquad \Re z > 0. \tag{1.1}$$

Fix a positive small ε, and define

$$\mathbb{C}_\varepsilon := \mathbb{C} \setminus \bigcup_{\nu=0}^{\infty} \{|z+\nu| < \varepsilon\}.$$

The series (1.1) converges uniformly in $\mathbb{C}_\varepsilon$, and each term of that series is holomorphic. Then also the limit is holomorphic, as can be seen from the Theorem of Morera. Consequently, I_1 can be extended analytically to $\mathbb{C}_\varepsilon$.

Proposition 1.2. *The Γ function can be analytically extended to $\mathbb{C} \setminus (-\mathbb{N}_0)$, and we have the representation*

$$\Gamma(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu} + \int_{t=1}^{t=\infty} e^{-t} t^{z-1}\,\mathrm{d}t.$$

In the points $z = -n$, the Gamma function has poles of first order with residue $\frac{(-1)^n}{n!}$.

1.2 The Weierstrass Product Representation

For $m, n \in \mathbb{N}_+$, we can write

$$\begin{aligned}
\Gamma(m) = (m-1)! &= \frac{(m+n)!}{m(m+1)\cdot\ldots\cdot(m+n)} = \frac{n!(n+1)\cdot\ldots\cdot(n+m)}{m(m+1)\cdot\ldots\cdot(m+n)} \\
&= \frac{n!n^m \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdot\ldots\cdot\left(1+\frac{m}{n}\right)}{m(m+1)\cdot\ldots\cdot(m+n)} \\
&= \lim_{n\to\infty} \frac{n!n^m}{m(m+1)\cdot\ldots\cdot(m+n)},
\end{aligned}$$

since the LHS does not depend on n. Observe that m could be complex in the RHS (assuming that the limit exists).

Definition 1.3 (Gauss)**.** *For $z \in \mathbb{C} \setminus (-\mathbb{N}_0)$, we define*

$$\Gamma_G(z) := \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdot\ldots\cdot(z+n)},$$

under the assumption that the limit exists.

Next, we show two things:

- the limit exists, for $z \in \mathbb{C} \setminus (-\mathbb{N}_0)$,
- $\Gamma_G = \Gamma$.

Lemma 1.4. *The limit in Definition 1.3 exists.*

Proof. For $z \in \mathbb{C}$ and $n \in \mathbb{N}_+$, we can write

$$\begin{aligned}\frac{z(z+1)\cdot\ldots\cdot(z+n)}{n!n^z} &= z\frac{z+1}{1}\cdot\frac{z+2}{2}\cdot\ldots\cdot\frac{z+n}{n}\cdot n^{-z}\\ &= z\left(1+\frac{z}{1}\right)\cdot\left(1+\frac{z}{2}\right)\cdot\ldots\cdot\left(1+\frac{z}{n}\right)n^{-z}\\ &= z\exp\left(z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)\right)\prod_{\nu=1}^{n}\left(1+\frac{z}{\nu}\right)\exp\left(-\frac{z}{\nu}\right).\end{aligned}$$

It is known that $\lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n)=\gamma=0.5772\ldots$, the Euler–Mascheroni constant. Therefore,

$$\lim_{n\to\infty} z\exp\left(z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)\right)=z\exp(\gamma z).$$

All that remains is to prove that $\prod_{\nu=1}^{n}(1+\frac{z}{\nu})\exp(-\frac{z}{\nu})$ has a limit, for $n\to\infty$.
Let $|z|\le N\in\mathbb{N}$, and suppose $n\ge 2N$. Trivially,

$$\prod_{\nu=1}^{n}\left(1+\frac{z}{\nu}\right)\exp\left(-\frac{z}{\nu}\right)=\prod_{\nu=1}^{2N-1}\left(1+\frac{z}{\nu}\right)\exp\left(-\frac{z}{\nu}\right)\prod_{\nu=2N}^{n}\left(1+\frac{z}{\nu}\right)\exp\left(-\frac{z}{\nu}\right),$$

with a first factor $\prod_{\nu=1}^{2N-1}\ldots$ which is entire in z and has simple zeroes for $z=-\nu$. Write the second factor as

$$\prod_{\nu=2N}^{n}\left(1+\frac{z}{\nu}\right)\exp\left(-\frac{z}{\nu}\right)=\exp\left(\sum_{\nu=2N}^{n}\log\left(1+\frac{z}{\nu}\right)-\frac{z}{\nu}\right),$$

where $\log w=\ln|w|+\mathrm{i}\arg w$, and $\arg w\in(-\pi,\pi)$. We have to show that

$$\lim_{n\to\infty}\sum_{\nu=2N}^{n}\left(\log\left(1+\frac{z}{\nu}\right)-\frac{z}{\nu}\right) \tag{1.2}$$

exists. But this is almost trivial: we know $|\frac{z}{\nu}|\le\frac{1}{2}$ and the expansion of the logarithm about the point 1:

$$\log(1+w)=w-\frac{w^2}{2}+\frac{w^3}{3}\mp,\qquad |w|<1.$$

Majorizing with the geometric series, we see that

$$\left|\log\left(1+\frac{z}{\nu}\right)-\frac{z}{\nu}\right|\le 2\left|\frac{z}{\nu}\right|^2\le 2\frac{N^2}{\nu^2}.$$

Therefore, the series in (1.2) converges absolutely and uniformly, for $|z|\le N$. The limit is then an analytic function.
Hence we conclude that

$$\frac{1}{\Gamma_G(z)}=ze^{\gamma z}\prod_{\nu=1}^{\infty}\left(1+\frac{z}{\nu}\right)\exp\left(-\frac{z}{\nu}\right),\qquad z\in\mathbb{C}.$$

This is the famous Weierstrass product for the Gamma function. □

Lemma 1.5. *If $z=x>0$, then $\Gamma(z)=\Gamma_G(z)$.*

Proof. Put $p = x + 1 + n$ with $n \in \mathbb{N}_0$, and let $t > 0$. Then we have

$$\exp\left(\frac{t}{p}\right) > 1 + \frac{t}{p} \Longrightarrow \exp(t) > \left(1 + \frac{t}{p}\right)^p \Longrightarrow \exp(-t) < \left(1 + \frac{t}{p}\right)^{-p}.$$

Plugging this estimate into the integral definition of Γ, we see that

$$\begin{aligned}\Gamma(x) &= \int_{t=0}^{t=\infty} e^{-t} t^{x-1}\,\mathrm{d}t < \int_{t=0}^{t=\infty} \left(1 + \frac{t}{p}\right)^{-p} t^{x-1}\,\mathrm{d}t && (\text{set } 1 + \tfrac{t}{p} = \tfrac{1}{\tau})\\ &= p^x \int_{\tau=0}^{\tau=1} \tau^{p-1-x}(1-\tau)^{x-1}\,\mathrm{d}\tau = (x+1+n)^x \int_{\tau=0}^{\tau=1} \tau^n (1-\tau)^{x-1}\,\mathrm{d}\tau\\ &= (x+1+n)^x \frac{n!}{x(x+1)\cdot\ldots\cdot(x+n)} && (\text{repeated partial integration}).\end{aligned}$$

We obtain for all $x > 0$ and all $n \in \mathbb{N}_0$:

$$\Gamma(x) < (x+1+n)^x \frac{n!}{x(x+1)\cdot\ldots\cdot(x+n)}.$$

Conversely, we have for $0 \le \tau < 1$ and $0 \le t < n$:

$$e^\tau \le \frac{1}{1-\tau} \Longrightarrow \exp\left(-\frac{t}{n}\right) \ge 1 - \frac{t}{n} \Longrightarrow e^{-t} \ge \left(1 - \frac{t}{n}\right)^n.$$

Plugging this into the integral definition of Γ, we see:

$$\begin{aligned}\Gamma(x) &= \int_{t=0}^{t=\infty} e^{-t} t^{x-1}\,\mathrm{d}t > \int_{t=0}^{t=n} \left(1 - \frac{t}{n}\right)^n t^{x-1}\,\mathrm{d}t && (\text{set } 1 - \tfrac{t}{n} = \tau)\\ &= n^x \int_{\tau=0}^{\tau=1} \tau^n (1-\tau)^{x-1}\,\mathrm{d}\tau = \frac{n^x n!}{x(x+1)\cdot\ldots\cdot(x+n)}.\end{aligned}$$

Both estimates together then yield:

$$\Gamma(x) \cdot \left(\frac{n}{n+x+1}\right)^x < \frac{n^x n!}{x(x+1)\cdot\ldots\cdot(x+n)} < \Gamma(x).$$

All that remains is to send n to ∞. □

1.3 Properties of the Gamma Function

Proposition 1.6. *For $z \in \mathbb{C} \setminus \mathbb{Z}$, the following identity holds:*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Sketch of proof. It suffices to prove the identity for $z = x \in (0,1)$. Then we have

$$\Gamma(x)\Gamma(1-x) = \int_{t=0}^{t=\infty} e^{-t} t^{x-1}\,\mathrm{d}t \cdot \int_{s=0}^{s=\infty} e^{-s} s^{-x}\,\mathrm{d}s = \int_0^\infty \int_0^\infty e^{-(t+s)} t^{x-1} s^{-x}\,\mathrm{d}s\,\mathrm{d}t.$$

Now we substitute:

$$u := t + s, \qquad v := \frac{t}{s}, \qquad s = \frac{u}{1+v}, \qquad t = \frac{uv}{1+v}, \qquad \mathrm{d}s\,\mathrm{d}t = \frac{u}{(1+v)^2}\,\mathrm{d}u\,\mathrm{d}v.$$

Then it follows that

$$\begin{aligned}\Gamma(1-x)\Gamma(x) &= \int_0^\infty \int_0^\infty e^{-u} \left(\frac{u}{1+v}\right)^{-x} \left(\frac{uv}{1+v}\right)^{x-1} \frac{u}{(1+v)^2}\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^\infty \int_0^\infty e^{-u} \frac{v^{x-1}}{1+v}\,\mathrm{d}u\,\mathrm{d}v = \int_{u=0}^{u=\infty} e^{-u}\,\mathrm{d}u \cdot \int_{v=0}^{v=\infty} \frac{v^{x-1}}{1+v}\,\mathrm{d}v\\ &= \int_{v=0}^{v=\infty} \frac{v^{x-1}}{1+v}\,\mathrm{d}v.\end{aligned}$$

You will find the value $\frac{\pi}{\sin(\pi x)}$ for the last integral, if you evaluate it by the residue theorem:

Choose a small $\varepsilon > 0$ and a big $R > 0$. Consider a curve C like this: a straight line from $\varepsilon + \mathrm{i}0$ to $R + \mathrm{i}0$, then a big circle with radius R to $R - \mathrm{i}0$, then a straight line to $\varepsilon - \mathrm{i}0$, then a small circle with radius ε to $\varepsilon + \mathrm{i}0$. The value of the integral along that curve is $-2\pi\mathrm{i}\exp(\pi\mathrm{i}x)$, by the residue theorem. For $x \in (\varepsilon + \mathrm{i}0, R + \mathrm{i}0)$, you have $\frac{v^{x-1}}{1+v} = \frac{|v|^{x-1}}{1+|v|}$, and for $x \in (\varepsilon - \mathrm{i}0, R - \mathrm{i}0)$, you have $\frac{v^{x-1}}{1+v} = \frac{(|v|\exp(2\pi\mathrm{i}))^{x-1}}{1+|v|}$. The integrals over the circles vanish for $\varepsilon \to +0$ and $R \to +\infty$. And so on. □

Corollary 1.7. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Corollary 1.8. *From the above reflection formula and the Weierstrass product representation, we deduce that*

$$\sin z = z \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{\nu^2\pi^2}\right).$$

Proposition 1.9. *For $z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$, the following formula holds:*

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi}2^{-2z+1}\Gamma(2z).$$

Proof. We use the Gauss definition:

$$\begin{aligned}
\frac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)} &= 2^{2z-1}\frac{\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdot\ldots\cdot(z+n)}\lim_{n\to\infty}\frac{n!n^{z+1/2}}{(z+1/2)\cdot\ldots\cdot(z+1/2+n)}}{\lim_{2n\to\infty}\frac{(2n)!(2n)^{2z}}{2z(2z+1)\cdot\ldots\cdot(2z+2n)}} \\
&= 2^{2z-1}\lim_{n\to\infty}\frac{n!n^z n!n^{z+1/2}(2z)(2z+1)\cdot\ldots(2z+2n)}{z(z+1)\cdot\ldots\cdot(z+n)(z+1/2)\cdot\ldots\cdot(z+1/2+n)(2n)!(2n)^{2z}} \\
&= 2^{-1}\lim_{n\to\infty}\frac{n!n!n^{1/2}}{(2n)!}\frac{(2z)(2z+1)\cdot\ldots\cdot(2z+2n)2^{2n+2}}{(2z)(2z+2)(2z+4)\cdot\ldots\cdot(2z+2n)(2z+1)(2z+3)\cdot\ldots\cdot(2z+1+2n)} \\
&= \lim_{n\to\infty}\frac{n!n!n^{1/2}}{(2n)!}\frac{2^{2n+1}}{(2z+1+2n)} \\
&= \lim_{n\to\infty}\frac{n!n!2^{2n+1}}{(2n)!\sqrt{n}}\frac{n}{(2z+1+2n)} \\
&= \lim_{n\to\infty}\frac{n!n!2^{2n}}{(2n)!\sqrt{n}},
\end{aligned}$$

and this does not depend on z. Hence it is a constant, which can determined by setting $z = \frac{1}{2}$. □

1.4 Asymptotic Expansions

We start with some heuristics.

The trapezoidal rule will give us a guess at the behavior of $\Gamma(x)$ for large positive x:

$$\begin{aligned}
\int_1^n \ln x \,\mathrm{d}x &\approx \frac{1}{2}\ln 1 + \ln 2 + \ln 3 + \ldots + \ln(n-1) + \frac{1}{2}\ln n \\
&= \ln((n-1)!) + \frac{1}{2}\ln n \\
&= \ln\Gamma(n) + \frac{1}{2}\ln n.
\end{aligned}$$

On the other hand,

$$\int_1^n \ln x \,\mathrm{d}x = (x\ln x - x)\Big|_{x=1}^{x=n} = n\ln n - n + 1,$$

which suggests something like

$$\ln\Gamma(x) \approx \left(x - \frac{1}{2}\right)\ln x - x + c, \qquad x \to +\infty.$$

In order to guess c, we exploit the doubling formula:

$$\begin{aligned}
&\ln\Gamma(z) + \ln\Gamma\left(z + \frac{1}{2}\right) = \frac{1}{2}\ln\pi + (-2z+1)\ln 2 + \ln\Gamma(2z),\\
&\left(z - \frac{1}{2}\right)\ln z - z + c + z\ln\left(z + \frac{1}{2}\right) - \left(z + \frac{1}{2}\right) + c\\
&\qquad \stackrel{!}{=} \frac{1}{2}\ln\pi + (1-2z)\ln 2 + \left(2z - \frac{1}{2}\right)\ln(2z) - 2z + c,\\
&z\ln z - \frac{1}{2}\ln z - 2z - \frac{1}{2} + 2c + z\ln\left(z\frac{1}{2}\right)\\
&\qquad \stackrel{!}{=} \frac{1}{2}\ln\pi - 2z\ln 2 + \ln 2 + \left(2z - \frac{1}{2}\right)\ln 2 + 2z\ln z - \frac{1}{2}\ln z - 2z + c,\\
&-\frac{1}{2} + c \stackrel{!}{=} \frac{1}{2}\ln\pi + \frac{1}{2}\ln 2 + z\ln\left(\frac{z}{z+1/2}\right) = \frac{1}{2}\ln(2\pi) + z\left(\frac{-1/2}{z+1/2}\right) + \mathfrak{O}(|z|^{-1}).
\end{aligned}$$

This suggests $c = \frac{1}{2}\ln(2\pi)$. Then our conjecture is the following:

$$\begin{aligned}
&\ln\Gamma(z) \approx \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi), \qquad z \to \infty,\\
&\Gamma(z) \approx z^{z-1/2}e^{-z}\sqrt{2\pi}, \qquad z \to \infty.
\end{aligned}$$

Example: *Take* $z = 11$*. Then* $\Gamma(11) = 10! = 3628800$*, and* $z^{z-1/2}e^{-}z\sqrt{2\pi} = 3601420.459$*. Take* $z = 41$*. Then* $\Gamma(41) = 40! = 8.1591528324789773E + 47$ *and* $z^{z-1/2}e^{-z}\sqrt{2\pi} = 8.1425863585161782E + 47$*.*

Of course, we should make sense of the "$\approx$". To this end, we define:

$$\mu(z) := \ln\Gamma(z) - \left(z - \frac{1}{2}\right)\ln z + z - \frac{1}{2}\ln(2\pi), \qquad z \in \mathbb{C} \setminus (-\mathbb{R}_{0,+}).$$

Our goal is to show that $|\mu(x)| \le \frac{C}{x}$ for large positive $x \in \mathbb{R}$.

Lemma 1.10. *For* $x \in (0,1)$*, the limit* $\lim_{n\to\infty}\mu(x+n)$ *exists.*

Proof. Define a function g by

$$g(x) := \mu(x) - \mu(x+1).$$

The recursion formula $\Gamma(x+1) = x\Gamma(x)$ then yields

$$g(x) = \left(x + \frac{1}{2}\right)\ln\left(1 + \frac{1}{x}\right) - 1.$$

By the series expansion $\ln(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \mathfrak{O}(\varepsilon^3)$, we then obtain

$$|g(x)| \le \frac{C}{|x|^2}, \qquad x \ge 2.$$

Now we have trivially

$$\mu(x+n) = \mu(x) - \sum_{\nu=0}^{n-1} g(x+\nu),$$

and the RHS has a limit for $n \to \infty$. $\square$

Lemma 1.11. *If $x \in (0,1)$, then $\lim_{n\to\infty}(\mu(x+n) - \mu(n)) = 0$. Here, $n \in \mathbb{N}$, as always.*

Proof. We compute:

$$\begin{aligned}
\mu(x+n) - \mu(n) &= \ln\left(\frac{\Gamma(x+n)}{\Gamma(n)}\right) - \left(x+n-\frac{1}{2}\right)\ln(x+n) + \left(n-\frac{1}{2}\right)\ln n + x \\
&= \ln\left(\frac{\Gamma(x+n)}{\Gamma(n)}\right) - x\ln(x+n) + \left(n-\frac{1}{2}\right)\ln\left(\frac{n}{x+n}\right) + x \\
&= \ln\left(\frac{\Gamma(x)x(x+1)\cdot\ldots\cdot(x+n-1)}{\Gamma(n)}\right) - x\ln(x+n) + \left(n-\frac{1}{2}\right)\frac{-x}{x+n} + x + \mathfrak{O}(n^{-1}) \\
&= \ln\left(\frac{\Gamma(x)x(x+1)\cdot\ldots\cdot(x+n-1)}{\Gamma(n)(n-1)^x}\right) + x\ln\left(\frac{n-1}{n+x}\right) + \mathfrak{O}(n^{-1}) \\
&= \ln\left(\frac{\Gamma(x)x(x+1)\cdot\ldots\cdot(x+n-1)}{(n-1)!(n-1)^x}\right) + \mathfrak{O}(n^{-1}),
\end{aligned}$$

and the first item converges to zero for $n \to \infty$, as follows from the GAUSS definition. □

Lemma 1.12. *We have $\lim_{x\to+\infty}\mu(x) = 0$.*

Proof. The existence of the limit follows from the last two lemmas. Making use of the doubling formula once more, we check that the value of the limit must be zero. □

Lemma 1.13. *For $x \in \mathbb{R}$, $x \geq 2$, the estimate $|\mu(x)| \leq \frac{C}{x}$ holds.*

Proof. We have

$$\mu(x) = \sum_{\nu=0}^{\infty} g(x+\nu)$$

and $|g(y)| \leq \frac{C}{y^2}$ for $y \geq 2$. □

Proposition 1.14 (STIRLING's formula). *The is a constant $C \in \mathbb{R}$ such that for $x \geq 2$ the following estimate holds:*

$$\left|\frac{\Gamma(x)}{x^{x-1/2}e^{-x}\sqrt{2\pi}} - 1\right| \leq \frac{C}{x}.$$

The proof should be obvious.

We can refine the above result a bit. First, we should introduce the BERNOULLI numbers and BERNOULLI polynomials.

The Bernoulli polynomials $B_n = B_n(x)$ are defined by

$$\begin{aligned}
&B_0(x) := 1, \\
&B_n'(x) = nB_{n-1}(x), \qquad n \geq 1, \\
&\int_{x=0}^{x=1} B_n(x)\,\mathrm{d}x = 0, \qquad n \geq 1.
\end{aligned}$$

For later use, we note that $B_1(x) = x - \frac{1}{2}$. The Bernoulli numbers B_n are then determined via

$$B_n := B_n(0).$$

For the convenience of the reader, we list some Bernoulli numbers:

$$B_0 = 1, \qquad B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42}, \qquad B_8 = -\frac{1}{30}, \qquad B_{10} = \frac{5}{66}.$$

The Bernoulli numbers with odd index vanish, except $B_1 = -\frac{1}{2}$.

We have the general relations

$$
\begin{aligned}
&B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j}, \qquad k \geq 1,\\
&B_k(0) = B_k(1), \qquad k \geq 2,\\
&\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0, \qquad n \geq 1.
\end{aligned} \tag{1.3}
$$

The last relation can be used for the recursive computation of the Bernoulli numbers. It turns out that $B_{2n+1} = 0$ for $n \geq 1$. They are famous as ingredients in important Taylor series:

$$
\begin{aligned}
&\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n,\\
&\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} 4^n (4^n - 1) x^{2n-1},\\
&\cot x = \frac{1}{x} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} 4^n x^{2n}.
\end{aligned}
$$

Moreover, we have the nice formula

$$
1^p + 2^p + \cdots + n^p = \frac{n^{p+1}}{p+1} + \frac{1}{2} n^p + \sum_{k=2}^{p} \frac{B_k}{k} \binom{p}{k-1} n^{p+1-k}, \qquad p \in \mathbb{N}_+.
$$

Now we can describe the function μ in another way:

$$
\begin{aligned}
\mu(x) &= \sum_{\nu=0}^{\infty} g(x+\nu) = \sum_{\nu=0}^{\infty} \int_{t=0}^{t=1} \frac{\frac{1}{2} - t}{x + \nu + t} \,\mathrm{d}t = -\sum_{\nu=0}^{\infty} \int_{t=0}^{t=1} (x+\nu+t)^{-1} B_1(t) \,\mathrm{d}t\\
&= -\frac{1}{2} \sum_{\nu=0}^{\infty} \int_{t=0}^{t=1} (x+\nu+t)^{-1} B_2'(t) \,\mathrm{d}t\\
&= -\frac{1}{2} \sum_{\nu=0}^{\infty} \left((x+\nu+t)^{-1} B_2(t) \Big|_{t=0}^{t=1} + \int_{t=0}^{t=1} (x+\nu+t)^{-2} B_2(t) \,\mathrm{d}t \right)\\
&= \frac{1}{2} \frac{B_2}{x} - \frac{1}{2} \sum_{\nu=0}^{\infty} \int_{t=0}^{t=1} (x+\nu+t)^{-2} B_2(t) \,\mathrm{d}t.
\end{aligned}
$$

Now we introduce a 1–periodic function $P_n \colon \mathbb{R} \to \mathbb{R}$ by the formula

$$
P_n(t) := B_n(t), \qquad 0 \leq t \leq 1.
$$

Then we can write

$$
\mu(x) = \frac{1}{12x} - \frac{1}{2} \int_{t=0}^{\infty} (x+\nu+t)^{-2} P_2(t) \,\mathrm{d}t.
$$

We have the recursion $P_n'(t) = nP_{n-1}(t)$. By partial integration, it then follows that

$$
-\frac{1}{k} \int_{t=0}^{t=\infty} \frac{P_k(t)}{(x+t)^k} \,\mathrm{d}t = \frac{B_{k+1}}{k(k+1)} \frac{1}{x^k} - \frac{1}{k+1} \int_{t=0}^{t=\infty} \frac{P_{k+1}(t)}{(x+t)^{k+1}} \,\mathrm{d}t.
$$

The numbers B_3, B_5, B_7, ... vanish; hence we conclude that

$$
\mu(x) = \sum_{\nu=1}^{n} \frac{B_{2\nu}}{(2\nu-1)2\nu} \frac{1}{x^{2\nu-1}} - \frac{1}{2n+1} \int_{t=0}^{t=\infty} \frac{P_{2n+1}(t)}{(x+t)^{2n+1}} \,\mathrm{d}t.
$$

In [12], the following remainder estimates can be found:

For all $n \in \mathbb{N}_+$, there is an M_n, such that for all $z \in \mathbb{C} \setminus (-\mathbb{R}_{+,0})$, the following estimate hold:

$$\left|\mu(z) - \sum_{\nu=1}^{n} \frac{B_{2\nu}}{(2\nu-1)2\nu} \frac{1}{z^{2\nu-1}}\right| \leq \frac{M_n}{\cos^{2n+2}(\varphi/2)} \frac{1}{|z|^{2n+1}}, \qquad \arg z = \varphi \in (-\pi, \pi).$$

In the right half-plane, this estimate can sometimes be improved:

$$\left|\mu(z) - \sum_{\nu=1}^{n} \frac{B_{2\nu}}{(2\nu-1)2\nu} \frac{1}{z^{2\nu-1}}\right| \leq \frac{B_{2n+2}}{(2n+1)(2n+2)} \cdot \frac{1}{|z|^{2n+1}}, \qquad |\arg z| \leq \frac{\pi}{4}.$$

Example: *Take $n = 4$ and $z = x \in \mathbb{R}$, $x \geq 10$. Then*

$$\frac{B_{2n+2}}{(2n+1)(2n+2)} \cdot \frac{1}{|z|^{2n+1}} \leq 8.417 \cdot 10^{-13}.$$

Remark 1.15. *It turns out that the recursion formula (1.3) is not numerically stable; and other formulas of B_{2n} for large n are advisable. One of these formulas follows from the Fourier series*

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2n}} = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}\left(\frac{x}{2\pi}\right)$$

upon setting $x = 0$:

$$B_{2n} = (-1)^{n+1} 2 \left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots\right) \prod_{m=1}^{2n} \left(\frac{m}{2\pi}\right), \qquad n = 1, 2, 3, \dots.$$

For $n = 17$, we have $2^{-2n} = 5.820766091346741 \cdot 10^{-11}$ and $3^{-2n} = 5.9962169748381 \cdot 10^{-17}$.

Now is a good point to introduce the **Euler summation**: take a function f, sufficiently smooth. Then we can write

$$\begin{aligned}
\int_{t=0}^{t=N} f(t)\,\mathrm{d}t &= \sum_{n=0}^{N-1} \int_{t=0}^{t=1} f(t+n)\,\mathrm{d}t = \sum_{n=0}^{N-1} \int_{t=0}^{t=1} f(t+n) B_1'(t)\,\mathrm{d}t \\
&= \sum_{n=0}^{N-1} \left(f(t+n) B_1(t) \Big|_{t=0}^{t=1} - \int_{t=0}^{t=1} f'(t+n) B_1(t)\,\mathrm{d}t \right) \\
&= \sum_{n=0}^{N-1} \frac{f(1+n) + f(n)}{2} - \frac{1}{2} \sum_{n=0}^{N-1} \int_{t=0}^{t=1} f'(t+n) B_2'(t)\,\mathrm{d}t \\
&= \frac{1}{2} f(0) + \sum_{n=1}^{N-1} f(n) + \frac{1}{2} f(N) - \frac{1}{2} \sum_{n=0}^{N-1} \left(f'(t+n) B_2(t) \Big|_{t=0}^{t=1} - \int_{t=0}^{t=1} f''(t+n) B_2(t)\,\mathrm{d}t \right) \\
&= \frac{1}{2} f(0) + \sum_{n=1}^{N-1} f(n) + \frac{1}{2} f(N) + \frac{B_2}{2} f'(t) \Big|_{t=0}^{t=N} - \frac{1}{2} \int_{t=0}^{t=N} f''(t) P_2(t)\,\mathrm{d}t \\
&= \frac{1}{2} f(0) + \sum_{n=1}^{N-1} f(n) + \frac{1}{2} f(N) + \left(\frac{B_2}{2!} f'(t) - \frac{B_3}{3!} f''(t) \right) \Big|_{t=0}^{t=N} + \frac{1}{3!} \int_{t=0}^{t=N} f'''(t) P_3(t)\,\mathrm{d}t.
\end{aligned}$$

Continuing in this style we find

$$\sum_{n=1}^{N} f(n) = \int_{t=0}^{t=N} f(t)\,\mathrm{d}t - \left(\frac{B_1}{1!} f(t) + \frac{B_2}{2!} f'(t) + \frac{B_4}{4!} f'''(t) + \dots + \frac{B_{2p}}{(2p)!} f^{(2p-1)}(t) \right) \Big|_{t=0}^{t=N} + R_p,$$

$$R_p = \frac{1}{(2p+1)!} \int_{t=0}^{t=N} f^{(2p+1)}(t) P_{2p+1}(t)\,\mathrm{d}t.$$

This will enable us to efficiently compute $\sum_{n=0}^{N} f(n)$, provided that we are able to integrate $\int_{t=0}^{t=N} f(t)\,\mathrm{d}t$. An important example is the zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. This function can be computed quite easily by Euler summation. We will come back to the zeta function in connection with the psi function below.

1.5 The Beta Function and the Psi Function

Definition 1.16 (Beta function). *For $\Re u > 0$ and $\Re v > 0$, we define the so-called* Beta function

$$B(u,v) = \int_{t=0}^{t=1} t^{u-1}(1-t)^{v-1}\,\mathrm{d}t.$$

Proposition 1.17. *For $u, v \in \mathbb{C}$ with $\Re u > 0$ and $\Re v > 0$, we have the identity*

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

(No proof.)

This result directly enables us to define the Beta function for $u, v \in \mathbb{C} \setminus (-\mathbb{N}_0)$. The resulting function is analytic everywhere where it is defined.

Definition 1.18 (Psi function). *For $z \in \mathbb{C} \setminus (-\mathbb{N}_0)$, we define*

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Trivially, we have $\psi(z+1) = \psi(z) + \frac{1}{z}$.

Next, we derive a power series for the psi function near $z = 1$:

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z e^{\gamma z} \prod_{\nu=1}^{\infty} \left(\left(1 + \frac{z}{\nu}\right) e^{-z/\nu} \right), \\
-\log\Gamma(z) &= \log(z) + \gamma z + \sum_{\nu=1}^{\infty} \left(\log\left(1 + \frac{z}{\nu}\right) - \frac{z}{\nu} \right), \\
-\psi(z) &= \frac{1}{z} + \gamma + \sum_{\nu=1}^{\infty} \frac{1}{\nu} \left(\frac{1}{1 + \frac{z}{\nu}} - 1 \right). \qquad (1.4)
\end{aligned}$$

Note that the last series is uniformly convergent for $|z| < 1$; therefore, the term-wise differentiation is allowed. Then,

$$-\psi(z) = \frac{1}{z} + \gamma + \sum_{\nu=1}^{\infty} \frac{1}{\nu} \sum_{n=1}^{\infty} \left(\frac{-z}{\nu} \right)^n, \qquad |z| < 1.$$

The absolute convergence allows to exchange the two summations. Then we get, with $\zeta(n) = \sum_{\nu=1}^{\infty} \nu^{-n}$,

$$\begin{aligned}
\psi(z+1) &= -\gamma - \sum_{n=1}^{\infty} (-z)^n \zeta(n+1), \\
\log\Gamma(z+1) &= -\gamma z + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) z^n, \qquad |z| < 1.
\end{aligned}$$

Gauss used a similar series 1812 to give a list of the values of $\Gamma(1+x)$, where $0 \le x \le 1$, with step-size 0.01.

For later use, we present some representations of the psi function:

Proposition 1.19. 1. *If $z \neq 0, -1, -2, \ldots$, then*

$$\psi(z) = -\frac{1}{z} - \gamma + \sum_{\nu=1}^{\infty} \frac{z}{\nu(z+\nu)}.$$

2. *If $\Re z > 0$, then*

$$\psi(z) = \int_{t=0}^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right) \mathrm{d}t.$$

For the proof, we make use of a simple lemma:

Lemma 1.20. *1. If $\Re p > 0$, then*

$$\frac{1}{p} = \int_{t=0}^{\infty} e^{-pt}\,\mathrm{d}t.$$

2. If $x > 1$, then

$$\ln x = \int_{t=0}^{\infty} \frac{e^{-t} - e^{-xt}}{t}\,\mathrm{d}t.$$

Proof. The first part is trivial, the second follows from

$$\ln x = \int_{p=1}^{x} \frac{1}{p}\,\mathrm{d}p = \int_{p=1}^{x}\int_{t=0}^{\infty} e^{-pt}\,\mathrm{d}t\,\mathrm{d}p = \int_{t=0}^{\infty}\int_{p=1}^{x} e^{-pt}\,\mathrm{d}p\,\mathrm{d}t = \int_{t=0}^{\infty} \frac{e^{-t} - e^{-xt}}{t}\,\mathrm{d}t.$$

□

Proof of Proposition 1.19. The first part is just a reformulation of (1.4).

The formula of the second part can be obtained as follows:

$$\begin{aligned}
\psi(z) &= -\frac{1}{z} - \gamma + \sum_{\nu=1}^{\infty} \frac{z}{\nu(z+\nu)} \\
&= -\frac{1}{z} - \left(\lim_{n\to\infty}\left(\sum_{\nu=1}^{n}\frac{1}{\nu} - \ln n\right)\right) + \sum_{\nu=1}^{\infty}\frac{z}{\nu(z+\nu)} \\
&= -\frac{1}{z} + \lim_{n\to\infty}\left(\ln n - \sum_{\nu=1}^{n}\frac{1}{\nu} + \sum_{\nu=1}^{n}\frac{z}{\nu(z+\nu)}\right) \\
&= -\frac{1}{z} + \lim_{n\to\infty}\left(\ln n - \sum_{\nu=1}^{n}\frac{1}{z+\nu}\right) = \lim_{n\to\infty}\left(\ln n - \sum_{\nu=0}^{n}\frac{1}{z+\nu}\right) \\
&= \lim_{n\to\infty}\left(\int_{t=0}^{\infty}\frac{e^{-t}-e^{-nt}}{t}\,\mathrm{d}t - \int_{t=0}^{\infty}\sum_{\nu=0}^{n} e^{-(z+\nu)t}\,\mathrm{d}t\right) \\
&= \lim_{n\to\infty}\left(\int_{t=0}^{\infty}\frac{e^{-t}-e^{-nt}}{t}\,\mathrm{d}t - \int_{t=0}^{\infty} e^{-zt}\frac{1-e^{-(n+1)t}}{1-e^{-t}}\,\mathrm{d}t\right) \\
&= \lim_{n\to\infty}\left(\int_{t=0}^{\infty}\left(\frac{e^{-t}}{t} - \frac{e^{-nt}}{t} - \frac{e^{-zt}}{1-e^{-t}} + \frac{e^{-zt}e^{-(n+1)t}}{1-e^{-t}}\right)\mathrm{d}t\right) \\
&= \lim_{n\to\infty}\left(\int_{t=0}^{\infty}\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} - e^{-nt}\left[\frac{1}{t} - \frac{e^{-(z+1)t}}{1-e^{-t}}\right]\mathrm{d}t\right).
\end{aligned}$$

We observe that the term $[\dots]$ is bounded over $\mathbb{R}_+$, which enables us to apply the theorem of Lebesgue.

□

Chapter 2

Homogeneous Linear Differential Equations

2.1 Existence and Uniqueness

We recall from the theory of real systems of differential equations:

Let $I = \{t \in \mathbb{R} \colon |t - t_0| \le R\}$ be an interval on $\mathbb{R}$, and let $A = A(t)$ be a continuous matrix function, $A \in C(I; \mathbb{R}^{n \times n})$. Then the initial value problem

$$\begin{cases} y'(t) = A(t)y(t) \\ y(t_0) = y_0 \end{cases}$$

has a unique solution $y \in C^1(I; \mathbb{R}^n)$. Note that the solution y exists on the whole interval I.

Now we consider a similar situation in the complex plane.

Proposition 2.1. *Let $z_0 \in \mathbb{C}$, $K = K_R(z_0) = \{z \in \mathbb{C} \colon |z - z_0| \le R\}$, and $A = A(z)$ be a function from $C(K, \mathbb{C}^{n \times n})$, holomorphic in K. Let $u_0 \in \mathbb{C}^n$ be a given complex vector. Then the initial value problem*

$$\begin{cases} u'(z) = A(z)u(z) \\ u(z_0) = u_0 \end{cases}$$

has a unique solution $u \in C(K; \mathbb{C}^n)$. This function is holomorphic in K.

Sketch of proof. It is very similar to the real case. Define a mapping $u \mapsto \tilde{u}$ by

$$\tilde{u} = \tilde{u}(z) := u_0 + \int_{\zeta = z_0}^{\zeta = z} A(\zeta)u(\zeta)\, \mathrm{d}\zeta,$$

where the integral is taken over any curve in K connecting z and z_0, for instance, the straight line. The proof is complete if you can show that this mapping $u \mapsto \tilde{u}$ has a fixed point. For details, see the real case. □

Note that the solution u is holomorphic in the whole ball $K = K_R(z_0)$.

As an application, consider the second order scalar differential equation

$$\begin{cases} w''(z) + p(z)w'(z) + q(z)w(z) = 0, \\ w(z_0) = w_0, \qquad w'(z_0) = w_1. \end{cases} \tag{2.1}$$

It is standard to transform it into a first order system upon setting $u = (u_1, u_2)^T = (w, w')^T$:

$$u'(z) = \begin{pmatrix} 0 & 1 \\ -q(z) & -p(z) \end{pmatrix} u(z), \qquad u(z_0) = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

We see that the initial value problem (2.1) has a unique holomorphic solution w in K, provided that the functions p and q are holomorphic in the ball K.

The concepts “linear independence” , “fundamental system” or “Wronski determinant” can be defined as in the real case.

Now we consider differential equations in an arbitrary domain G, under the assumption that the functions p and q are holomorphic in G.

Proposition 2.2. *If p and q are holomorphic in G, and $z_0 \in G$, then a solution w to (2.1) exists in a ball $K_R(z_0) \subset G$. This solution can be holomorphically extended along any path that remains in G; and any such analytic continuation is again a solution to (2.1).*

The proof should be obvious.

Remark 2.3.

1. *If G is simply connected, then different analytic continuations coincide, by the monodromy theorem.*
2. *If w_1 and w_2 form a fundamental system in $K_R(z_0)$, then also their analytic continuations form a fundamental system.*
3. *If p and q are rational functions, then any solution can be extended along any path that avoids the poles of p and q.*

2.2 Solutions Near Isolated Singularities of p and q

Now we are interested in the equation

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0, \qquad 0 < |z - z_0| < R, \tag{2.2}$$

where p and q are holomorph for $0 < |z - z_0| < R$:

$$p(z) = \sum_{\nu=-\infty}^{+\infty} a_\nu (z - z_0)^\nu, \qquad q(z) = \sum_{\nu=-\infty}^{+\infty} b_\nu (z - z_0)^\nu, \qquad 0 < |z - z_0| < R.$$

As an example, we take

$$w''(z) + \frac{1}{2z} w'(z) = 0, \qquad p(z) = \frac{1}{2z}, \quad q(z) = 0, \qquad z_0 = 0, \quad R = \infty.$$

It is easy to check that a fundamental system of solutions is given by $w_1(z) = 1$, $w_2(z) = \sqrt{z}$. Without loss of generality, we can choose the main branch of the square root function, with cut on $\mathbb{R}_-$.

The cut plane is a simply connected domain. If we choose a small circle about a point $a + bi$, with a, b negative, then w_1 and w_2 are a fundamental system of the differential equation in this ball. We can continue the solution analytically, going around the origin $z_0 = 0$ counterclockwise, until we arrive at the “upper bank” of the “river” $\mathbb{R}_-$. The extensions of w_1 and w_2 are 1 and $\sqrt{z}$. If we cross the cut and continue the solutions analytically, then we obtain extensions $w_1^+ = 1$ and $w_2^+(z) = -\sqrt{z}$.

We will now extend these ideas to the general case (2.2), and will find the following description of the solutions:

Proposition 2.4. *Let the functions p and q be holomorphic for $0 < |z-z_0| < R$. Then the equation (2.2) has a fundamental system of solutions (w_I, w_{II}) of the following form:*

$$w_I(z) = (z - z_0)^{\varrho_1} \sum_{\nu=-\infty}^{\infty} c_\nu^I (z - z_0)^\nu,$$

$$w_{II}(z) = (z - z_0)^{\varrho_2} \sum_{\nu=-\infty}^{\infty} c_\nu^{II} (z - z_0)^\nu + a w_I(z) \cdot \log(z - z_0).$$

If $\varrho_1 - \varrho_2 \notin \mathbb{Z}$, then $a = 0$.

Remark 2.5. *1. This is just an existence result.*

2. The exponents ε_α and the coefficients c_ν^α can be computed by comparing equal powers of $z - z_0$. In general, this is a nonlinear system for infinitely many unknowns, which cannot be solved. However, in almost all relevant cases, this system is solvable step by step.

Proof. We choose a ray starting in z_0, and cut the plane along this ray. Then we have a simply connected domain, in which the coefficients p and q are holomorphic functions. Hence we have a fundamental system (w_1, w_2) of solutions to (2.2). Every solution w can be written as linear combination of w_1 and w_2:

$$w = w(z) = C_1 w_1(z) + C_2 w_2(z) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix}.$$

If we extend these solutions crossing the ray (counter-clockwise), we obtain new solutions w_1^+, w_2^+, and w^+. They can be written in terms of w_1 and w_2:

$$\begin{pmatrix} w_1^+(z) \\ w_2^+(z) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix},$$
$$w^+(z) = C_1 w_1^+(z) + C_2 w_2^+(z) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix}.$$

The functions w_1^+ and w_2^+ must be linearly independent. Consequently,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0.$$

Now we are looking for solutions w_I and w_{II} with $w_I^+ = \lambda_1 w_I$ and $w_{II}^+ = \lambda_2 w_{II}$, for some λ_α. We make the ansatz $w_\alpha = C_1^\alpha w_1 + C_2^\alpha w_2$, and deduce that

$$\begin{pmatrix} C_1^\alpha & C_2^\alpha \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \lambda_\alpha \begin{pmatrix} C_1^\alpha & C_2^\alpha \end{pmatrix}.$$

Therefore, λ_1 and λ_2 are the eigenvalues of the matrix

$$A = \begin{pmatrix} a_{12} & a_{21} \\ a_{12} & a_{22} \end{pmatrix},$$

and $(C_1^\alpha, C_2^\alpha)^T$ are the eigenvectors to λ_α.

- The λ_α are not zero, since $\det A \neq 0$.
- The λ_α do not depend on the fundamental system (w_1, w_2), because similar matrices have the same eigenvalues.

Now two cases are possible.

Case 1: $\lambda_1 \neq \lambda_2$**.**

Then we have functions w_I and w_{II} with $w_I^+ = \lambda_1 w_I$ and $w_{II}^+ = \lambda_2 w_{II}$. These functions are linearly independent, since eigenvectors $(C_1, C_2)^T$ to different eigenvalues are linearly independent. Now we define

$$\varrho_\alpha := \frac{1}{2\pi \mathrm{i}} \log \lambda_\alpha, \qquad \alpha = 1, 2.$$

We can choose any branch of the logarithm. Choosing another branch will add an integer constant to ϱ_α. Consider the function $z \mapsto (z - z_0)^{\varrho_\alpha}$, defined in $\mathbb{C}$ with a cut along a ray starting from z_0. From $(z - z_0)^\varrho = \exp(\varrho \log(z - z_0))$, we see that analytic continuation over the cut will give the value $\lambda_\alpha (z - z_0)^{\varrho_\alpha}$. Then we conclude that the functions

$$\frac{w_I(z)}{(z - z_0)^{\varrho_1}}, \qquad \frac{w_{II}(z)}{(z - z_0)^{\varrho_2}}$$

do not jump if we cross the cut. Hence, they are analytic in the annular domain $\{z\colon 0 < |z-z_0| < R\}$. Consequently, they can be written as a Laurent series, and we obtain the desired representations

$$w_I(z) = (z-z_0)^{\varrho_1} \sum_{\nu=-\infty}^{\infty} c_\nu^I (z-z_0)^\nu,$$
$$w_{II}(z) = (z-z_0)^{\varrho_2} \sum_{\nu=-\infty}^{\infty} c_\nu^{II} (z-z_0)^\nu.$$

Remark 2.6. *1. A different choice of ϱ_α induces an index shift in the series $\sum_{\nu=-\infty}^{\infty}$.*

2. A different choice of $(z-z_0)^{\varrho_\alpha}$ induces a multiplication of the coefficients c_ν^α by a constant.

Case 2: $\lambda_1 = \lambda_2$.

As above, we find a solution $w_I(z) = (z-z_0)^{\varrho_1} \sum_{\nu=-\infty}^{\infty} c_\nu^I (z-z_0)^\nu$. Now we forget the old functions w_1 and w_2.

Take another arbitrary solution w_2, linearly independent to w_I. We choose (w_I, w_2) as fundamental system. By analytic continuation, we get functions w_I^+ and w_2^+ with

$$w_I^+ = \lambda_1 w_I + 0 w_2,$$
$$w_2^+ = a_{21} w_I + a_{22} w_2.$$

The number λ_1 must be a double eigenvalue of the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ a_{21} & a_{22} \end{pmatrix},$$

which implies $a_{22} = \lambda_1$. Then we have $w_2^+ = a_{21} w_I + \lambda_1 w_2$.

The function w_2/w_I has the following analytic continuation over the cut:

$$\frac{w_2^+}{\lambda_1 w_I} = \frac{a_{21} w_I + \lambda_1 w_2}{\lambda_1 w_I} = \frac{a_{21}}{\lambda_1} + \frac{w_2}{w_I}.$$

Then the function

$$\frac{w_2}{w_I} - \frac{a_{21}}{\lambda_1} \frac{1}{2\pi \mathrm{i}} \log(z-z_0)$$

is continuous when we cross the cut; hence it can be written as a Laurent series. Put $a = \frac{a_{21}}{\lambda_1}\frac{1}{2\pi \mathrm{i}}$. Then

$$\frac{w_2(z)}{w_I(z)} - a \log(z-z_0) = \sum_{\nu=-\infty}^{\infty} \tilde{c}_\nu (z-z_0)^\nu, \qquad 0 < |z-z_0| < R.$$

Multiplying with w_I, we get

$$w_2(z) = (z-z_0)^{\varrho_1} \sum_{\nu=-\infty}^{\infty} c_\nu^{II} (z-z_0)^\nu + a w_I(z) \log(z-z_0).$$

We have the freedom to simultaneously shift the summation and the exponent ϱ_1, giving us another exponent ϱ_2 with $\varrho_1 - \varrho_2 \in \mathbb{Z}$. □

2.3 Inessential Singular Points

Definition 2.7. *A number $z_0 \in \mathbb{C}$ is called* inessential singular point[1] *of the differential equation $w'' + pw' + qw = 0$ if the main part of the Laurent serieses of the solutions w_I and w_{II} contains only a finite number of terms. Otherwise it is called* essential singular point[2].

[1] außerwesentlich singulärer Punkt oder regulärer singulärer Punkt

[2] wesentlich singulärer Punkt oder irregulärer singulärer Punkt

Essential singular points are nasty. That is the reason why we want to exclude them.

Proposition 2.8. *If z_0 is an inessential singular point of the differential equation $w'' + pw' + qw = 0$, where the coefficients p and q are holomorphic in $\{z \colon 0 < |z - z_0| < R\}$, then this differential equation has a fundamental system of the following form:*

$$w_I(z) = (z - z_0)^{\varrho_1} \sum_{\nu=0}^{\infty} c_\nu^I (z - z_0)^\nu, \tag{2.3}$$

$$w_{II}(z) = (z - z_0)^{\varrho_2} \sum_{\nu=0}^{\infty} c_\nu^{II} (z - z_0)^\nu + a w_I(z) \cdot \log(z - z_0), \tag{2.4}$$

where c_0^I and c_0^{II} do not vanish. If $\varrho_1 - \varrho_2 \notin \mathbb{Z}$, then $a = 0$.

Proof. Just shift the numbers ϱ_1 and ϱ_2 by integers. □

Proposition 2.9. *A number z_0 is an inessential singularity for (2.2) if and only if the following three conditions are satisfied:*

- *The functions p and q have an isolated singularity at z_0 (or are regular there).*
- *The function p is regular at z_0, or it has a pole of order 1 there.*
- *The function q is regular at z_0, or it has a pole of order at most 2 there.*

These conditions are satisfied in all differential equations which we care about.

Sketch of proof. **The conditions on p and q are necessary:**
Let (w_1, w_2) be a fundamental system of the form (2.3), (2.4). Suppose $z_0 = 0$. Then we can write

$$w_1(z) = z^{\varrho_1} P_1(z),$$
$$w_2(z) = z^{\varrho_2} P_2(z) + a w_1(z) \log(z),$$

where P_1 and P_2 are power series with $P_1(0) \neq 0$, $P_2(0) \neq 0$. Then we have

$$w_1'' + pw_1' + qw_1 = 0,$$
$$w_2'' + pw_2' + qw_2 = 0,$$

which is a linear system for the functions p and q. By Cramer's rule,

$$p = -\frac{w_2'' w_1 - w_1'' w_2}{w_2' w_1 - w_1' w_2}, \qquad q = -\frac{w_1''}{w_1} - p\frac{w_1'}{w_1}.$$

Now we just have to count the multiplicities of the zero (or pole) at $z_0 = 0$ of the numerators and denominators. Observe that

$$\frac{w_2'' w_1 - w_1'' w_2}{w_2' w_1 - w_1' w_2} = \partial_z \log\left(w_2' w_1 - w_1' w_2\right),$$
$$w_2' w_1 - w_1' w_2 = w_1^2 \partial_z \left(\frac{w_2}{w_1}\right).$$

It turns out that the order of a pole of p at $z_0 = 0$ can be at most one, and the order of a pole of q at $z_0 = 0$ can be at most two. The details are left to the reader.

The conditions on p and q are sufficient:
We have

$$p(z) = \frac{1}{z} \sum_{\nu=0}^{\infty} a_\nu z^\nu, \qquad q(z) = \frac{1}{z^2} \sum_{\nu=0}^{\infty} b_\nu z^\nu,$$

and make the following ansatz for the unknown solution w:

$$w(z) = z^\varrho \sum_{\nu=0}^{\infty} c_\nu z^\nu, \qquad c_0 = 1.$$

Then we obtain

$$\begin{aligned} 0 &= w'' + pw' + qw \\ &= \sum_{\nu=0}^{\infty} c_\nu(\nu+\varrho)(\nu+\varrho-1)z^{\nu+\varrho-2} + \left(\frac{1}{z}\sum_{\kappa=0}^{\infty} a_\kappa z^\kappa\right)\left(\sum_{\mu=0}^{\infty} c_\mu(\mu+\varrho)z^{\mu+\varrho-1}\right) \\ &\quad + \left(\frac{1}{z^2}\sum_{\kappa=0}^{\infty} b_\kappa z^\kappa\right)\left(\sum_{\mu=0}^{\infty} c_\mu z^{\mu+\varrho}\right). \end{aligned}$$

Set

$$f(\varrho) := \varrho(\varrho-1) + a_0\varrho + b_0 = \varrho^2 + (a_0-1)\varrho + b_0.$$

Comparing equal powers of z then yields the system of equations

$$f(\varrho) = 0, \tag{2.5}$$

$$c_\nu f(\varrho+\nu) = -\sum_{\mu=0}^{\nu-1} \left((\varrho+\mu)a_{\nu-\mu} + b_{\nu-\mu}\right) c_\mu, \qquad \nu \geq 1. \tag{2.6}$$

Equation (2.5) is called *indicial equation.* It has two solutions, ϱ_1 and ϱ_2. If $f(\varrho_j+\nu)$ never vanishes, then the coefficients c_ν can be computed recursively.

Case 1: $\varrho_1 - \varrho_2 \notin \mathbb{Z}$**.** Then $f(\varrho_j+\nu)$ is never zero, and the c_ν are uniquely determined.

Case 2: $\varrho_1 - \varrho_2 \in \mathbb{Z}$**.** Then, without loss of generality, $\varrho_1 = \varrho_2 + m$, with $m \in \mathbb{N}_0$. As a consequence, $f(\varrho_1+\nu)$ never vanishes, and we can construct a solution w_I of the desired form. The other solution can then contain a logarithmic term; maybe the ansatz must be changed.

The further proof comprises the following steps:

The series $\sum_\nu c_\nu^\alpha (z-z_0)^\nu$ **converges.** We have to show that the radius of convergence is greater than zero. To this end, we have to estimate how fast the coefficients c_ν can grow with increasing ν.

The radius of convergence is R**.** Assume that the radius of convergence were smaller than R. Then the boundary of the disk of convergence must contain a singularity of the solution. But this is impossible, since the solution must exist as a holomorphic function wherever the coefficients are regular.

The solutions are linearly independent. This is quite easy for $\varrho_1 - \varrho_2 \notin \mathbb{Z}$, and a bit harder in the exceptional case.

The details are intentionally omitted. □

Remark 2.10. *The numbers ϱ_1 and ϱ_2 can be computed as follows:*

put $a_0 = \lim_{z\to z_0}(z-z_0)p(z)$ and $b_0 = \lim_{z\to z_0}(z-z_0)^2 q(z)$. Then ϱ_1 and ϱ_2 are the zeroes of the polynomial $f(\varrho) = \varrho(\varrho-1) + a_0\varrho + b_0$.

Remark 2.11. *Consider a regular point z_0 of the differential equation. Then p and q are regular at z_0, and we can prescribe $w(z_0)$ and $w'(z_0)$ as initial values. Choosing them as 1 and 0, we find two solutions $w_I(z) = P_I(z)$, $w_{II}(z) = (z-z_0)P_{II}(z)$ with holomorphic functions P_I and P_{II} satisfying $P_I(z_0) = P_{II}(z_0) = 1$. On the other hand, the leading coefficients of p and q are $a_0 = b_0 = b_1 = 0$, by the formulas from the previous remark. The indicial equation then is $f(\varrho) = \varrho(\varrho-1) = 0$. This gives $\varrho_1 = 0$ and $\varrho_2 = 1$, as can be seen in the formulas for w_I and w_{II}.*

Conclusion: regular points can be interpreted as inessential singular points.

2.4 Example: the Bessel Differential Equation

The differential equation

$$w''(z) + \frac{1}{z}w'(z) + \left(1 - \frac{\nu^2}{z^2}\right) w(z) = 0, \qquad \nu \in \mathbb{C}, \quad z \in \mathbb{C} \setminus \{0\} \tag{2.7}$$

is called *Bessel differential equation.* We have $p = p(z) = \frac{1}{z}$, $q = q(z) = 1 - \frac{\nu^2}{z^2}$, which are holomorphic functions for $0 < |z| < \infty$. The point $z_0 = 0$ is an inessential singular point. We have

$$a_0 = 1, \quad a_1 = a_2 = \cdots = 0, \qquad b_0 = -\nu^2, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = \cdots = 0,$$

which leads to the indicial equation $f(\varrho) = \varrho(\varrho - 1) + a_0\varrho + b_0 = 0$, hence $\varrho^2 - \nu^2 = 0$, with solutions

$$\varrho_1 = \nu, \qquad \varrho_2 = -\nu.$$

Case 1: $\varrho_1 - \varrho_2 \notin \mathbb{Z}$. Then we have a fundamental system (w_I, w_{II}) with $w_I(z) = z^\nu \sum_{k=0}^\infty c_k^I z^k$, $w_{II}(z) = z^{-\nu} \sum_{k=0}^\infty c_k^{II} z^k$. The recursion formula for the coefficients is

$$c_k f(\varrho + k) = -\sum_{\mu=0}^{k-1} ((\varrho + \mu)a_{k-\mu} + b_{k-\mu})\, c_\mu, \qquad k \geq 1.$$

For $k = 1$, we obtain $c_1 f(\varrho + 1) = -(\varrho a_1 + b_1)c_0 = 0$, hence $c_1 = 0$.

For $k \geq 2$, we find $c_k f(\varrho + k) = -c_{k-2}$ instead. Therefore,

$$c_k = -\frac{c_{k-2}}{(\varrho + k)^2 - \nu^2}.$$

Since $c_1 = 0$, all c_k with odd k vanish, too. The other c_k are given by

$$c_{2m} = (-1)^m \frac{c_0}{\prod_{\mu=1}^m f(\varrho + 2\mu)}.$$

Observe that $f(\varrho + 2\mu) = (\varrho + 2\mu)^2 - \varrho^2 = 4\mu(\varrho + \mu)$. Then it follows that

$$c_{2m}^I = \frac{(-1)^m c_0^I}{\prod_{\mu=1}^m (4\mu(\nu + \mu))} = \frac{(-1)^m c_0^I}{4^m m!} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + m + 1)},$$
$$c_{2m}^{II} = \frac{(-1)^m c_0^{II}}{\prod_{\mu=1}^m (4\mu(-\nu + \mu))} = \frac{(-1)^m c_0^{II}}{4^m m!} \frac{\Gamma(-\nu + 1)}{\Gamma(-\nu + m + 1)}.$$

It is custom to put

$$c_0^I = \frac{1}{2^\nu \Gamma(\nu + 1)}, \qquad c_0^{II} = \frac{1}{2^{-\nu}\Gamma(-\nu + 1)}.$$

Then we have as fundamental system of solutions (w_I, w_{II}) the functions J_ν and $J_{-\nu}$, where

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^\infty (-1)^m \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma(m + \nu + 1)}, \qquad \nu \in \mathbb{C},$$

and the formula for $J_{-\nu}$ is given by replacing $\nu \mapsto -\nu$. This series converges everywhere in $\mathbb{C}$, and the function J_ν is called *Bessel function* or *Cylinder function of first kind.*

These functions are always solutions, even for $\varrho_1 - \varrho_2 \in \mathbb{Z}$.

Case 2: $\varrho_1 - \varrho_2$ is an odd integer. Then we have $\nu = k + \frac{1}{2}$, where $k \in \mathbb{N}_0$, without loss of generality. The functions $J_{k+1/2}$ and $J_{-k-1/2}$ are solutions of the Bessel equation, and they are linearly independent, because: the coefficient of the first term of the series does not vanish. Therefore, the series converges to $1/\Gamma(k + 3/2)$ if $z \to 0$. Then $J_{k+1/2}(z)$ goes to zero, and $J_{-k-1/2}(z)$ goes to ∞ for z tending to zero.

Case 3: $\varrho_1 - \varrho_2$ is an even integer. Now, we have $\nu = k$ with $k \in \mathbb{N}_0$.

Obviously, $J_\nu = J_{-\nu}$ for $k = 0$. For $k \in \mathbb{N}_+$, we have

$$\begin{aligned}
J_{-k}(z) &= \left(\frac{z}{2}\right)^{-k} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma(m-k+1)} \\
&= \left(\frac{z}{2}\right)^{-k} \sum_{m=k}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m}}{m!(m-k)!} \\
&= \left(\frac{z}{2}\right)^{+k} (-1)^k \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m}}{m!(m+k)!} \\
&= (-1)^k J_k(z).
\end{aligned}$$

Therefore, the functions J_ν and $J_{-\nu}$ *do not* form a fundamental system of solutions of the Bessel equation if $\nu \in \mathbb{Z}$.

In this case, we make the ansatz

$$w_{II}(z) = z^{-k} \sum_{m=0}^{\infty} c_m z^m + a \log(z) J_k(z), \qquad z \neq 0.$$

Put $L = \partial_z^2 + \frac{1}{z}\partial_z + (1 - \frac{k^2}{z^2})$. Then the Bessel differential equation can be written as $Lw = 0$. Now we compute:

$$\begin{aligned}
L(z^{m-k}) &= z^{m-k-2}\left((m-k)(m-k-1) + (m-k) + z^2 - k^2\right) = z^{m-k-2}(m^2 - 2mk + z^2), \\
L\left(\sum_{m=0}^{\infty} c_m z^{m-k}\right) &= \sum_{m=0}^{\infty} c_m m(m-2k) z^{m-k-2} + \sum_{m=0}^{\infty} c_m z^{m-k}, \\
L\left(\log(z) J_k(z)\right) &= (\log(z)) L J_k(z) + (\partial_z^2 \log(z)) J_k + 2\left(\partial_z \log(z)\right) \partial_z J_k(z) + \left(\frac{1}{z}\partial_z \log(z)\right) J_k(z) \\
&= \frac{2}{z}\partial_z J_k(z), \\
0 = L(w_{II}(z)) &= \sum_{m=1}^{\infty} c_m m(m-2k) z^{m-k-2} + \sum_{m=0}^{\infty} c_m z^{m-k} + \frac{2a}{z}\partial_z J_k(z) \\
&= \sum_{m=1}^{\infty} c_m m(m-2k) z^{m-k-2} + \sum_{m=2}^{\infty} c_{m-2} z^{m-k-2} + \frac{2a}{z}\partial_z \left(\sum_{\mu=0}^{\infty} \left(\frac{z}{2}\right)^{k+2\mu} \frac{(-1)^{2\mu}}{\mu!(\mu+k)!}\right) \\
&= c_1(1-2k) z^{-k-1} + \sum_{m=2}^{\infty} \left(c_m m(m-2k) + c_{m-2}\right) z^{m-k-2} + \\
&\qquad + 2a \sum_{\mu=0}^{\infty} (-1)^{2\mu} 2^{-k-2\mu} (k+2\mu) \frac{1}{\mu!(\mu+k)!} z^{k+2\mu-2}.
\end{aligned}$$

Multiply the equation with z^{k+2}. Then

$$\begin{aligned}
0 = c_1(1-2k)z &+ \sum_{m=2}^{\infty} \left(c_m m(m-2k) + c_{m-2}\right) z^m + \\
&+ 2a \sum_{\mu=0}^{\infty} (-1)^{2\mu} 2^{-k-2\mu} (k+2\mu) \frac{1}{\mu!(\mu+k)!} z^{2k+2\mu}.
\end{aligned}$$

Now we compare equal powers of z. The last sum can not contain odd powers of z. Therefore, $c_1 = 0$, and all the other c_{2n+1} vanish, too, because $m - 2k \neq 0$ for odd m. Then the index m should be even, $m = 2\lambda$. We obtain

$$0 = \sum_{\lambda=1}^{\infty} \left(4c_{2\lambda}\lambda(\lambda-k) + c_{2\lambda-2}\right) z^{2\lambda} + 2a \sum_{\mu=0}^{\infty} (-1)^{2\mu} 2^{-k-2\mu} (k+2\mu) \frac{1}{\mu!(\mu+k)!} z^{2k+2\mu}.$$

Choose $c_0 = 1$, and compute $c_2, c_4, \ldots, c_{2k-2}$, by setting $4c_{2\lambda}\lambda(\lambda - k) + c_{2\lambda-2} = 0$. For $\lambda = k$, we obtain $c_{2k-2}z^{2k}$ from the first sum, which can be eliminated by an appropriate choice of a. Choose c_{2k} arbitrary, and compute $c_{2k+2}, c_{2k+4}, \ldots$, by another recursion formula, which contains the contributions from the second sum.

The details are left to the reader.

Another choice of the fundamental system is as follows: put

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \qquad \nu \notin \mathbb{Z}.$$

Then (J_ν, Y_ν) form a fundamental system of solutions for $\nu \notin \mathbb{Z}$. And if $\nu = n \in \mathbb{Z}$, define

$$Y_n(z) = \lim_{\nu \to n} Y_\nu(z).$$

It is possible to show that (J_n, Y_n) are a fundamental system of solutions (we will not prove this).

Evaluation of Bessel Functions

We give some remarks on how to evaluate $J_\nu(x)$ for $\nu, x \in \mathbb{R}_+$.

x and ν are small: use the power series

x is small and ν is large: use the following recursion formula either forwards or backwards:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z).$$

x is large (greater than ν): use the asymptotic expansion

$$\begin{aligned}
J_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left(P(\nu, z)\cos\chi - Q(\nu, u)\sin\chi \right), \\
\chi &= z - \left(\frac{\nu}{2} + \frac{1}{4} \right)\pi, \\
P(\nu, z) &\sim 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4} \mp \ldots, \qquad z \to \infty, \\
Q(\nu, z) &\sim \frac{\mu - 1}{8z} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} \pm \ldots, \qquad z \to \infty, \\
\mu &= 4\nu^2.
\end{aligned}$$

The series for P and Q do not converge. The error does not exceed the $(k+1)$th term and is of the same sign provided that $k > \nu/2$ and z is from $\mathbb{R}_+$.

Details can be found in [14].

Roughly, the asymptotic behavior of $J_\nu(z)$ for $z \to \infty$ can be explained as follows: if $J_\nu = w$ is a solution to

$$w''(z) + \frac{1}{z}w'(z) + \left(1 - \frac{\nu^2}{z^2}\right) w(z) = 0,$$

then $u(z) = z^{1/2}w(z)$ is a solution to

$$u''(z) + \left(1 - \frac{\nu^2 - 1/4}{z}\right) u(z) = 0,$$

which looks like the equation $u''(z) + u(z) = 0$ for large $|z|$.

2.5 Differential Equations of Fuchs class

We consider the differential equation

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0, \qquad R < |z|, \tag{2.8}$$

where we assume that p and q are holomorphic functions for $|z| > R$. Our goal is to study the behavior of w for $z \to \infty$. Therefore, it is natural to transform the equation:

$$z = \frac{1}{\zeta}, \qquad w(z) = \omega(\zeta), \qquad w'(z) = -\zeta^2\omega'(\zeta), \qquad w''(z) = \omega''(\zeta)\zeta^4 + 2\omega'(\zeta)\zeta^3,$$

$$\omega''(\zeta) + \left(\frac{2}{\zeta} - \frac{1}{\zeta^2}p(\zeta^{-1})\right)\omega'(\zeta) + \frac{1}{\zeta^4}q(\zeta^{-1})\omega(\zeta) = 0, \qquad 0 < |\zeta| < \frac{1}{R},$$

$$\varphi(\zeta) := \frac{2}{\zeta} - \frac{1}{\zeta^2}p(\zeta^{-1}), \qquad \chi(\zeta) := \frac{1}{\zeta^4}q(\zeta^{-1}).$$

Definition 2.12. *We say that $z_0 = \infty$ is an* inessential singularity *of the equation (2.8) if $\zeta_0 = 0$ is an inessential singularity of the transformed equation $\omega''(\zeta) + \varphi(\zeta)\omega'(\zeta) + \chi(\zeta)\omega(\zeta) = 0$.*

We say that $z_0 = \infty$ is a regular point *of the equation (2.8) if $\zeta_0 = 0$ is a regular point of the transformed equation.*

We know that $\zeta_0 = 0$ is an inessential singularity of the transformed equation if and only if

$$\varphi(\zeta) = \frac{1}{\zeta}\sum_{k=0}^{\infty}\alpha_k\zeta^k, \qquad \chi(\zeta) = \frac{1}{\zeta^2}\sum_{k=0}^{\infty}\beta_k\zeta^k, \qquad 0 < |\zeta| < \frac{1}{R}.$$

Reformulating in terms of p and q, we find

$$2z - z^2p(z) = z\sum_{k=0}^{\infty}\alpha_k z^{-k}, \qquad z^4q(z) = z^2\sum_{k=0}^{\infty}\beta_k z^{-k},$$

$$p(z) = \frac{1}{z}\left((2-\alpha_0) - \sum_{k=1}^{\infty}\alpha_k z^{-k}\right), \qquad q(z) = \frac{1}{z^2}\sum_{k=0}^{\infty}\beta_k z^{-k}, \qquad |z| > R.$$

This means that p must have a first order zero at $z = \infty$, and q must have a second order zero at $z = \infty$. Additionally, both coefficients must be holomorphic functions near infinity.

Remark 2.13. *The point $z_0 = \infty$ is a regular point if and only if $\alpha_0 = 2$ and $\beta_0 = \beta_1 = 0$. That means, p has a second order zero at infinity, and q has a fourth order zero at infinity.*

Remark 2.14. *The indicial equation for $\omega = \omega(\zeta)$ is*

$$\varrho(\varrho - 1) + \alpha_0\varrho + \beta_0 = 0,$$

with $\alpha_0 = 2 - \lim_{z\to\infty} zp(z)$ and $\beta_0 = \lim_{z\to\infty} z^2q(z)$.

Remark 2.15. *The Bessel differential equation (2.7) has an* essential *singularity at infinity.*

Definition 2.16. *We say that a differential equation $w'' + pw' + qw = 0$ belongs to the* Fuchs *class if all its singular points (including, possibly, infinity) are inessential singularities.*

A differential equation of the Fuchs class can have only a finite number of singular points (otherwise, the cluster point would be a non-isolated singularity).

Let $z_1, \ldots, z_n \in \mathbb{C}$ denote the (mutually distinct) inessential singular points of the differential equation. Then the Theorem of Mittag-Leffler guarantees that p and q have the form

$$p(z) = \sum_{k=1}^{n}\frac{A_k}{z - z_k} + p^*(z), \qquad q(z) = \sum_{k=1}^{n}\left(\frac{B_k}{(z-z_k)^2} + \frac{C_k}{z - z_k}\right) + q^*(z), \qquad z \in \mathbb{C},$$

where p^* and q^* are entire functions.

From $\lim_{z\to\infty} p(z) = \lim_{z\to\infty} q(z) = 0$ we obtain $\lim_{z\to\infty} p^*(z) = \lim_{z\to\infty} q^*(z) = 0$. Then the LIOUVILLE theorem implies $p^* \equiv q^* \equiv 0$. Since q must have a double zero at infinity, it follows that

$$\sum_{k=1}^{n} C_k = 0.$$

Proposition 2.17. *The differential equation $w''(z) + p(z)w'(z) + q(z)w(z) = 0$ belongs to the Fuchs class if and only if there is a finite collection of mutually distinct complex numbers $z_1, \ldots, z_n$ such that*

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - z_k}, \qquad q(z) = \sum_{k=1}^{n} \left(\frac{B_k}{(z - z_k)^2} + \frac{C_k}{z - z_k} \right), \qquad \sum_{k=1}^{n} C_k = 0.$$

Proof. The necessity has been shown above, and we skip the proof of the sufficiency. □

Remark 2.18. *The indicial equation at z_k reads*

$$\varrho(\varrho - 1) + A_k \varrho + B_k = 0.$$

The indicial equation at $z = \infty$ (with respect to $\zeta = \frac{1}{z}$) reads

$$\varrho(\varrho - 1) + \alpha_{0,\infty} \varrho + \beta_{0,\infty} = 0$$

with

$$\begin{aligned} \alpha_{0,\infty} &= 2 - \lim_{z\to\infty} zp(z) = 2 - \sum_{k=1}^{n} A_k, \\ \beta_{0,\infty} &= \lim_{z\to\infty} z^2 q(z) = \sum_{k=1}^{n} B_k + \lim_{z\to\infty} z \sum_{k=1}^{n} \frac{C_k}{1 - z_k/z} \\ &= \sum_{k=1}^{n} B_k + \lim_{z\to\infty} z \sum_{k=1}^{n} C_k \left(1 + \frac{z_k}{z} + \left(\frac{z_k}{z}\right)^2 + \ldots \right) = \sum_{k=1}^{n} (B_k + C_k z_k). \end{aligned}$$

2.6 The Hypergeometric Differential Equation

Definition 2.19. *Let $a, b, c \in \mathbb{C}$. The differential equation*

$$z(1 - z)w''(z) + (c - (a + b + 1)z)w'(z) - abw(z) = 0$$

is called hypergeometric differential equation.

This differential equation can be written as $w''(z) + p(z)w'(z) + q(z)w(z) = 0$ with

$$\begin{aligned} p(z) &= \frac{c - (a + b + 1)z}{z(1 - z)} = \frac{c}{z} + \frac{a + b + 1 - c}{z - 1}, \\ q(z) &= \frac{ab}{z(z - 1)} = \frac{-ab}{z} + \frac{ab}{z - 1}, \\ z_1 &= 0, \qquad z_2 = 1, \\ A_1 &= c, \qquad A_2 = a + b + 1 - c, \\ B_1 &= 0, \qquad B_2 = 0, \\ C_1 &= -ab, \qquad C_2 = ab. \end{aligned}$$

Remark 2.20. *The differential equation is symmetric in a and b.*

The indicial equations are:

$$\begin{aligned}
z_1 = 0: &\quad \varrho(\varrho-1) + A_1\varrho + B_1 = 0, && \Longrightarrow \quad \varrho_1^{(1)} = 0, && \varrho_2^{(1)} = 1-c,\\
z_2 = 1: &\quad \varrho(\varrho-1) + A_2\varrho + B_2 = 0, && \Longrightarrow \quad \varrho_1^{(2)} = 0, && \varrho_2^{(2)} = c-a-b,\\
z_3 = \infty: &\quad \varrho(\varrho-1) + (2-A_1-A_2)\varrho + (C_1z_1 + C_2z_2) = 0, && \Longrightarrow \quad \varrho_1^{(3)} = a, && \varrho_2^{(3)} = b.
\end{aligned}$$

Then we can write down the fundamental systems of solutions:

In the domain $0 < |z| < 1$: Our general theory gives us

$$w_I(z) = z^0\sum_{j=0}^{\infty} c_j^I z^j, \qquad w_{II}(z) = z^{1-c}\sum_{j=0}^{\infty} c_j^{II} z^j,$$

provided that $1-c \notin \mathbb{Z}$. If $c = 1, 2, 3, \dots$, then we have solutions of the form

$$w_I(z) = z^0\sum_{j=0}^{\infty} c_j^I z^j, \qquad w_{II}(z) = z^{1-c}\sum_{j=0}^{\infty} c_j^{II} z^j + \varepsilon w_I(z)\log(z).$$

And if $c = 0, -1, -2, \dots$, then we have solutions of the form

$$w_I(z) = z^0\sum_{j=0}^{\infty} c_j^I z^j + \varepsilon w_{II}(z)\log(z), \qquad w_{II}(z) = z^{1-c}\sum_{j=0}^{\infty} c_j^{II} z^j.$$

According to our general theory, these series converge for $0 < |z| < 1$. A simple calculation shows that, for $c \neq 0, -1, -2, \dots$,

$$w_I(z) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdot\ldots\cdot(a+k-1)b(b+1)\cdot\ldots\cdot(b+k-1)}{c(c+1)\cdot\ldots\cdot(c+k-1)k!} z^k.$$

This series is called the *hypergeometric series*, since it becomes the geometric series for $a = 1$ and $b = c$. Note that we obtain polynomial solutions for $a = 0, -1, -2, \dots$.

To simplify notation, we introduce the Pochhammer symbol:

$$(a)_k = \begin{cases} 1 & : k = 0,\\ a(a+1)\cdot\ldots\cdot(a+k-1) & : k \in \mathbb{N}_+. \end{cases}$$

Then we can write, for $c \neq 0, -1, -2, \dots$,

$$w_I(z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k =: {}_2\mathrm{F}_1(a, b; c; z) =: F(a, b; c; z)$$

We know that the hypergeometric series can be analytically extended to $\mathbb{C}\setminus\{1\}$. In general, this extension is not unique if we go around the singularity at 1. It is custom to cut the plane along $\mathbb{R}_+\setminus[0,1]$, i.e., to define the analytic continuation only for $|\arg(1-z)| < \pi$.

For $c \notin \mathbb{Z}$, we obtain

$$w_{II}(z) = z^{1-c}\,{}_2\mathrm{F}_1(a+1-c, b+1-c; 2-c; z),$$

with two cuts along $\mathbb{R}_+\setminus[0,1]$ and $\mathbb{R}_-$.

In the domain $0 < |z-1| < 1$: We have one solution

$$w_{III}(z) = {}_2\mathrm{F}_1(a, b; 1+a+b-c; 1-z)$$

under the assumption $1+a+b-c \neq 0, -1, -2, \dots$, and a second solution

$$w_{IV}(z) = (1-z)^{c-b-a}\,{}_2\mathrm{F}_1(c-b, c-a; 1+c-b-a; 1-z)$$

in case of $1+a+b-c \neq 1, 2, 3, \dots$. For the continuation of w_{III}, we cut the plane along $\mathbb{R}_-$; and for the continuation of w_{IV}, we cut the plane along $\mathbb{R}_-$ and $\mathbb{R}_+\setminus[0,1]$.

In the domain $|z| > 1$: We have the solution

$$w_V(z) = z^{-a}\,{}_2\mathrm{F}_1(a, 1+a-c; 1+a-b; z^{-1})$$

under the assumption $1+a-b \neq 0, -1, -2, \ldots$; and the solution

$$w_{VI}(z) = z^{-b}\,{}_2\mathrm{F}_1(b, 1+b-c; 1+b-a; z^{-1})$$

under the assumption $1+a-b \neq 1, 2, 3, \ldots$. For the continuation of each solution, we cut the plane along $\mathbb{R}_+$.

Quite a lot of classical functions can be written as hypergeometric functions:

$$\begin{aligned}
&{}_2\mathrm{F}_1(1, b; b; z) = \frac{1}{1-z},\\
&{}_2\mathrm{F}_1(a, 0; c; z) = 1,\\
&{}_2\mathrm{F}_1(-n, 1; 1; 1-z) = z^n,\\
&{}_2\mathrm{F}_1(1/2, 1; 1; z) = (1-z)^{-1/2},\\
&{}_2\mathrm{F}_1(1, 1; 2; z) = -\frac{1}{z}\log(1-z),\\
&z\,{}_2\mathrm{F}_1(1/2, 1; 3/2; -z^2) = \arctan(z),\\
&z\,{}_2\mathrm{F}_1(1/2, 1/2; 3/2; z^2) = \arcsin(z),\\
&{}_2\mathrm{F}_1(-n, n+1; 1; (1-z)/2) = P_n(z), \qquad \text{(Legendre polynomial)}.
\end{aligned}$$

Next we give an explicit expression for the analytic continuation of ${}_2\mathrm{F}_1(a, b; c; z)$ outside the unit circle. Observe that

$$\begin{aligned}
\frac{(a)_k}{k!} &= (-1)^k \binom{-a}{k},\\
\frac{(b)_k}{(c)_k} &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b+k)\Gamma(c-b)}{\Gamma(c+k)}\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b+k, c-b) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{t=0}^{1} t^{b+k-1}(1-t)^{c-b-1}\,\mathrm{d}t,
\end{aligned}$$

for $\Re(b+k-1) > -1$ and $\Re(c-b-1) > -1$. A sufficient condition is $\Re c > \Re b > 0$. Then we can write

$$\begin{aligned}
{}_2\mathrm{F}_1(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \int_{t=0}^{1} t^{b+k-1}(1-t)^{c-b-1}\,\mathrm{d}t\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{t=0}^{1} t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k z^k t^k\,\mathrm{d}t\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{t=0}^{1} t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t
\end{aligned}$$

provided that $|zt| < 1$, which is satisfied for $|z| < 1$. We see that the integral on the RHS exists for all $z \in \mathbb{C} \setminus \mathbb{R}_+$ and $\Re c > \Re b > 0$. The integral converges uniformly with respect to z if $0 < \varepsilon < |z| < R$ and $|\arg(1-z)| < \pi - \delta$. Then the integral is a holomorphic function of $z \in \mathbb{C}$ with the exception of real $z \geq 1$.

We also see that ${}_2\mathrm{F}_1(a, b; c; z)$ is a holomorphic function of a, b, c for $a \in \mathbb{C}$ and $\Re c > \Re b > 0$. Note that the roles of a and b can be switched. Additionally, we remark that

$$\partial_z\,{}_2\mathrm{F}_1(a, b; c; z) = \frac{ab}{c}\,{}_2\mathrm{F}_1(a+1, b+1; c+1; z).$$

From such integral representation formulas and similar formulas, it is possible to show that

$$\frac{1}{\Gamma(c)}\,{}_2\mathrm{F}_1(a, b, c; z)$$

is an entire analytic function of $a, b, c \in \mathbb{C}$.

Proposition 2.21. *Every differential equation of the Fuchs class with at most three inessential singular points (including, possibly, infinity) can be transformed to a hypergeometric differential equation.*

Sketch of proof. We distinguish two cases.

Case 1: ∞ is a regular point. Then we have (at most) three inessential singularities at $z_1, z_2, z_3 \in \mathbb{C}$, and

$$p(z) = \sum_{k=1}^{3} \frac{A_k}{z - z_k}, \qquad q(z) = \sum_{k=1}^{3} \left(\frac{B_k}{(z - z_k)^2} + \frac{C_k}{z - z_k} \right)$$

with $\sum_{k=1}^{3} C_k = 0$. Additionally, we know that p behaves like $\frac{2}{z}$ for $z \to \infty$; and q has a fourth order zero at infinity. We get $\sum_{k=1}^{3} A_k = 2$. Let α_k, β_k denote the solutions of the indicial equation at z_k. We find

$$A_k = 1 - \alpha_k - \beta_k, \qquad B_k = \alpha_k \beta_k.$$

By a longer calculation, you can express C_k in terms of the α, β, and z_j.

Then we transform the variables:

$$t = \frac{z_2 - z_3}{z_2 - z_2} \cdot \frac{z - z_1}{z - z_3}, \qquad W(t) = t^{-\alpha_1}(1 - t)^{-\alpha_2} w(z(t)).$$

We have the correspondences

$$\begin{aligned} z &= z_1 & &\longleftrightarrow & t &= 0, \\ z &= z_2 & &\longleftrightarrow & t &= 1, \\ z &= z_3 & &\longleftrightarrow & t &= \infty. \end{aligned}$$

It is possible to show that W solves a hypergeometric differential equation with the parameters

$$\begin{aligned} a &= \alpha_1 + \alpha_2 + \alpha_3, \\ b &= \alpha_1 + \alpha_2 + \beta_3, \\ c &= 1 + \alpha_1 - \beta_1. \end{aligned}$$

Note that $\sum_{k=1}^{3} (\alpha_k + \beta_k) = 1$. Therefore, β_2 does not appear.

Case 2: ∞ is an inessential singularity. Then we have

$$p(z) = \sum_{k=1}^{2} \frac{A_k}{z - z_k}, \qquad q(z) = \sum_{k=1}^{2} \left(\frac{B_k}{(z - z_k)^2} + \frac{C_k}{z - z_k} \right)$$

with $C_1 + C_2 = 0$, $A_k = 1 - \alpha_k - \beta_k$, and $B_k = \alpha_k \beta_k$. Expressing C_k in terms of α_k, β_k and z_k, we find

$$q(z) = \frac{1}{(z - z_1)(z - z_2)} \left(\alpha_3 \beta_3 + \frac{(z_1 - z_2)\alpha_1 \beta_1}{z - z_1} + \frac{(z_2 - z_1)\alpha_2 \beta_2}{z - z_2} \right).$$

Then we transform the variables:

$$t = \frac{z - z_1}{z_2 - z_1}, \qquad W(t) = t^{-\alpha_1}(1 - t)^{-\alpha_2} w(z(t)),$$

with the correspondences

$$\begin{aligned} z &= z_1 & &\longleftrightarrow & t &= 0, \\ z &= z_2 & &\longleftrightarrow & t &= 1, \\ z &= \infty & &\longleftrightarrow & t &= \infty. \end{aligned}$$

The function W is a solution to a hypergeometric equation with the same parameters as in the first case.

□

We can permutate the points $0, 1, \infty$ and obtain again solutions to the hypergeometric differential equation. This way, we can show that

$$\begin{aligned} {}_2\mathrm{F}_1(a, b; c; z) &= (1-z)^{c-a-b}\, {}_2\mathrm{F}_1(c-a, c-b; c; z) = (1-z)^{-a}\, {}_2\mathrm{F}_1(a, c-b; c; z/(z-1)) \\ &= (1-z)^{-b}\, {}_2\mathrm{F}_1(c-a, b; c; z/(z-1)). \end{aligned}$$

Similar relations can be written down for $w_{II}, \ldots, w_{VI}$, giving you 24 relations, the so-called KUMMER relations.

Chapter 3

General Properties of Special Functions

3.1 Differential Equations of Hypergeometric Type

A hypergeometric DE has the form

$$w''(z) + \frac{\gamma - (\alpha+\beta+1)z}{z(1-z)} w'(z) - \frac{\alpha\beta}{z(1-z)} w(z) = 0.$$

A *differential equation of hypergeometric type* has the form

$$w''(z) + \frac{\tau(z)}{\sigma(z)} w'(z) + \frac{\lambda}{\sigma(z)} w(z) = 0, \tag{3.1}$$

where $\tau = \tau(z)$ and $\sigma = \sigma(z)$ are polynomials in of degree at most 1 and 2, respectively.
A *differential equation of generalized hypergeometric type* has the form

$$w''(z) + \frac{\tau(z)}{\sigma(z)} w'(z) + \frac{\gamma(z)}{\sigma^2(z)} w(z) = 0, \tag{3.2}$$

where τ and σ are as above, and $\gamma = \gamma(z)$ is a polynomial of degree at most 2.
Our goal is to transform these differential equations into each other. Let $w = w(z)$ be a solution to (3.2). Set $w(z) = \Phi(z)u(z)$ with a certain holomorphic function Φ. Then we have

$$u''(z) + \left(2\frac{\Phi'}{\Phi} + \frac{\tau}{\sigma}\right) u'(z) + \left(\frac{\Phi''}{\Phi} + \frac{\Phi'}{\Phi} \cdot \frac{\tau}{\sigma} + \frac{\gamma}{\sigma^2}\right) u(z) = 0.$$

Then the following lemma is proved easily:

Lemma 3.1. *If $\varphi = \varphi(z)$ is a polynomial of degree at most 1 and $\Phi'(z)/\Phi(z) = \varphi(z)/\sigma(z)$, then a differential equation of generalized hypergeometric type is transformed into a differential equation of the same type.*

By simple calculation, one can show:

Lemma 3.2. *A differential equation (3.2), where σ is neither a constant nor a quadratic polynomial with a double zero, can be transformed into an equation of the form (3.1), by setting $w = \Phi u$ and choosing Φ suitably.*

Example 3.3. *Consider the Bessel equation*

$$w''(z) + \frac{1}{z} w'(z) + \left(1 - \frac{\nu^2}{z^2}\right) w(z) = 0.$$

We have $\tau(z) = 1$, $\sigma(z) = z$ *and* $\gamma(z) = z^2 - \nu^2$. *Put* $\Phi(z) = \exp(\mathrm{i}z)z^\nu$ *and* $\lambda = \mathrm{i}(2\nu + 1)$. *Then* $u = u(z) = w(z)/\Phi(z)$ *solves*

$$u''(z) + \frac{2\mathrm{i}z + 2\nu + 1}{z}u'(z) + \frac{\mathrm{i}(2\nu + 1)}{z}u(z) = 0.$$

If σ is a constant or a second order polynomial with double zero, then other transformations are possible which lead to differential equations of hypergeometric type. Details can be found in [10].

Definition 3.4. *Any solution to an equation of hypergeometric type is called* function of hypergeometric type.

3.2 Polynomials of Hypergeometric Type

Hypergeometric functions ${}_2\mathrm{F}_1(a, b; c; z)$ are polynomials if a is a negative integer.

We look for polynomial solutions to the differential equation of hypergeometric type

$$\sigma(z)w''(z) + \tau(z)w'(z) + \lambda w(z) = 0. \tag{3.3}$$

Lemma 3.5. *If w solves (3.3), then the derivatives $w_k(z) := \partial_z^k w(z)$ solve*

$$\sigma(z)w_k''(z) + \tau_k(z)w_k'(z) + \mu_k w_k(z) = 0, \tag{3.4}$$
$$\tau_k(z) = \tau(z) + k\sigma'(z), \qquad \mu_k = \lambda + k\tau'(z) + \frac{k(k-1)}{2}\sigma''(z) \in \mathbb{C}.$$

Proof. This is clear for $k = 0$. Differentiating (3.4), we obtain

$$\sigma w_{k+1}'' + \sigma' w_{k+1}' + \tau_k w_{k+1}' + \tau_k' w_{k+1} + \mu_k w_{k+1} = 0.$$

This is the differential equation for w_{k+1}. □

This can be reversed, under an additional assumption:

Lemma 3.6. *If a function w_1 solves (3.4) with $k = 1$ and $\lambda \neq 0$, then a function w with $w' = w_1$ exists which solves (3.3).*

Proof. Define

$$w(z) = -\frac{1}{\lambda}(\sigma(z)w_1'(z) + \tau(z)w_1(z)).$$

We proceed in two steps:

Step 1: $w' = w_1$. This is easy:

$$\begin{aligned} w' &= -\frac{1}{\lambda}\left(\sigma' w_1' + \sigma w_1'' + \tau' w_1 + \tau w_1'\right) \\ &= -\frac{1}{\lambda}\left(\sigma' w_1' - \tau_1 w_1' - \mu_1 w_1 + \tau' w_1 + \tau w_1'\right) = -\frac{1}{\lambda}(-\lambda w_1) = w_1. \end{aligned}$$

Step 2: w solves (3.3). We have $w_1' = w''$, hence

$$w = -\frac{1}{\lambda}\left(\sigma w'' + \tau w'\right),$$

which is equivalent to (3.3).

□

Corollary 3.7. *If w_k is a solution of (3.4) and $\lambda, \mu_1, \dots, \mu_{k-1} \neq 0$, then a function w with $w_k = \partial_z^k w$ exists which is a solution to (3.3).*

Proposition 3.8. *Let $n \in \mathbb{N}_+$ with $\lambda(n) := -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z) \neq 0$, and assume that*

$$\mu_k = \lambda(n) + k\tau'(z) + \binom{k}{2}\sigma''(z) \neq 0, \qquad k = 0, 1, 2, \dots, n-1.$$

Then (3.3) with $\lambda = \lambda(n)$ has a solution w which is a polynomial of degree n.

Proof. We have $\mu_n = 0$. Then (3.4) with $k = n$ reads

$$\sigma(z)w_n''(z) + \tau_n(z)w_n'(z) = 0$$

which has a constant $w_n \equiv c \in \mathbb{C}$ as solution. Then a function w exists with $w_n = \partial_z^n w(z)$ which solves (3.3). This function is a polynomial of degree n. □

Definition 3.9. *These polynomials are called* polynomials of hypergeometric type.

Proposition 3.10 (Formula of Rodriguez). *Let $\lambda = \lambda(n)$. Define a function $\varrho = \varrho(z)$ by $\partial_z(\sigma(z)\varrho(z)) = \tau(z)\varrho(z)$, i.e.,*

$$\varrho(z) = \frac{1}{\sigma(z)} \exp\left(\int \frac{\tau(\zeta)}{\sigma(\zeta)}\, \mathrm{d}\zeta \right).$$

Then polynomial solutions to (3.3) are given by the formula

$$P_n(z) = \frac{B_n}{\varrho(z)} \partial_z^n \left(\varrho(z)\sigma^n(z) \right),$$

where $B_n \in \mathbb{C}$ are free constants.

Proof. Put $\varrho_k(z) = \varrho(z)\sigma^k(z)$. Then $(\sigma\varrho_k)' = \tau_k\varrho_k$. The differential equations (3.3) and (3.4) can be written in self-adjoint form:

$$\begin{aligned}
&(\varrho\sigma w')' + \lambda(n)\varrho w = 0,\\
&(\varrho_k\sigma w_k')' + \mu_k\varrho_k w_k = 0.
\end{aligned}$$

We know $\partial_z^k w = w_k$ for $k = 1, 2, \dots, n-1$. Write $w_0 = w$. Then we have

$$\varrho_k w_k = -\frac{1}{\mu_k}(\varrho_{k+1}w_{k+1})', \qquad k = 0, 1, 2, \dots, n-1,$$

because of $(\varrho_{k+1}w_{k+1})' = (\varrho_k\sigma w_k')' = -\mu_k\varrho_k w_k$, for $0 \leq k \leq n-1$.

Plugging these equations into each other, we get

$$\varrho w_0 = \left(\frac{-1}{\mu_0}\right) \cdot \ldots \cdot \left(\frac{-1}{\mu_{n-1}}\right) \partial_z^n(\varrho_n w_n).$$

We know that w_n is a constant, which completes the proof. □

Corollary 3.11. *Put*

$$A_k = (-1)^k \prod_{l=0}^{k-1} \mu_l, \qquad A_0 = 1.$$

Then the polynomial solutions of the Rodriguez formula satisfy

$$\begin{aligned}
P_n(z) &= \frac{1}{\varrho(z)} \frac{1}{A_n} (\partial_z^n P_n) \partial_z^n \left(\varrho(z)\sigma^n(z) \right),\\
\partial_z^k P_n(z) &= \frac{1}{\varrho(z)\sigma^k(z)} \frac{A_k}{A_n} (\partial_z^n P_n) \partial_z^{n-k} (\varrho(z)\sigma^n(z)).
\end{aligned}$$

Especially, we have $B_n = \frac{1}{A_n}\partial_z^n P_n$.

3.2.1 Legendre Polynomials

The Legendre differential equation is

$$(1 - z^2)w''(z) - 2zw'(z) + \lambda w(z) = 0.$$

We have $\sigma(z) = 1 - z^2$, $\tau(z) = -2z$, $\mu_n = \lambda + n\tau' + \binom{n}{2}\sigma'' = \lambda - 2n - n(n-1)$, which vanishes for $\lambda = \lambda(n) = n(n+1)$, $n \in \mathbb{N}_0$. We may choose $\varrho = \varrho(z) \equiv 1$ and get

$$P_n(z) = B_n \partial_z^n((1 - z^2)^n)$$

as polynomial solution to the Legendre differential equation for $\lambda = n(n+1)$. It is custom to set $B_n = \frac{(-1)^n}{2^n n!}$.

3.2.2 Chebyshev Polynomials

The Chebyshev differential equation is

$$(1 - z^2)w''(z) - zw'(z) + \lambda w(z) = 0.$$

We have $\sigma(z) = 1 - z^2$, $\tau(z) = -z$, $\mu_n = \lambda + n\tau' + \binom{n}{2}\sigma'' = \lambda - n - n(n-1)$, which vanishes for $\lambda = \lambda(n) = n^2$, $n \in \mathbb{N}_0$. By computation, we find

$$\varrho(z) = (1 - z^2)^{-1/2}, \qquad P_n(z) = B_n \sqrt{1 - z^2}\, \partial_z^n \left((1 - z^2)^{n-1/2} \right).$$

The famous Chebsyhev polynomials

$$T_n(x) = \cos(n \arccos x)$$

are obtained via

$$T_n(x) = \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \sqrt{1 - x^2}\, \partial_x^n \left((1 - x^2)^{n-1/2} \right), \qquad |x| < 1.$$

3.2.3 Hermite Polynomials

The Hermite differential equation is

$$w''(z) - zw'(z) + \lambda w(z) = 0,$$

with $\sigma = \sigma(z) = 1$ and $\tau = \tau(z) = -z$. Then we find $\mu_n = \lambda + n\tau' + \binom{n}{2}\sigma'' = \lambda - n$, which vanishes for $\lambda = \lambda(n) = n$, $n \in \mathbb{N}_0$. Then we find

$$\varrho(z) = \exp(-z^2/2), \qquad P_n(z) = B_n \exp(z^2/2) \partial_z^n(\exp(-z^2/2)).$$

The known Hermite polynomials He_n are obtained for $B_n = (-1)^n$.

3.2.4 Jacobi Polynomials

The Jacobi differential equation is

$$z(1 - z)w''(z) + (\gamma - (\alpha + 1)z)w'(z) + \lambda w(z) = 0,$$

with $\sigma = \sigma(z) = z(1 - z)$ and $\tau = \tau(z) = \gamma - (\alpha + 1)z$. We obtain $\mu_n = \lambda - n(\alpha + 1) - n(n-1)$, which vanishes for $\lambda = \lambda(n) = n(\alpha + n)$. We then find

$$\varrho(z) = \frac{1}{z(1-z)} \left(z^\gamma (z - 1)^{1+\alpha-\gamma} \right).$$

Polynomial solutions are the Jacobi polynomials,

$$J_n(\alpha,\gamma;z) = {}_2\mathrm{F}_1(-n,\alpha+n;\gamma;z).$$

The Legendre polynomials and Chebyshev polynomials can be written in terms of Jacobi polynomials:

$$P_n(z) = J_n\left(1,1;\frac{1-z}{2}\right) = {}_2\mathrm{F}_1\left(-n,n+1;1;\frac{1-z}{2}\right),$$
$$T_n(z) = J_n\left(0,\frac{1}{2};\frac{1-z}{2}\right) = {}_2\mathrm{F}_1\left(n,-n;\frac{1}{2};\frac{1-z}{2}\right).$$

Another approach to the Jacobi polynomials is the transformation of the variable z, $z=\frac{1+t}{2}$. The exceptional points $z=0,1$ then transform to $t=\pm 1$. Transforming the variable $z\mapsto t$, and writing z again (abusing notation), we find

$$(1-z^2)w''(z) + (b-a-(a+b+2)z)w'(z) + \lambda w(z) = 0,$$

with certain constants $a,b\in\mathbb{C}$ and a new λ. We assume

$$\Re a > -1, \qquad \Re b > -1,$$

and then find $\varrho(z) = (1-z)^a(1+z)^b$. Polynomial solutions exist for

$$\lambda = \lambda(n) = (a+b+2)n + n(n-1) = n(a+b+n+1).$$

These polynomial solutions have the form

$$P_n(z) = \frac{B_n}{1}(1-z)^{-a}(1+z)^{-b}\partial_z^n\left((1-z)^{n+a}(1+z)^{n+b}\right).$$

The custom setting $B_n = \frac{(-1)^n}{2^n n!}$ leads to the Jacobi polynomials $P_n^{(a,b)}$.
We obtain polynomials of Legendre type for $a=b=0$, and polynomials of Chebyshev type for $a=b=-\frac{1}{2}$.
From Corollary 3.11, it quickly follows that

$$\partial_z P_n^{(\alpha,\beta)}(z) = \frac{\alpha+\beta+n+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(z).$$

3.2.5 Laguerre Polynomials

The Laguerre differential equation is

$$zw''(z) + (\alpha+1-z)w'(z) + \lambda w(z) = 0,$$

with $\sigma=\sigma(z)=z$, $\tau=\tau(z)=\alpha+1-z$, hence $\mu_n = \lambda - n + \binom{n}{2}\cdot 0$ which vanishes for $\lambda=\lambda(n)=n$. Then we find

$$\varrho(z) = z^\alpha\exp(-z), \qquad P_n(z) = B_n e^z z^{-\alpha}\partial_z^n\left(e^{-z}z^{\alpha+n}\right).$$

The Laguerre polynomials are obtained for $B_n = \frac{1}{n!}$,

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-z)^k}{k!}.$$

We can also write

$$nL_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_1\mathrm{F}_1(-n;\alpha+1;z).$$

There is a connection to the Hermite polynomials:

$$He_{2n}(z) = (-2)^n n! L_n^{(-1/2)}\left(\frac{z^2}{2}\right),$$
$$He_{2n+1}(z) = (-2)^n n! L_n^{(1/2)}\left(\frac{z^2}{2}\right).$$

3.2.6 Solutions to Bessel Equations

We know that the Bessel differential equation $z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0$ can be transformed by setting

$$w(z) = \Phi(z)u(z) = e^{\mathrm{i}z}z^\nu u(z),$$

giving us the following differential equation for u:

$$zu''(z) + (2\mathrm{i}z + 2\nu + 1)u'(z) + \mathrm{i}(2\nu + 1)u(z) = 0,$$

with $\sigma(z) = z$, $\tau(z) = 2\mathrm{i}z + (2\nu + 1)$, and $\lambda_0 = \mathrm{i}(2\nu + 1)$. We find $\lambda(n) = -2\mathrm{i}n$ as values that give polynomial solutions, hence

$$\nu = -n - \frac{1}{2}, \qquad n \in \mathbb{N}_0.$$

As a result, $\mu_k = 2\mathrm{i}(k - n) \neq 0$ for $k \neq n$. It turns out that

$$\varrho(z) = e^{2\mathrm{i}z}z^{-2n-1},$$
$$P_n(z) = \frac{B}{e^{2\mathrm{i}z}}z^{2n+1}\partial_z^n\left(e^{2\mathrm{i}z}z^{-n-1}\right).$$

The derivative $\partial_z^n(\dots)$ can be evaluated by the Leibniz formula. Going back to the solution w, we then find

$$w(z) = B_n e^{\mathrm{i}z}\frac{1}{\sqrt{z}}\frac{(2\mathrm{i})^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+k)!\left(\frac{\mathrm{i}}{2z}\right)^k.$$

It is custom to choose

$$B_n = \sqrt{\frac{2}{\pi}}\frac{1}{2^n},$$

which then yields $w = H^{(1)}_{-n-1/2}$, the HANKEL *function of first kind.*

Remark 3.12. *Obviously, there must be a relation*

$$H^{(1)}_{-n-1/2}(z) = C_1J_{-n-1/2}(z) + C_2J_{n+1/2}(z).$$

One can show that $C_1 = 1$ *and* $C_2 = (-1)^n\mathrm{i}$.

Remark 3.13. *We have the following asymptotic behavior for* $z \to \infty$*:*

$$H^{(1)}_{-n-1/2}(z) \sim \sqrt{\frac{2}{\pi z}}e^{\mathrm{i}z}\mathrm{i}^n.$$

Remark 3.14. *The general definition of* $H^{(1)}_\nu(z)$ *is*

$$H^{(1)}_\nu(z) = \begin{cases} \sqrt{\frac{2}{\pi}}z^\nu\frac{\exp(\mathrm{i}(z-\nu\pi/2-\pi/4))}{\Gamma(\nu+1/2)}\int_{s=0}^\infty e^{-sz}s^{\nu-1/2}\left(1+\frac{\mathrm{i}s}{2}\right)^{\nu-1/2}\mathrm{d}s & : \Re\nu \geq 0, \\ e^{-\mathrm{i}\nu\pi}H^{(1)}_{-\nu}(z) & : \Re\nu < 0. \end{cases}$$

As a consequence,

$$H^{(1)}_{-n-1/2}(z) = e^{-\mathrm{i}(-n-1/2)\pi}\sqrt{\frac{2}{\pi}}z^{n+1/2}\frac{\exp(\mathrm{i}(z-(n+1/2)\pi/2-\pi/4))}{n!}\int_{s=0}^\infty e^{-sz}s^n\left(1+\frac{\mathrm{i}s}{2}\right)^n\mathrm{d}s.$$

3.3 Integral Representations

We know that a DE of hypergeometric type (3.3) has a polynomial solution

$$P_n(z) = \frac{1}{\varrho(z)} \partial_z^n (\varrho(z)\sigma(z)^n)$$

provided that $n \in \mathbb{N}$ and $\lambda = \lambda(n) = -n\tau' - \binom{n}{2}\sigma''$. By the Cauchy integral formula, we then have

$$P_n(z) = \frac{1}{\varrho(z)} \frac{n!}{2\pi \mathrm{i}} \int_{|t-z|=\varepsilon} \frac{\sigma^n(t)\varrho(t)}{(t-z)^{n+1}} \,\mathrm{d}t,$$

assuming that z is not a zero of σ; otherwise, ϱ would not be holomorphic.
This suggests the following ansatz: if $\lambda \in \mathbb{C}$ and there is a number $\nu \in \mathbb{C}$ with $\lambda = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma''$, and if a curve $\Gamma \subset \mathbb{C}$ is chosen suitably, then the function w_ν defined by

$$w_\nu(z) := \frac{u(z)}{\varrho(z)}, \qquad u(z) := \int_\Gamma \frac{\sigma(t)^\nu \varrho(t)}{(t-z)^{\nu+1}} \,\mathrm{d}t \tag{3.5}$$

should be a solution to (3.3) with that constant λ.

Proposition 3.15. *We make the following assumptions concerning (3.3):*

- *there is a $\nu \in \mathbb{C}$ with $\lambda = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma''$,*
- *the function u from (3.5) is holomorphic, and can be differentiated twice under the integral sign,*
- *if the endpoints of Γ are denoted by $t_1, t_2 \in \mathbb{C} \cup \{\infty\}$, then*

$$\left.\frac{\sigma^{\nu+1}(t)\varrho(t)}{(t-z)^{\nu+2}}\right|_{t=t_1}^{t=t_2} = 0.$$

Then the function $w_\nu = \frac{u}{\varrho}$ is a solution to (3.3).

Remark 3.16. *The endpoint condition is satisfied in the following cases:*

- *t_1 and t_2 are zeros of $\sigma = \sigma(t)$ with $\Re\nu > 0$,*
- *$t_1 = z$ and $t_2 = z$ with $\Re(\nu + 2) < 0$,*
- *$t_1 = \infty$ and $t_2 = \infty$ with*

$$\lim_{t\to\infty, t\in\Gamma} \frac{\sigma^{\nu+1}(t)\varrho(t)}{(t-z)^{\nu+1}} = 0.$$

Proof. We start with a differential equation for u: if $\sigma w'' + \tau w' + \lambda w = 0$ and $(\sigma\varrho)' = \tau\varrho$ and $u = \varrho w$, then:

$$\begin{aligned}
&(\sigma\varrho w')' + \lambda\varrho w = 0,\\
&\sigma\varrho w' = (\sigma\varrho w)' - (\sigma\varrho)' w = (\sigma u)' - \tau\varrho w = (\sigma u)' - \tau u,\\
&\implies ((\sigma u)' - \tau u)' + \lambda u = 0,\\
&\implies \sigma u'' + (2\sigma' - \tau)u' + (\sigma'' - \tau' + \lambda)u = 0.
\end{aligned}$$

This reasoning can be reversed.
If $\lambda = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma''$, then

$$\sigma'' - \tau' + \lambda = -(\nu+1)\left(\frac{\nu-2}{2}\sigma'' + \tau'\right).$$

Now let $u = u(z)$ be given by (3.5). We compute:

$$\begin{aligned}
I &= \sigma(z)u''(z) + (2\sigma'(z) - \tau(z))u(z) - (\nu+1)\left(\frac{\nu-2}{2}\sigma'' + \tau'\right)u(z) \\
&= (\nu+1)\int_\Gamma \sigma^\nu(t)\varrho(t)\left(\frac{(\nu+2)\sigma(z)}{(t-z)^{\nu+3}} + \frac{2\sigma'(z)-\tau(z)}{(t-z)^{\nu+2}} - \frac{\frac{\nu-2}{2}\sigma''+\tau'}{(t-z)^{\nu+1}}\right)\mathrm{d}t \\
&= (\nu+1)\int_\Gamma \frac{\sigma^\nu(t)\varrho(t)}{(t-z)^{\nu+3}}\left((\nu+2)\sigma(z) + (2\sigma'(z)-\tau(z))(t-z) - \left(\frac{\nu-2}{2}\sigma''+\tau'\right)(t-z)^2\right)\mathrm{d}t.
\end{aligned}$$

By Taylor's formula, we get

$$\begin{aligned}
&\sigma(z) = \sigma(t) - \sigma'(t)(t-z) + \frac{1}{2}\sigma''\cdot(t-z)^2, \\
&\sigma'(z) = \sigma(t) - \sigma''\cdot(t-z), \\
&\tau(z) = \tau(t) - \tau'\cdot(t-z).
\end{aligned}$$

Recall that σ'' and τ' are constants.

Inserting these expressions, we find, after some calculation,

$$I = (\nu+1)\int_\Gamma \frac{\sigma^\nu(t)\varrho(t)}{(t-z)^{\nu+3}}\left((\nu+2)\sigma(t) - \nu\sigma'(t)(t-z) - \tau(t)(t-z)\right)\mathrm{d}t. \tag{3.6}$$

Now we observe that

$$(\sigma^{\nu+1}\varrho)' = (\sigma^\nu\cdot\varrho\sigma)' = \nu\sigma^{\nu-1}\sigma'\varrho\sigma + \sigma^\nu(\varrho\sigma)' = \nu\sigma^\nu\varrho\sigma' + \sigma^\nu\tau\varrho.$$

Then it is easy to check that

$$\begin{aligned}
\partial_t\left(\frac{\sigma^{\nu+1}(t)\varrho(t)}{(t-z)^{\nu+2}}\right) &= \frac{1}{(t-z)^{\nu+3}}\left(-(\nu+2)\sigma^{\nu+1}\varrho + (\nu\sigma^\nu\varrho\sigma' + \sigma^\nu\tau\varrho)(t-z)\right) \\
&= \frac{\sigma^\nu(t)\varrho(t)}{(t-z)^{\nu+3}}\left(-(\nu+2)\sigma(t) + \nu\sigma'(t)(t-z) + \tau(t)(t-z)\right).
\end{aligned}$$

Modulo the sign, this is exactly the integrand from (3.6). Consequently, we get

$$I = -(\nu+1)\left(\frac{\sigma^{\nu+1}(t)\varrho(t)}{(t-z)^{\nu+2}}\right)\Bigg|_{t=t_1}^{t=t_2} = 0.$$

Therefore, the function u is a solution of the differential equation under consideration. □

3.3.1 Representations for the Legendre Functions

For the Legendre DE, we have $\sigma = \sigma(z) = 1 - z^2$, $\tau = \tau(z) = -2z$ and $\varrho = \varrho(z) = 1$. We then obtain $\lambda = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma'' = \nu(\nu+1)$. The equation

$$\nu^2 + \nu - \lambda = 0$$

has two solutions $\nu_1, \nu_2 \in \mathbb{C}$ with $\nu_1 + \nu_2 = -1$, $\nu_1\nu_2 = -\lambda$. Choose such a ν_j. Then we get the solution representation

$$w_\nu(z) = \int_\Gamma \frac{(1-t^2)^\nu}{(t-z)^{\nu+1}}\,\mathrm{d}t,$$

for $\Re\nu > -1$, and (for instance) $\Gamma = [-1,1]$, $z \notin [-1,1]$.

The Legendre differential equation can be transformed, by setting $\zeta = (1-z)/2$, to the equation

$$\zeta(1-\zeta)\partial_\zeta^2 w(\zeta) + (1-2\zeta)\partial_\zeta w(\zeta) + \nu(\nu+1)w(\zeta) = 0,$$

which is a hypergeometric DE with $a = -\nu$, $b = \nu + 1$ and $c = 1$. Consequently, the function

$$w = w(z) = P_\nu(z) := {}_2\mathrm{F}_1\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right), \qquad |1-z| < 2,$$

is a solution to the Legendre differential equation.

Another transformation of the Legendre DE is to set $\zeta = z^{-2}$ and $w = z^{-\nu-1}v$, leading to ([8]):

$$\zeta(1-\zeta)\partial_\zeta^2 v(\zeta) + \left(\left(\nu + \frac{3}{2}\right) - \left(\nu + \frac{5}{2}\right)\zeta\right)\partial_\zeta v(\zeta) - \left(\frac{\nu}{2}+1\right)\left(\frac{\nu}{2}+\frac{1}{2}\right)v(\zeta) = 0,$$

which is again a hypergeometric DE with $a = \frac{\nu}{2} + 1$, $b = \frac{\nu}{2} + \frac{1}{2}$, $c = \nu + \frac{3}{2}$. Consequently, we have a solution

$$w(z) = Q_\nu(z) := \frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma(\nu+3/2)(2z)^{\nu+1}}\, {}_2\mathrm{F}_1\left(\frac{\nu}{2}+1, \frac{\nu}{2}+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right), \qquad |z| > 1, \qquad \nu \neq -1, -2, \ldots.$$

In the definition of P_ν and Q_ν, we have to assume that $z \notin (-\infty, -1]$ and $z \notin (-\infty, 1]$, respectively. Both functions can be analytically extended to the cut complex plane.

It can be shown that Q_ν has a logarithmic pole at $z = 1$. Therefore, the functions P_ν and Q_ν must be linearly independent. A deeper analysis shows that

$$Q_\nu(z) = \frac{1}{2^{\nu+1}} \int_{t=-1}^{t=1} \frac{(1-t^2)\nu}{(z-t)^{\nu+1}}\,\mathrm{d}t, \qquad \Re\nu > -1, \qquad z \in \mathbb{C} \setminus (-\infty, 1].$$

This is exactly our above integral representation formula.

Another type of integral representation can be derived as follows ([8]):

We start with

$$\frac{2}{\pi}\int_{\varphi=0}^{\pi/2} \sin^{2k}(\varphi)\,\mathrm{d}\varphi = \frac{(1/2)_k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

Then we can write

$$\begin{aligned}
P_\nu(z) = {}_2\mathrm{F}_1(-\nu, \nu+1; 1; (1-z)/2) &= \sum_{k=0}^{\infty} \frac{(-\nu)_k(\nu+1)_k}{(k!)^2}\left(\frac{1-z}{2}\right)^k \\
&= \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-\nu)_k(\nu+1)_k}{(1/2)_k k!}\left(\frac{1-z}{2}\right)^k \int_{\varphi=0}^{\pi/2} \sin^{2k}(\varphi)\,\mathrm{d}\varphi \\
&= \frac{2}{\pi}\int_{\varphi=0}^{\infty} {}_2\mathrm{F}_1\left(-\nu, \nu+1; \frac{1}{2}; \frac{1-z}{2}\sin^2\varphi\right)\mathrm{d}\varphi.
\end{aligned}$$

We can compute the last hypergeometric function. Our claim is:

$${}_2\mathrm{F}_1(-\nu, \nu+1; 1/2; -u) = f_\nu(u) := \frac{(\sqrt{1+u}+\sqrt{u})^{2\nu+1} + (\sqrt{1+u}+\sqrt{u})^{-2\nu-1}}{2\sqrt{1+u}}, \qquad |u| < 1.$$

This is proved in several steps:

- The function f_ν is first defined for $|u| < 1$, $|\arg u| < \pi$, and obviously analytic there.
- The function f_ν does *not* jump when crossing the line $(-1, 0)$. Note that $\sqrt{1+u} + \sqrt{u}$ is a complex number with modulus one for u in the interval $(-1, 0)$.
- Therefore, the function f_ν is regular for $|u| < 1$. We have $f_\nu(0) = 1$.
- The function f_ν solves

$$\sqrt{u}\left(\sqrt{u}\sqrt{1+u}\left(\sqrt{1+u}f_\nu\right)'\right)' - (\nu+1/2)^2 f_\nu = 0, \qquad |u| < 1, \qquad |\arg u| < \pi.$$

- Then we conclude that f_ν solves

$$u(1+u)f_\nu'' + \left(\frac{1}{2}+2u\right)f_\nu' - \nu(\nu+1)f_\nu = 0,$$

which can be transformed into a hypergeometric DE upon replacing u by $-u$. The parameters of this DE are $\alpha = -\nu$, $\beta = \nu+1$, and $\gamma = \frac{1}{2}$.

- Such a hypergeometric DE has exactly one solution that is regular at the origin and approaches 1 for $u \to 0$, namely ${}_2\mathrm{F}_1(\alpha,\beta;\gamma;-u)$. The other base solution $(-u)^{1-\gamma}\,{}_2\mathrm{F}_1(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;-u)$ is singular for $u \to 0$.

- This proves the identity.

Therefore, we have

$$P_\nu(z) = \frac{2}{\pi}\int_{\varphi=0}^{\pi/2} f_\nu\left(\frac{z-1}{2}\sin^2\varphi\right) \mathrm{d}\varphi, \qquad |1-z|<2.$$

We know that P_ν is a regular function of z for $z \in \mathbb{C}\setminus(-\infty,-1]$. The same holds for the RHS of this representation formula. Consequently, this integral representation holds for $z \in \mathbb{C}\setminus(-\infty,-1]$ and all ν.

Now we assume $z = \cosh\alpha$ with $\alpha \geq 0$ and substitute $\sinh\frac{\theta}{2} = \sinh\frac{\alpha}{2}\sin\varphi$. After some calculation, we then find

$$P_\nu(\cosh\alpha) = \frac{2}{\pi}\int_0^\alpha \frac{\cosh\left(\nu+\frac{1}{2}\right)\theta}{\sqrt{2\cosh\alpha - 2\cosh\theta}}\,\mathrm{d}\theta, \qquad \alpha \geq 0, \qquad \nu \in \mathbb{C}.$$

Be careful when analyzing this integral—the integrand has a pole for $\theta = \alpha$.

Another representation can be found for $-1 < z \leq 1$. Then we can write $z = \cos\beta$ with $0 \leq \beta < \pi$, and substitute $\sin\frac{\theta}{2} = \sin\frac{\beta}{2}\sin\varphi$, giving us the famous Dirichlet–Mehler formula

$$P_\nu(\cos\beta) = \frac{2}{\pi}\int_0^\beta \frac{\cos\left(\nu+\frac{1}{2}\right)\theta}{\sqrt{2\cos\theta - 2\cos\beta}}\,\mathrm{d}\theta,$$

for any $\nu \in \mathbb{C}$. Also this integrand has a pole at the end of the integration interval.

3.3.2 Representations for the Bessel Functions

If w is a solution to the Bessel equation with parameter μ, then $u = u(z)$, defined by $u(z) = e^{-\mathrm{i}z}z^{-\mu}w(z)$ is a solution to

$$zu''(z) + (2\mathrm{i}z + 2\mu + 1)u'(z) + \mathrm{i}(2\mu+1)u(z) = 0,$$

with $\sigma(z) = z$, $\tau(z) = 2\mathrm{i}z + (2\mu+1)$ and $\lambda = \mathrm{i}(2\mu+1)$. We find $\varrho(z) = e^{2\mathrm{i}z}z^{2\mu}$, and make the ansatz

$$u(z) = \frac{1}{\varrho(z)}\int_\Gamma \frac{\sigma^\nu(t)\varrho(t)}{(t-z)^{\nu+1}}\,\mathrm{d}t = e^{-2\mathrm{i}z}z^{-2\mu}\int_\Gamma \frac{t^\nu e^{2\mathrm{i}t}t^{2\mu}}{(t-z)^{\nu+1}}\,\mathrm{d}t,$$

where ν is an not yet known number, related to $\lambda = \mathrm{i}(2\mu+1)$ via

$$\lambda = -\nu\tau' - \frac{\nu(\nu+1)}{2}\sigma'' = -2\nu\mathrm{i}, \qquad \Longrightarrow \qquad \nu = -\mu - \frac{1}{2}.$$

The result is

$$u(z) = e^{-2\mathrm{i}z}z^{-2\mu}\int_\Gamma \frac{t^{\mu-\frac{1}{2}}e^{2\mathrm{i}t}}{(t-z)^{-\mu+\frac{1}{2}}}\,\mathrm{d}t,$$

or, reformulated in terms of w (modulo a constant phase factor),

$$w(z) = e^{-\mathrm{i}z}z^{-\mu}\int_\Gamma t^{\mu-\frac{1}{2}}(z-t)^{\mu-\frac{1}{2}}e^{2\mathrm{i}t}\,\mathrm{d}t.$$

In determining the endpoints of Γ, several choices are possible:

- $t = 0$ if $\Re(\mu + \frac{1}{2}) > 0$,
- $t = z$ if $\Re(\mu - \frac{3}{2}) > 0$,
- $t = \infty$ if $\Im t \to +\infty$.

Three possible paths come to mind, assuming $\Re\mu > \frac{3}{2}$ and $z \in \mathbb{R}_+$:

Γ_0: from 0 to z

Γ_1: from z "vertically" to $+\infty$,

Γ_2: from 0 "vertically" to $+\infty$.

By the Cauchy integral theorem, we see

$$\int_{\Gamma_0} f(t)\,\mathrm{d}t + \int_{\Gamma_1} f(t)\,\mathrm{d}t = \int_{\Gamma_2} f(t)\,\mathrm{d}t,$$

for any holomorphic function f.

We fix the arguments of the power functions as follows:

$t \in \Gamma_0$	:	$\arg t = 0$,	$\arg(z-t) = 0$,
$t \in \Gamma_1$	:	$\arg t \in [0, \pi/2)$,	$\arg(z-t) = -\pi/2$,
$t \in \Gamma_2$	:	$\arg t = \pi/2$,	$\arg(z-t) \in (-\pi/2, 0]$.

In other words, we cut the t-plane along $(-\infty, 0]$ and $[z, +\infty)$.

Now we compute the three integrals:

Along Γ_0: Parametrizing the interval $[0, z] \subset \mathbb{R}$ by $t = z \cdot \frac{1+s}{2}$ with $-1 \le s \le 1$, we obtain

$$\begin{aligned} w_\mu(z) &= e^{-\mathrm{i}z} z^{-\mu} \int_{s=-1}^{1} e^{\mathrm{i}z(1+s)} \left(z\left(\frac{1+s}{2}\right)\left(z - z\left(\frac{1+s}{2}\right)\right)\right)^{\mu-1/2} \frac{z}{2}\,\mathrm{d}s \\ &= z^\mu \frac{1}{2^{2\mu}} \int_{s=-1}^{1} e^{\mathrm{i}zs}(1-s^2)^{\mu-1/2}\,\mathrm{d}s = \frac{z^\mu}{2^{2\mu-1}} \int_{s=0}^{1} \cos(zs)(1-s^2)^{\mu-1/2}\,\mathrm{d}s, \end{aligned}$$

for $\Re\mu > \frac{3}{2}$ and $z > 0$. By analyticity, this integral represents a solution of the Bessel DE in the cut plane $\mathbb{C}_z$.

We expand the Cosine into its power series, exchange summation and integration, use

$$\int_{s=0}^{1} s^{2m}(1-s^2)^{\mu-1/2}\,\mathrm{d}s = \frac{1}{2}\int_{x=0}^{1} x^{m-1/2}(1-x)^{\mu-1/2}\,\mathrm{d}x = \frac{1}{2}B(m+1/2, \mu+1/2),$$

exploit the doubling formula of the Gamma function, and obtain the POISSON integral representation of the Bessel function:

$$J_\mu(z) = \frac{2}{\sqrt{\pi}\Gamma(\mu+1/2)}\left(\frac{z}{2}\right)^\mu \int_{s=0}^{1} \cos(zs)(1-s^2)^{\mu-1/2}\,\mathrm{d}s, \qquad \Re\mu > \frac{3}{2}. \tag{3.7}$$

Along Γ_1: We parametrize the integration interval as $t = z(1 + \frac{\mathrm{i}s}{2})$ with $0 \le s \le \infty$, and obtain, for $\Re\mu > \frac{3}{2}$,

$$\begin{aligned} w_\mu(z) &= e^{-\mathrm{i}z} z^{-\mu} \int_{s=0}^{\infty} e^{2\mathrm{i}z - sz} \left(z\left(1 + \frac{\mathrm{i}s}{2}\right)\left(z - z - z\frac{\mathrm{i}s}{2}\right)\right)^{\mu-1/2} \frac{\mathrm{i}z}{2}\,\mathrm{d}s \\ &= e^{\mathrm{i}z} z^\mu \frac{\mathrm{i}}{2} \int_{s=0}^{\infty} e^{-sz} \left(\left(1 + \frac{\mathrm{i}s}{2}\right)\left(-\frac{\mathrm{i}s}{2}\right)\right)^{\mu-1/2} \mathrm{d}s \\ &= -e^{\mathrm{i}z} \frac{1}{\sqrt{2}} \left(\frac{z}{2}\right)^\mu e^{-\mathrm{i}(\mu+1/2)\pi/2} \int_{s=0}^{\infty} e^{-sz} s^{\mu-1/2} \left(1 + \frac{\mathrm{i}s}{2}\right)^{\mu-1/2} \mathrm{d}s. \end{aligned}$$

Choosing a constant factor, we get

$$H_\mu^{(1)}(z) = \sqrt{\frac{2}{\pi}} z^\mu \frac{e^{\mathrm{i}(z-\mu\pi/2-\pi/4)}}{\Gamma(\mu+1/2)} \int_{s=0}^{\infty} e^{-sz} s^{\mu-1/2} \left(1+\frac{\mathrm{i}s}{2}\right)^{\mu-1/2} \mathrm{d}s, \tag{3.8}$$

the POISSON representation of the first HANKEL function. This is a solution to the Bessel DE for $z>0$ and $\Re\mu > 3/2$, but the integral exists for any z with $\Re z > 0$ and is holomorphic there.

One can show that $H_\mu^{(1)}$ is an entire analytic function of μ with $H_{-\mu}^{(1)}(z) = e^{\mathrm{i}\mu\pi} H_\mu^{(1)}(z)$.

Along Γ_2: We parametrize with $t = \frac{\mathrm{i}zs}{2}$, and obtain, by similar calculations,

$$H_\mu^{(2)}(z) = \sqrt{\frac{2}{\pi}} z^\mu \frac{e^{-\mathrm{i}(z-\mu\pi/2-\pi/4)}}{\Gamma(\mu+1/2)} \int_{s=0}^{\infty} e^{-sz} s^{\mu-1/2} \left(1-\frac{\mathrm{i}s}{2}\right)^{\mu-1/2} \mathrm{d}s, \tag{3.9}$$

for $\Re\mu > 3/2$ and $\Re z > 0$. This function is called the *second* HANKEL *function*. Formally, we have replaced i by $-$i everywhere.

One can show that first and second Hankel function are linearly independent, and that

$$J_\mu(z) = \frac{1}{2}\left(H_\mu^{(1)}(z) + H_\mu^{(2)}(z)\right), \qquad \Re\mu > \frac{3}{2}.$$

The so-called NEUMANN function, or cylinder function of second kind, is defined by

$$N_\mu(z) = \frac{1}{2\mathrm{i}}\left(H_\mu^{(1)}(z) - H_\mu^{(2)}(z)\right).$$

Sometimes it is denoted by Y. We have the identities

$$H_\mu^{(1)}(z) = J_\mu(z) + \mathrm{i}N_\mu(z),$$
$$H_\mu^{(2)}(z) = J_\mu(z) - \mathrm{i}N_\mu(z).$$

3.4 Recursion Formulas

We know that the equation

$$\sigma w'' + \tau w' + \lambda(\nu) w = 0, \qquad \lambda(\nu) = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma'',$$

has a solution

$$w_\nu(z) = \frac{1}{\varrho(z)} \int \frac{\sigma^\nu(t)\varrho(t)}{(t-z)^{\nu+1}} \,\mathrm{d}t, \qquad (\sigma\varrho)' = \tau\varrho,$$

provided that the path of integration satisfies some conditions. Define

$$\Phi_{\nu\mu}(z) := \int \frac{\sigma^\nu(t)\varrho(t)}{(t-z)^{\mu+1}} \,\mathrm{d}t.$$

Proposition 3.17. *Any three functions $\Phi_{\nu_i\mu_i}$, $i=1,2,3$, satisfy a linear relation*

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i\mu_i}(z) = 0,$$

where the A_i are polynomials, and we assume that the following conditions are met:

- $\nu_i - \nu_j$, $\mu_i - \mu_j \in \mathbb{Z}$, *for* $i,j = 1,2,3$,

- *for all $m \in \mathbb{N}_0$, the expressions*

$$\frac{\sigma^{\nu_0+1}(t)\varrho(t)}{(t-z)^{\mu_0}} t^m \Big|_{t=t_1}^{t=t_2}$$

vanish, where t_1 and t_2 are the endpoints of the integration curve, and $\nu_0 \in \{\nu_1, \nu_2, \nu_3\}$, $\mu_0 \in \{\mu_1, \mu_2, \mu_3\}$, with $\Re\nu_0 \le \Re\nu_i$, $\Re\mu_0 \ge \Re\mu_i$.

Sketch of proof. Let a_i, $i = 1, 2, 3$, be functions of z. Then we have

$$\sum_{i=1}^{3} a_i(z)\Phi_{\nu_i\mu_i}(z) = \int \frac{\sigma^{\nu_0}(t)\varrho(t)}{(t-z)^{\mu_0+1}} \left(\sum_{i=1}^{3} a_i(z)\sigma^{\nu_i-\nu_0}(t)(t-z)^{\mu_0-\mu_i} \right) \mathrm{d}t =: \int \frac{\sigma^{\nu_0}(t)\varrho(t)}{(t-z)^{\mu_0+1}} P(t,z)\, \mathrm{d}t$$

We look for rational functions a_i and a polynomial Q with the property that

$$\frac{\sigma^{\nu_0}(t)\varrho(t)}{(t-z)^{\mu_0+1}} P(t,z) = \partial_t \left(Q(t) \frac{\sigma^{\nu_0+1}(t)\varrho(t)}{(t-z)^{\mu_0}} \right).$$

Then the integral vanishes, and it remains to multiply with the denominators of the a_i.

Of course, the hard part of the proof is to show that such functions a_i and Q really exist. □

As an example, we remark that

$$\left(\tau' + \frac{\nu-1}{2}\sigma'' \right) \Phi_{\nu,\nu-1} + (\tau - \sigma')\Phi_{\nu,\nu} - (\nu+1)\sigma\Phi_{\nu,\nu+1} = 0.$$

In this case, we have $a_i = A_i$ and $Q = 1$.

The following result gives the connection between the $\Phi_{\nu\mu}$ and the derivatives of w_ν:

Proposition 3.18. *Under the assumptions of Propositions 3.15 and 3.17, we have*

$$\partial_z^k w_\nu(z) = \prod_{j=1}^{k} C_\nu^{(j)} \Phi_{\nu,\nu-k}(z) \cdot \frac{1}{\sigma^k(z)\varrho(z)},$$

where $C_\nu^{(j)} = \tau' + \frac{\nu+j-2}{2}\sigma''$.

Proof. (Long.) Use the above relation between $\Phi_{\nu,\nu-1}$, $\Phi_{\nu,\nu}$, $\Phi_{\nu,\nu+1}$; and the fact that derivatives of functions of hypergeometric type are again functions of hypergeometric type (and solve a certain second order differential equation). Do the induction over k. □

Example 3.19. *Let w_μ stand for J_μ or $H_\mu^{(1)}$ or $H_\mu^{(2)}$. Then we have the recursion formulas*

$$w'_\mu(z) + \frac{\mu}{z} w_\mu(z) - w_{\mu-1}(z) = 0, \qquad \Re\mu > \frac{5}{2}, \qquad z > 0,$$

$$w_{\mu-2}(z) + 2\frac{1-\mu}{z} w_{\mu-1}(z) + w_\mu(z) = 0, \qquad \Re\mu > \frac{7}{2}, \qquad z > 0.$$

Since J_μ is analytic in $\mathbb{C} \setminus \{0\}$, and since $H_\mu^{(1)}$, $H_\mu^{(2)}$ are analytic for $\Re z > 0$, these recursion formulas extend to $\mathbb{C} \setminus \{0\}$ and the right half-plane, respectively.

Since we have no integral representation (not even a definition) of the Hankel functions for $\Re\mu \le \frac{3}{2}$, we define them by the second recursion formula in this case. Then the Hankel functions are holomorphic functions of z for all $\mu \in \mathbb{C}$.

From the integral representations, it can be seen that the Hankel functions are holomorphic functions of μ, for $\Re\mu > \frac{3}{2}$ and $z > 0$. By analytic continuation, the Hankel functions are holomorphic functions of μ for all $\mu \in \mathbb{C}$ and $\Re z > 0$.

Integral representations of the $H_\mu^{(j)}(z)$ for z in the left half-plane can be found starting from the identities (3.8) and (3.9), and carefully bending the integration path from the real half-axis $(0,\infty)$ to a ray $(0,\infty(\alpha))$ with $|\alpha| < \frac{\pi}{2}$. The result then is

$$H_\mu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\frac{\exp(\mathrm{i}(z - \frac{\mu\pi}{2} - \frac{\pi}{4}))}{\Gamma(\mu + \frac{1}{2})}\int_{t=0}^{\infty(\alpha_1)} e^{-t}t^{\mu-1/2}\left(1 + \frac{\mathrm{i}t}{2z}\right)^{\mu-1/2}\mathrm{d}t,$$
$$-\frac{\pi}{2} + \alpha_1 < \arg z < \frac{3\pi}{2} + \alpha_1, \qquad \Re\mu > -\frac{1}{2},$$
$$H_\mu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\frac{\exp(-\mathrm{i}(z - \frac{\mu\pi}{2} - \frac{\pi}{4}))}{\Gamma(\mu + \frac{1}{2})}\int_{t=0}^{\infty(\alpha_2)} e^{-t}t^{\mu-1/2}\left(1 - \frac{\mathrm{i}t}{2z}\right)^{\mu-1/2}\mathrm{d}t,$$
$$-\frac{3\pi}{2} + \alpha_2 < \arg z < \frac{\pi}{2} + \alpha_2, \qquad \Re\mu > -\frac{1}{2}.$$

The numbers α_1 and α_2 must satisfy $|\alpha_j| < \frac{\pi}{2}$, and are chosen depending on z.

We have two integral representations in case of $-\pi < \arg z < \pi$.

Another sometimes useful recursion formula is

$$\left(\frac{1}{z}\partial_z\right)^n (z^\mu w_\mu(z)) = z^{\mu-n}w_{\mu-n}(z).$$

Example 3.20. *Write the Legendre polynomial P_{n+1} as*

$$P_{n+1}(z) = \frac{B_{n+1}}{\varrho(z)}\partial_z^{n+1}\left(\sigma^{n+1}(z)\varrho(z)\right).$$

A careful analysis of the RHS then shows

$$(1 - z^2)P_n'(z) = (n+1)\left(zP_n(z) - P_{n+1}(z)\right).$$

Example 3.21. *The Laguerre polynomials satisfy*

$$\partial_z L_n^{(\alpha)}(z) = -L_{n-1}^{(\alpha+1)}(z).$$

3.5 Generating Functions

We are given a collection of polynomial solutions

$$P_n(z) := \frac{B_n}{\varrho(z)}\partial_z^n\left(\varrho(z)\sigma(z)^n\right)$$

to the equation $\sigma w'' + \tau w' + \lambda(n)w = 0$, where $\lambda(n) = -n\tau' - \frac{n(n-1)}{2}\sigma''$.

And we are looking for an explicit formula for the expression

$$\Phi(t,z) = \sum_{n=0}^{\infty}\frac{P_n(z)}{B_n n!}t^n,$$

where $|t|$ is small. This function Φ is called *generating function.*

By Cauchy's integral formula, we have

$$P_n(z) = \frac{B_n}{\varrho(z)}\frac{n!}{2\pi\mathrm{i}}\oint_{|s-z|=r}\frac{\sigma(s)^n\varrho(s)}{(s-z)^{n+1}}\mathrm{d}s.$$

The zeroes of σ are, in general, singularities of ϱ. Therefore, the radius r must be chosen so small that these zeroes of σ are outside the circle of integration.

If $|s-z|=r$, and $|t|$ is small, then the series

$$\sum_{n=0}^{\infty}\left(\frac{t\varrho(s)}{s-z}\right)^n$$

converges, and takes the value

$$\frac{1}{1-\frac{t\sigma(s)}{s-z}}=\frac{s-z}{(s-z)-t\sigma(s)}.$$

Then we can write

$$\begin{aligned}\Phi(t,z)&=\sum_{n=0}^{\infty}\frac{1}{2\pi\mathrm{i}\varrho(z)}\oint_{|s-z|=r}\frac{\varrho(s)}{(s-z)}\left(\frac{t\varrho(s)}{s-z}\right)^n\mathrm{d}s\\&=\frac{1}{2\pi\mathrm{i}\varrho(z)}\oint_{|s-z|=r}\frac{\varrho(s)}{s-z}\cdot\frac{s-z}{s-z-t\sigma(s)}\mathrm{d}s=\frac{1}{2\pi\mathrm{i}\varrho(z)}\oint_{|s-z|=r}\frac{\varrho(s)}{s-z-t\sigma(s)}\mathrm{d}s.\end{aligned}$$

This integral can be evaluated by the residue theorem. The only singularities of the integrand are the zeroes of the denominator, which we denote by $N(s)$,

$$N(s)=s-z-\sigma(s)t.$$

Let the radius r be fixed. Then Vieta's theorem shows that there is exactly one zero of N inside the circle about z with radius r, for small $|t|$. Call that zero s_0. We find that

$$\Phi(t,z)=\frac{1}{\varrho(z)}\mathrm{Res}_{s_0}\left(\frac{\varrho(s)}{s-z-\sigma(s)t}\right)=\frac{1}{\varrho(z)}\frac{\varrho(s_0)}{1-\sigma'(s_0)t}.$$

[1]

Hence, we have proved:

Proposition 3.22. *Let the functions P_n, $n\in\mathbb{N}_0$,*

$$P_n(z)=\frac{B_n}{\varrho(z)}\partial_z^n\left(\varrho(z)\sigma(z)^n\right)$$

be a polynomial solution to the equation $\sigma w''+\tau w'+\lambda(n)w=0$, where $\lambda(n)=-n\tau'-\frac{n(n-1)}{2}\sigma''$, and $(\sigma\varrho)'=\tau\varrho$. Then we have the identity

$$\Phi(t,z):=\sum_{n=0}^{\infty}\frac{P_n(z)}{B_n n!}t^n=\frac{\varrho(s_0)}{\varrho(z)}\frac{1}{1-\sigma'(s_0)t},\qquad |t|\ll 1,$$

where $s_0=s_0(z,t)$ is that zero of $N(s)=s-z-\sigma(s)t$ which is closest to z.

Example 3.23. *For the Legendre polynomials, we have*

$$\begin{aligned}&P_n(z)=B_n\partial_z^n\left((1-z^2)^n\right),\qquad B_n=\frac{(-1)^n}{2^n n!},\\&\sigma(z)=1-z^2,\qquad \varrho(z)\equiv 1,\\&N(s)=s-z-(1-s^2)t,\\&0=ts_0^2+s_0-z-t,\\&s_0=\frac{1}{2t}\left(-1+\sqrt{1+4t(t+z)}\right),\\&1-\sigma'(s_0)t=1+2s_0t=\sqrt{1+4t(t+z)}.\end{aligned}$$

We write t instead of $-2t$, and find

$$\sum_{n=0}^{\infty}P_n(z)t^n=\frac{1}{\sqrt{1-2tz+t^2}},\qquad |t|\ll 1.$$

There are several applications of this generating function:

[1] We have to take the derivative of the denominator.

- *Put $z = 1$:*

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} P_n(1)t^n \qquad \Longrightarrow \qquad P_n(1) = 1.$$

- *Put $z = -1$:*

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} P_n(-1)t^n \qquad \Longrightarrow \qquad P_n(-1) = (-1)^n.$$

- *Differentiate with respect to t:*

$$\sum_{n=0}^{\infty} P_n(z)nt^{n-1} = \Phi(t,z)\cdot\left(-\frac{1}{2}\right)\frac{1}{1-2tz+t^2}\cdot 2(t-z),$$

from which we deduce that

$$(z-t)\sum_{n=0}^{\infty} P_n(z)t^n = (1-2tz+t^2)\sum_{n=0}^{\infty} P_n(z)nt^{n-1}.$$

Comparing equal powers of t reveals the famous formula

$$(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0. \tag{3.10}$$

- *Differentiate with respect to z:*

$$\sum_{n=0}^{\infty} P_n'(z)t^n = \Phi(t,z)\cdot\frac{t}{1-2tz+t^2},$$

which implies

$$t\sum_{n=0}^{\infty} P_n(z)t^n = (1-2tz+t^2)\sum_{n=0}^{\infty} P_n'(z)t^n.$$

Comparing equal powers of t then gives

$$P_{n+1}'(z) - 2zP_n'(z) + P_{n-1}'(z) - P_n(z) = 0. \tag{3.11}$$

Eliminating P_{n+1}' from (3.11) and putting it into the derived version of (3.10), we conclude that

$$zP_n'(z) - P_{n-1}'(z) - nP_n(z) = 0.$$

Similar results can be proved endlessly.

Example 3.24. *We consider the Hermite polynomials H_n, which are defined by*

$$H_n(z) = (-1)^n \exp(z^2)\partial_z^n\left(\exp(-z^2)\right).$$

Note that $H_n \neq He_n$. We have $B_n = (-1)^n$, $\varrho(z) = \exp(-z^2)$, $\sigma(z) \equiv 1$, and $N(s) = s - z - \sigma(s)t = s - z - t$, hence $s_0 = z + t$. It follows that

$$\sum_{n=0}^{\infty}\frac{H_n(z)}{(-1)^n n!}t^n = \frac{\varrho(s_0)}{\varrho(z)}\cdot\frac{1}{1-\sigma'(s_0)t} = \exp(-2tz - t^2).$$

We replace $-t$ by t, and obtain

$$\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n = \exp(2tz - t^2).$$

As applications, we obviously have:

- *Put $z = 0$: we get*

$$H_n(0) = \begin{cases} 0 & : n \text{ even}, \\ (-1)^m \frac{(2m)!}{m!} & : n = 2m. \end{cases}$$

- *Deriving with respect to t, we find $H_n'(z) = 2nH_{n-1}(z)$.*
- *Deriving with respect to z, we conclude that*

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0.$$

Example 3.25. *Finally, consider the Laguerre polynomials,*

$$L_n^{(\alpha)}(z) = \frac{1}{n!} e^z z^{-\alpha} \partial_z^n \left(z^{\alpha+n} e^{-z} \right).$$

We have $B_n = \frac{1}{n!}$, $\varrho(z) = e^{-z} z^\alpha$, $\sigma(z) = z$, which leads us to $N(s) = s - z - \sigma(s)t = s - z - st$, hence

$$s_0 = \frac{z}{1-t}.$$

Then it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha)}(z) t^n &= \frac{\varrho(s_0)}{\varrho(z)} \frac{1}{1 - \sigma'(s_0)t} = \frac{e^{-s_0} s_0^\alpha}{e^{-z} z^\alpha} \cdot \frac{1}{1-t} = e^{z-s_0} \left(\frac{s_0}{z} \right)^\alpha \cdot \frac{1}{1-t} \\ &= (1-t)^{-\alpha-1} \exp\left(-\frac{zt}{1-t} \right). \end{aligned}$$

Going the known way, we easily find

$$\partial_z L_n^{(\alpha)}(z) - \partial_z L_{n-1}^{(\alpha)} + L_{n-1}^{(\alpha)}(z) = 0.$$

3.6 Orthogonality

3.6.1 Special Orthogonal Polynomials

Let $(a, b) \subset \overline{\mathbb{R}}$ be an interval of integration, and let w_ν, w_μ be solutions to the equations of hypergeometric type

$$\begin{aligned} \sigma w_\nu'' + \tau w_\nu' + \lambda(\nu) w_\nu &= 0, \\ \sigma w_\mu'' + \tau w_\mu' + \lambda(\mu) w_\mu &= 0, \end{aligned}$$

which can be written in self-adjoint form

$$\begin{aligned} (\sigma \varrho w_\nu')' + \lambda(\nu) \varrho w_\nu &= 0, \\ (\sigma \varrho w_\mu')' + \lambda(\mu) \varrho w_\mu &= 0. \end{aligned}$$

We multiply the first equation with w_μ, the second with w_ν, and subtract:

$$\left(\sigma \varrho (w_\mu w_\nu' - w_\nu w_\mu') \right)' + (\lambda(\nu) - \lambda(\mu)) \varrho w_\nu w_\mu = 0.$$

We assume that this equation can be integrated over (a, b). Then we get

$$(\sigma \varrho (w_\mu w_\nu' - w_\nu w_\mu')) \Big|_a^b = (\lambda(\mu) - \lambda(\nu)) \int_{x=a}^{x=b} \varrho(x) w_\nu(x) w_\mu(x) \, \mathrm{d}x.$$

We wish the LHS to vanish. This will happen if a and b are zeroes of $\sigma\varrho$, and w_μ, w_ν as well as their derivatives stay bounded at the points a and b.

However, the functions w_ν and w_μ will have singularities at the zeroes of σ, in general. The exceptional case are polynomial solutions.

Proposition 3.26. *Let P_n, P_m with $n \neq m$ be polynomial solutions to the equation*

$$\sigma w'' + \tau w' + \lambda w = 0, \qquad (\lambda = \lambda(n) \text{ or } \lambda = \lambda(m)),$$

of hypergeometric type, and assume that

$$\sigma(x)\varrho(x)x^k \Big|_{x=a}^{x=b} = 0, \qquad k = 0, 1, 2, \dots.$$

Then the functions P_n and P_m are orthogonal in $L^2((a,b), \varrho\,\mathrm{d}x)$:

$$\int_{x=a}^{b} P_n(x)P_m(x)\varrho(x)\,\mathrm{d}x = 0.$$

Proof. See above. □

Example 3.27. *Take the Jacobi polynomials $P_n^{(\alpha,\beta)}$. We have $\sigma(x) = 1-x^2$ and choose $(a,b) = (-1,1)$. The function ϱ is $\varrho(x) = (1-x)^\alpha(1+x)^\beta$, where we assume $\Re \leq> -1$, $\Re\beta > -1$. Then it follows*

$$\int_{x=-1}^{x=1} (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,\mathrm{d}x = 0, \qquad n \neq m.$$

We obtain orthogonality relations for the Legendre polynomials and Chebyshev polynomials upon setting $\alpha = \beta = 0$ and $\alpha = \beta = -\frac{1}{2}$, respectively.

Example 3.28. *Take the Laguerre polynomials $L_n^{(\alpha)}$. Then we have $\sigma(x) = x$ and $\varrho(x) = x^\alpha e^{-x}$. We assume $\alpha > -1$ and choose $(a,b) = (0,\infty)$, leading us to*

$$\int_{x=0}^{\infty} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)x^\alpha e^{-x}\,\mathrm{d}x = 0, \qquad n \neq m.$$

Example 3.29. *Take the Hermite polynomials. There we have $\sigma \equiv 1$ and $\varrho(x) = \exp(-x^2)$. We choose $(a,b) = (-\infty,\infty)$ and obtain*

$$\int_{x=-\infty}^{x=\infty} H_n(x)H_m(x)e^{-x^2}\,\mathrm{d}x = 0.$$

It remains to evaluate the integrals $\int_a^b |P_n(x)|^2\varrho(x)\,\mathrm{d}x$.

To this end, we recall that $w = w(z)$ and $w_k := \partial_z^k w$ solve

$$\begin{aligned}
&\sigma w'' + \tau w' + \lambda(n)w = 0,\\
&\sigma w_k'' + \tau_k w' + \mu_k w_k = 0,\\
&\tau_k(z) = \tau(z) + k\sigma'(z), \qquad \mu_k = \lambda(n) + k\tau' + \frac{k(k-1)}{2}\sigma'' = \lambda(n) - \lambda(k),
\end{aligned}$$

which can be brought into self-adjoint form:

$$\begin{aligned}
&(\varrho\sigma w')' + \lambda(n)\varrho w = 0,\\
&(\varrho\sigma^{k+1}w_k')' + (\lambda(n) - \lambda(k))\varrho\sigma^k w_k = 0.
\end{aligned}$$

Now we put $P_n = w$ and $P_n^{(k)} = w_k$, and see that

$$\varrho P_n = -\partial_z\left(\frac{1}{\lambda(n)}\varrho\sigma P_n'\right), \qquad \varrho\sigma^k P_n^{(k)} = -\partial_z\left(\frac{1}{\lambda(n)-\lambda(k)}\varrho\sigma^{k+1}P_n^{(k+1)}\right).$$

These representations enable us to calculate $\int_a^b \varrho |P_n|^2 \,\mathrm{d}x$ by repeated partial integration:

$$\begin{aligned}
&\int_{x=a}^{b} \varrho(x) P_n(x) \cdot P_n(x) \,\mathrm{d}x \\
&\quad = \left(-\frac{1}{\lambda(n)} \varrho(x)\sigma(x)P_n'(x)P_n(x) \right) \Big|_{x=a}^{x=b} - \int_{x=a}^{b} -\frac{1}{\lambda(n)} \varrho(x)\sigma(x)P_n'(x)P_n'(x)\,\mathrm{d}x \\
&\quad = \frac{1}{\lambda(n)} \int_{x=a}^{b} \varrho(x)\sigma(x)P_n^{(1)}(x) \cdot P_n^{(1)}(x)\,\mathrm{d}x \\
&\quad = \frac{1}{\lambda(n)(\lambda(n)-\lambda(1))} \int_{x=a}^{b} \varrho(x)\sigma^2(x)P_n^{(2)}(x) \cdot P_n^{(2)}(x)\,\mathrm{d}x \\
&\quad = \dots \\
&\quad = \frac{(P_n^{(n)})^2}{\mu_0\mu_1 \cdot \ldots \cdot \mu_{n-1}} \int_{x=a}^{b} \sigma^n(x)\varrho(x)\,\mathrm{d}x.
\end{aligned}$$

We discuss some examples.

Jacobi polynomials: We know already that $\partial_z P_n^{(\alpha,\beta)} = \frac{\alpha+\beta+n+1}{2} P_{n-1}^{(\alpha,\beta)}$. By induction, we then find

$$\partial_z^n P_n^{(\alpha,\beta)} = (\alpha+\beta+n+1)(\alpha+\beta+n+1) \cdot \ldots \cdot (\alpha+\beta+n+n) P_0^{(\alpha+n,\beta+n)} \frac{1}{2^n}.$$

The Rodriguez formula representation tells us that $P_0^{(\alpha+n,\beta+n)} = 1$, from which we conclude that

$$\partial_z^n P_n^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+\beta+n+1)2^n}.$$

Further, we have $\mu_k = \lambda(n) - \lambda(k) = (n-k)(\alpha+\beta+n+k+1)$, which implies

$$\mu_0 \cdot \ldots \cdot \mu_{n-1} = n! \frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+\beta+n+1)}.$$

Finally,

$$\begin{aligned}
\int_{x=a}^{b} \sigma^n(x)\varrho(x)\,\mathrm{d}x &= \int_{x=-1}^{1} (1-x^2)^n (1-x)^\alpha (1+x)^\beta \,\mathrm{d}x \\
&= \int_{x=-1}^{1} (1-x)^{\alpha+n}(1+x)^{\beta+n}\,\mathrm{d}x \qquad (x = 2\xi - 1) \\
&= 2^{\alpha+\beta+2n+1} \int_{\xi=0}^{1} (1-\xi)^{\alpha+n}\xi^{\beta+n}\,\mathrm{d}\xi = 2^{\alpha+\beta+2n+1} B(\alpha+n+1, \beta+n+1) \\
&= 2^{\alpha+\beta+2n+1} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)}.
\end{aligned}$$

Altogether, we obtain

$$\int_a^b \varrho(x)|P_n^{(\alpha,\beta)}(x)|^2\,\mathrm{d}x = 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}.$$

Legendre polynomials: The Legendre polynomials are Jacobi polynomials with $\alpha = \beta = 0$, hence

$$\int_{x=-1}^{1} |P_n(x)|^2\,\mathrm{d}x = \frac{2}{n+1}.$$

Chebyshev polynomials: The Chebyshev polynomials T_n can be written as

$$T_n(z) = \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{(-1/2,-1/2)}(z) = \frac{n!\Gamma(1/2)}{\Gamma(n+1/2)} P_n^{(-1/2,-1/2)}(z).$$

Then it follows that

$$\int_{x=-1}^{1} \frac{1}{\sqrt{1-x^2}} |T_n(x)|^2\,\mathrm{d}x = \left(\frac{n!\Gamma(1/2)}{\Gamma(n+1/2)} \right)^2 2^0 \frac{\Gamma(n+1/2)\Gamma(n+1/2)}{2n\Gamma(n)n!} = \frac{\pi}{2}.$$

Laguerre polynomials: By similar computations, we find

$$\int_{x=0}^{\infty} e^{-x} x^{-\alpha} \left(L_n^{(\alpha)}(x)\right)^2 \,\mathrm{d}x = \frac{\Gamma(n+\alpha+1)}{n!}.$$

Hermite polynomials: We have

$$\int_{x=-\infty}^{\infty} e^{-x^2} \left(H_n(x)\right)^2 \,\mathrm{d}x = 2^n n! \sqrt{\pi}.$$

3.6.2 General Orthogonal Polynomials

Now let $(a, b) \subset \mathbb{R}$ be any (non-empty) interval, and ϱ be an integrable weight function from (a, b) into $\mathbb{R}$ that does not change its sign.

Definition 3.30. *Let $\langle \cdot, \cdot \rangle$ denote the following scalar product:*

$$\langle f, g \rangle = \int_{x=a}^{b} \varrho(x) f(x) g(x) \,\mathrm{d}x.$$

As usual, we define a norm $\|f\|$ by setting $\|f\| = \sqrt{\langle f, f \rangle}$.

If we apply the GRAM–SCHMIDT orthogonalization procedure to the sequence of polynomials $1, x, x^2, \dots,$ we obtain polynomials

$$p_0, p_1, p_2, \dots$$

with $\langle p_i, p_j \rangle = 0$ for $i \neq j$ and $\deg p_j = j$.

These polynomials p_j are uniquely determined, up to multiplicative constants (which may depend on j).

We do not assume that these functions
solve differential functions of hypergeometric or whatever type !

Proposition 3.31. *Every polynomial q of degree n can be written as a linear combination*

$$q(x) = \sum_{j=0}^{n} c_j p_j(x),$$

where the coefficients c_j are uniquely determined and given by

$$c_j = \frac{\langle q, p_j \rangle}{\langle p_j, p_j \rangle}.$$

Proof. Elementary linear algebra. □

Proposition 3.32. *Let q be a polynomial of degree $k < n$. Then $\langle q, p_n \rangle = 0$.*

Proof. Trivial because of $q = \sum_{j=0}^{k} c_j p_j$. □

Example 3.33. *We know that*

$$\int_{\varphi=0}^{\pi} \cos(n\varphi) \cos(m\varphi) \,\mathrm{d}\varphi = \begin{cases} 0 & : n \neq m, \\ \frac{\pi}{2} & : n = m > 0. \end{cases}$$

MOIVRE*'s formula* $\cos(n\varphi) + \mathrm{i} \sin(n\varphi) = (\cos\varphi + \mathrm{i}\sin\varphi)^n$ *yields*

$$\cos(n\varphi) = \sum_{j=0}^{n} \alpha_j \cos^j \varphi, \qquad \cos(m\varphi) = \sum_{l=0}^{m} \beta_l \cos^l \varphi.$$

We substitute $x = \cos\varphi$, and obtain

$$\cos(n \arccos x) = \sum_{j=0}^{n} \alpha_j x^j =: P_n(x), \qquad \cos(m \arccos x) = \sum_{l=0}^{m} \beta_l x^l =: P_m(x),$$

and the integral identities

$$\int_{x=-1}^{1} P_n(x) P_m(x) \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \begin{cases} 0 & : n \neq m, \\ \frac{\pi}{2} & : n = m > 0. \end{cases}$$

But these are exactly the orthogonality relations of the Chebyshev polynomials. The uniqueness of the family of orthogonal polynomials (for a given scalar product) then implies

$$T_n(x) = \cos(n \arccos x), \qquad n \in \mathbb{N}_0, \qquad -1 \leq x \leq 1,$$

where T_n are the Chebyshev polynomials.
Using this new representation and addition theorems for the cosine, we obtain the recursion relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \qquad n \in \mathbb{N}_+, \qquad -1 \leq x \leq 1.$$

Clearly, $T_0(x) = 1$ and $T_1(x) = x$, which (together with the recursion relation) tells us that the leading coefficient of T_n is 2^{n-1}:

$$T_n(x) = 2^{n-1} x^n + b_n x^{n-1} + \dots.$$

Chebyshev polynomials play a certain role in numerical mathematics: let q be a polynomial of degree n with leading coefficient equal to 2^{n-1}. Then

$$\max_{-1 \leq x \leq 1} |q(x)| \geq 1,$$

and equality occurs exactly for $q = T_n$. You can prove this by assuming $\max_{-1 \leq x \leq 1} |q(x)| < 1$ and then counting how often the sign of $q - T_n$ changes. You will find that $q - T_n$ has at least n zeroes which is impossible because the degree of this polynomial is at most $n - 1$.

Now we have enough knowledge to count the zeroes of p_n:

Proposition 3.34. *Let $(p_n)_n$ be a family of orthogonal polynomials. Then the polynomial p_n has n mutually distinct zeroes in the interval (a, b).*

Proof. The polynomial p_0 is a non-zero constant. Since $\langle p_n, p_0 \rangle = 0$ for $n \geq 1$ and the weight function ϱ does not change its sign, the polynomial p_n must have a zero with sign change in (a, b).
Let $(x_1, \dots, x_k)$ denote all zeroes of p_n in (a, b) where p_n changes its sign. Clearly, $k \leq n$. Put

$$q(x) = (x - x_1) \cdot \dots \cdot (x - x_k).$$

Then $\langle q, p_n \rangle \neq 0$, because the integrand in this scalar product never changes its sign in (a, b). If $k < n$, then $\langle q, p_n \rangle = 0$, because of Proposition 3.32. This is a contradiction, hence $k = n$. □

Proposition 3.35. *There are constants α_n, β_n and γ_n such that the family of orthogonal polynomials $(p_n)_n$ satisfies the following orthogonality relation:*

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \qquad n \geq 0, \qquad p_{-1} \equiv 0.$$

Proof. Since $xp_n(x)$ is a polynomial of degree $n + 1$, there are constants c_j with the property that

$$xp_n(x) = \sum_{j=0}^{n+1} c_j p_j(x), \qquad c_j = \frac{\langle xp_n, p_j \rangle}{\langle p_j, p_j \rangle}.$$

We observe that $\langle xp_n, p_j \rangle = \langle p_n, xp_j \rangle = 0$ for $n > 1 + j$, due to Proposition 3.32. Put $\alpha_n = c_{n+1}$, $\beta_n = c_n$ and $\gamma_n = c_{n-1}$. □

Proposition 3.36. *Assume that the polynomials of the orthogonal family $(p_n)_n$ have the form*

$$p_n(x) = a_n x^n + b_n x^{n-1} + \dots .$$

Then the numbers α_n, β_n, γ_n of the previous proposition are given by

$$\alpha_n = \frac{a_n}{a_{n+1}}, \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \qquad \gamma_n = \frac{a_{n-1}}{a_n} \frac{\|p_n\|^2}{\|p_{n-1}\|^2}.$$

Proof. By the recursion relation, we have

$$a_n x^{n+1} + b_n x^n + \dots = \alpha_n (a_{n+1} x^{n+1} + b_{n+1} x^n + \dots) + \beta_n (a_n x^n + b_n x^{n-1} + \dots) + \gamma_n (a_{n-1} x^{n-1} + \dots).$$

Comparing equal powers of x, we find

$$a_n = \alpha_n a_{n+1}, \qquad b_n = \alpha_n b_{n+1} + \beta_n a_n.$$

Finally, the number γ_n is obtained via

$$\begin{aligned}
\gamma_n = c_{n-1} &= \frac{\langle x p_n, p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle} = \frac{1}{\|p_{n-1}\|^2} \langle p_n, x p_{n-1}\rangle \\
&= \frac{1}{\|p_{n-1}\|^2} \langle p_n, \alpha_{n-1} p_n + \beta_{n-1} p_{n-1} + \gamma_{n-1} p_{n-2}\rangle = \frac{1}{\|p_{n-1}\|^2} \langle p_n, \alpha_{n-1} p_n\rangle \\
&= \frac{\|p_n\|^2}{\|p_{n-1}\|^2} \frac{a_{n-1}}{a_n}.
\end{aligned}$$

□

Proposition 3.37 (Darboux–Christoffel **formula).** *Let the polynomials p_n and the numbers a_n as above. Then we have the identity*

$$\sum_{k=0}^{n} \frac{p_k(x) p_k(y)}{\|p_k\|^2} = \frac{1}{\|p_n\|^2} \frac{a_n}{a_{n+1}} \frac{p_{n+1}(x) p_n(y) - p_{n+1}(y) p_n(x)}{x - y}, \qquad x \neq y.$$

Proof. We know that

$$\begin{aligned}
x p_k(x) &= \alpha_k p_{k+1}(x) + \beta_k p_k(x) + \alpha_{k-1} \|p_k\|^2 \frac{1}{\|p_{k-1}\|^2} p_{k-1}(x), \\
y p_k(y) &= \alpha_k p_{k+1}(y) + \beta_k p_k(y) + \alpha_{k-1} \|p_k\|^2 \frac{1}{\|p_{k-1}\|^2} p_{k-1}(y).
\end{aligned}$$

Multiply the first equation with $\frac{1}{\|p_k\|^2} p_k(y)$ and the second with $\frac{1}{\|p_k\|^2} p_k(x)$, and subtract:

$$\frac{(x-y) p_k(x) p_k(y)}{\|p_k\|^2} = \frac{\alpha_k}{\|p_k\|^2} \left(p_{k+1}(x) p_k(y) - p_{k+1}(y) p_k(x)\right) - \frac{\alpha_{k-1}}{\|p_{k-1}\|^2} \left(p_k(x) p_{k-1}(y) - p_k(y) p_{k-1}(x)\right).$$

It remains to sum up over $k = 0, 1, 2, \dots, n$. □

This formula enables us to locate the zeroes of the p_n even better:

Proposition 3.38. *Every interval of consecutive zeroes of p_n contains exactly one zero of p_{n+1}, and vice versa.*

Proof. Write the Darboux–Christoffel formula as

$$\sum_{k=0}^{n} \frac{p_k(x) p_k(y)}{\|p_k\|^2} = \frac{1}{\|p_n\|^2} \frac{a_n}{a_{n+1}} \left(\frac{p_{n+1}(x) - p_{n+1}(y)}{x - y} p_n(y) + \frac{p_n(y) - p_n(x)}{x - y} p_{n+1}(y) \right),$$

and send y to x:

$$\sum_{k=0}^{n} \frac{|p_k(x)|^2}{\|p_k\|^2} = \frac{1}{\|p_n\|^2} \frac{a_n}{a_{n+1}} \left(p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right).$$

The LHS is always non-negative; hence the term $p'_{n+1}p_n - p'_n p_{n+1}$ never changes its sign.

Now let x_j and x_{j+1} be two consecutive zeroes of p_n. Then we have

$$p_n(x_j) = p_n(x_{j+1}) = 0, \qquad p'_n(x_j) \cdot p'_n(x_{j+1}) < 0,$$

since these zeroes are single and there is no other zero of p_n between them. We know that the expressions

$$0 - p'_n(x_j)p_{n+1}(x_j) \qquad \text{and} \qquad 0 - p'_n(x_{j+1})p_{n+1}(x_{j+1})$$

have the same sign. It follows that $p_{n+1}(x_j)$ and $p_{n+1}(x_{j+1})$ must have differing signs, from which we conclude that there is an odd number of zeroes of p_{n+1} between x_j and x_{j+1}.

In a similar fashion, we can deduce that each interval of consecutive zeroes of p_{n+1} contains an odd number of zeroes of p_n. A simple counting argument completes the proof. □

Chapter 4

Numerical Aspects of Special Functions

4.1 Asymptotic Expansions

We start with an example: how to evaluate efficiently the integral $f(x) = \int_{t=0}^{\infty} e^{-xt} \frac{\mathrm{d}t}{1+t}$, for $x > 0$. Writing

$$\frac{1}{1+t} = 1 - t + t^2 \mp \cdots + (-1)^n t^n + \frac{(-1)^{n+1}}{1+t} t^{n+1},$$

we get

$$f(x) = \sum_{k=0}^{n} (-1)^k \int_{t=0}^{\infty} e^{-xt} t^k \,\mathrm{d}t + (-1)^{n+1} \int_{t=0}^{\infty} e^{-xt} \frac{t^{n+1}}{1+t} \,\mathrm{d}t.$$

Call the remainder term $r_n(x)$. Substituting $xt = \tau$ and recalling the definition of the Gamma function, we find

$$f(x) = \sum_{k=0}^{n} (-1)^k \frac{k!}{x^{k+1}} + r_n(x).$$

The remainder can be trivially estimated as follows:

$$|r_n(x)| \le \int_{t=0}^{\infty} e^{-xt} t^{n+1} \,\mathrm{d}t = \frac{(n+1)!}{x^{n+2}}.$$

Comparing this remainder estimate with the last term of the sum, we get

$$\left| \frac{r_n(x) x^{n+1}}{n!} \right| \le \frac{n+1}{x},$$

which goes to zero for fixed n and $x \to +\infty$.

However, we can not write $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{x^{k+1}}$, because this series converges nowhere. Such a series is called *asymptotic expansion.*

Take $x = 10$ and $n = 3$, then $|r_n(x)| \le \frac{24}{10^5} = 0.00024$. The optimal error estimate is attained for $n = 9$: $|r_n(x)| \le 3.6288 \cdot 10^{-5}$.

The main difference between a converging series and an asymptotic expansion is:

converging series: given a fixed x, we can calculate $\sum_n f_n(x)$ as precise as we wish (assuming an exact arithmetic).

asymptotic expansion: given a fixed x, only a limited precision can be obtained (even if we had an exact arithmetic).

Nevertheless, asymptotic expansions are useful tools for many purposes.

Definition 4.1. *Let $S \subset \mathbb{C}$ be a sector in the complex plane, and $\lambda_0, \lambda_1, \ldots$ be complex numbers with*

$$0 \leq \Re\lambda_0 < \Re\lambda_1 < \ldots.$$

We say that a function f has the asymptotic expansion *at infinity*

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^{\lambda_k}}, \qquad z \in S, \qquad z \to \infty$$

if $f(z) = \sum_{k=0}^{n} \frac{a_k}{z^{\lambda_k}} + r_n(z)$ with $r_n(z) = \mathfrak{O}(|z^{-\lambda_{n+1}}|)$ for $z \to \infty$, $z \in S$.

Definition 4.2. *Let $S \subset \mathbb{C}$ be a sector in the complex plane, and $\lambda_0, \lambda_1, \ldots$ be complex numbers with*

$$0 \leq \Re\lambda_0 < \Re\lambda_1 < \ldots.$$

We say that a function f has the asymptotic expansion *at zero*

$$f(z) \sim \sum_{k=0}^{\infty} a_k z^{\lambda_k}, \qquad z \in S, \qquad z \to 0$$

if $f(z) = \sum_{k=0}^{n} a_k z^{\lambda_k} + r_n(z)$ with $r_n(z) = \mathfrak{O}(|z^{\lambda_{n+1}}|)$ for $z \to 0$, $z \in S$.

Remark 4.3.
- *An asymptotic expansion need not converge.*
- *If it converges, it need not converge to f (but the limit function has the same asymptotic expansion).*
- *Two functions that differ by an exponentially decaying function will have the same asymptotic expansion (if they have one).*
- *A common observation is: the terms of the series decay first in modulus, then they grow again. A standard numerical algorithm is to sum up the terms as long as they are falling, wait for the smallest term, stop the summation there, and hope for the best.*
- *You can add, subtract, and multiply two asymptotic expansions; even divide if the denominator never vanishes. Term-wise integration of an asymptotic expansion is also allowed, but not term-wise differentiation.*

Theorem 4.4 (Watson). *Let $g\colon \mathbb{R}_+ \to \mathbb{C}$ be a continuous function with the asymptotic expansion $g(t) \sim \sum_{k=0}^{\infty} \alpha_k t^{\lambda_k - 1}$ for $t \to +0$, where $0 < \Re\lambda_0 < \Re\lambda_1 < \ldots$. Suppose that the integral*

$$f(z) = \int_{t=0}^{\infty} e^{-zt} g(t)\,\mathrm{d}t$$

converges for $z = z_0$ absolutely (then it also converges for all $z \in \mathbb{C}$ with $\Re z \geq \Re z_0$). Additionally, we assume that g has at most exponential growth at infinity:

$$|g(t)| \leq Ce^{C_0 t}, \qquad 0 < t < \infty.$$

Then f has the following asymptotic expansion at infinity:

$$f(z) \sim \sum_{k=0}^{\infty} \alpha_k \frac{\Gamma(\lambda_k)}{z^{\lambda_k}}, \qquad z \to \infty, \quad \Re z > \Re z_0, \quad \Re z > C_0, \quad |\arg z| \leq \frac{\pi}{2} - \delta, \quad \delta > 0.$$

Proof. Write $g(t) = \sum_{k=0}^{n-1} \alpha_k t^{\lambda_k - 1} + s_{n-1}(t)$. We know that $|s_{n-1}(t)| \le C_n |t^{\lambda_n - 1}|$ for $0 < t < c$, where c may depend on n. And for $c < t < \infty$, we have $|s_{n-1}(t)| \le Ce^{C_0 t}$.

Then we find

$$\begin{aligned} f(z) &= \sum_{k=0}^{n-1} \int_{t=0}^{\infty} e^{-zt} \alpha_k t^{\lambda_k - 1}\,\mathrm{d}t + \int_{t=0}^{\infty} e^{-zt} s_{n-1}(t)\,\mathrm{d}t \\ &= \sum_{k=0}^{n-1} \alpha_k \frac{\Gamma(\lambda_k)}{z^{\lambda_k}} + \int_{t=0}^{c} e^{-zt} s_{n-1}(t)\,\mathrm{d}t + \int_{t=c}^{\infty} e^{-zt} s_{n-1}(t)\,\mathrm{d}t, \end{aligned}$$

by the substitution $zt = \tau$. This is possible for $|\arg z| \le \frac{\pi}{2} - \delta$, where $\delta > 0$ can be chosen arbitrarily small.

The two remainder integrals can be estimated as follows:

$$\left| \int_{t=0}^{c} e^{-zt} s_{n-1}(t)\,\mathrm{d}t \right| \le C_n \int_{t=0}^{c} \exp(-(\Re z)t) t^{\Re\lambda_n - 1}\,\mathrm{d}t \le C_n \frac{\Gamma(\Re\lambda_n)}{(\Re z)^{\Re\lambda_n}} \le C\left|z^{-\lambda_n},\right|,$$

since $|\Im z| \le C|\Re z|$.

Finally, for $\Re z > C_0$, we get

$$\left| \int_{t=c}^{\infty} e^{-zt} s_{n-1}(t)\,\mathrm{d}t \right| \le C \int_{t=c}^{\infty} \exp(-(\Re z)t + C_0 t)\,\mathrm{d}t = \frac{C}{(\Re z) - C_0} \exp((C_0 - \Re z)c),$$

which is an exponentially decaying term for $z \to \infty$, $|\arg z| \le \frac{\pi}{2} - \delta$. □

4.1.1 The Asymptotic Expansion of the Gamma Function

Our goal is to reproduce the Stirling asymptotic of the Gamma function. We start with some formulas from Proposition 1.19 and Lemma 1.20:

$$\begin{aligned} \psi(z) &= \int_{t=0}^{\infty} \frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\,\mathrm{d}t, \\ -\ln z &= -\int_{t=0}^{\infty} \frac{e^{-t} - e^{-zt}}{t}\,\mathrm{d}t, \\ \frac{1}{2z} &= \int_{t=0}^{\infty} \frac{1}{2} e^{-zt}\,\mathrm{d}t. \end{aligned}$$

Summing up, we find

$$\frac{\Gamma'(z)}{\Gamma(z)} - \ln z + \frac{1}{2z} = \int_{t=0}^{\infty} e^{-zt} \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) \mathrm{d}t =: \int_{t=0}^{\infty} e^{-zt} \gamma(t)\,\mathrm{d}t.$$

Integrating this from 1 to z_0, we find

$$\begin{aligned} &\ln\Gamma(z_0) - \ln\Gamma(1) - (z\ln z - z)\Big|_{z=1}^{z=z_0} + \frac{1}{2}\ln z_0 - \frac{1}{2}\ln 1 = \int_{t=0}^{\infty} \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) \int_{z=1}^{z_0} e^{-zt}\,\mathrm{d}z\,\mathrm{d}t, \\ &\ln\Gamma(z_0) - z_0 \ln z_0 + z_0 - 1 + \frac{1}{2}\ln z_0 = \int_{t=0}^{\infty} \frac{\gamma(t)}{t} (e^{-t} - e^{-tz_0})\,\mathrm{d}t. \end{aligned}$$

Now we write z again instead of z_0. Observe that

$$\begin{aligned} \frac{\gamma(t)}{t} &= \frac{1}{t^2} \left(\frac{t}{2} + 1 - \frac{te^t}{e^t - 1} \right) = \frac{1}{t^2} \left(\frac{t}{2} + 1 - \frac{te^t - t + t}{e^t - 1} \right) = \frac{1}{t^2} \left(-\frac{t}{2} + 1 - \frac{t}{e^t - 1} \right) \\ &= \frac{1}{t^2} \left(-\frac{t}{2} + 1 - \frac{te^{-t}}{1 - e^{-t}} \right) = \frac{1}{t^2} \left(-\frac{t}{2} + 1 - \frac{-te^{-t}}{e^{-t} - 1} \right) = \frac{\gamma(-t)}{t^2} \end{aligned}$$

is even, continuous and bounded near $t = 0$, and recall that

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k,$$

where the B_k are the Bernoulli numbers. Then we deduce that

$$\frac{\gamma(t)}{t} = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k-2},$$

and we can apply Watson's Theorem with $\alpha_k = \frac{B_{2k}}{(2k)!}$ and $\lambda_k = 2k - 1$, for $k \geq 1$. The result is

$$\ln\Gamma(z) - z\ln z + z - 1 + \frac{1}{2}\ln z \sim \int_{t=0}^{\infty} \frac{e^{-t}\gamma(t)}{t}\, \mathrm{d}t + \sum_{k=1}^{\infty} \frac{B_{2k}(2k-2)!}{(2k)! z^{2k-1}}.$$

One can show that

$$1 + \int_{t=0}^{\infty} \frac{\gamma(t)}{t} e^{-t}\, \mathrm{d}t = -\frac{1}{2}\ln(2\pi),$$

which leads us again to Stirling's expansion.

4.1.2 The Asymptotic Expansions of the Bessel Functions

The Hankel function $H_\mu^{(1)}$ has the integral representation

$$H_\mu^{(1)}(z) = \sqrt{\frac{2}{\pi}} z^\mu \exp(\mathrm{i}(z - \mu\pi/2 - \pi/4)) \frac{1}{\Gamma(\mu + 1/2)} \int_{t=0}^{\infty} e^{-tz} t^{\mu-1/2} \left(1 + \frac{\mathrm{i}t}{2}\right)^{\mu-1/2} \mathrm{d}t$$

provided that $\Re z > 0$ and $\Re\mu > -\frac{1}{2}$. We can apply Watson's theorem immediately, with

$$g(t) = t^{\mu-1/2}\left(1 + \frac{\mathrm{i}t}{2}\right)^{\mu-1/2} = t^{\mu-1/2} \sum_{k=0}^{\infty} \binom{\mu - 1/2}{k} \frac{\exp(\mathrm{i}\pi k/2)}{2^k} t^k, \qquad |t| < 1.$$

We have

$$\alpha_k = \binom{\mu - 1/2}{k} \frac{\exp(\mathrm{i}\pi k/2)}{2^k}, \qquad \lambda_k = \mu - \frac{1}{2} + k + 1,$$

and obtain

$$\begin{aligned} H_\mu^{(1)}(z) &\sim \sqrt{\frac{2}{\pi}} z^\mu \frac{\exp(\mathrm{i}(z - \mu\pi/2 - \pi/4))}{\Gamma(\mu + 1/2)} \sum_{k=0}^{\infty} \frac{\exp(\mathrm{i}\pi k/2)}{2^k} \binom{\mu - 1/2}{k} \frac{\Gamma(\mu + k + 1/2)}{z^{\mu+k+1/2}} \\ &= \sqrt{\frac{2}{\pi z}} \exp(\mathrm{i}(z - \mu\pi/2 - \pi/4)) \sum_{k=0}^{\infty} \frac{\exp(\mathrm{i}\pi k/2)\Gamma(\mu + k + 1/2)}{2^k k! \Gamma(\mu - k + 1/2)} z^{-k}. \end{aligned}$$

We get a similar expansion for $H_\mu^{(2)}$ immediately, just replace i by $-$i everywhere.
Then an expansion of $J_\mu(z)$ for $z \to \infty$ follows directly, via

$$J_\mu(z) = \frac{1}{2}\left(H_\mu^{(1)}(z) + H_\mu^{(2)}(z)\right).$$

4.1.3 A Variation of Watson's Theorem

We consider the function

$$f(x) = \int_{t=a}^{b} e^{-xp(t)} g(t)\,\mathrm{d}t,$$

where $p = p(t)$ is real–valued and monotonically increasing. For $p(t) = t$ and $[a, b) = (0, \infty)$, we recover the situation of Watson's theorem.

Following [11], we assume the asymptotic expansions

$$p(t) \sim p(a) + \sum_{j=0}^{\infty} p_j (t-a)^{j+\mu}, \qquad t \to a+0,$$

$$g(t) \sim \sum_{j=0}^{\infty} g_j (t-a)^{j+\lambda-1}, \qquad t \to a+0.$$

Additionally, we assume the following[1] expansion of p':

$$p'(t) \sim \sum_{j=0}^{\infty} p_j (j+\mu)(t-a)^{j+\mu-1}, \qquad t \to a+0.$$

If p is monotonically increasing, then $p_0 > 0$. In typical cases, we have $\mu = \lambda = 1$.

We introduce a new integration variable τ by $\tau := p(t) - p(a)$. Our goal is to rewrite the integral $\int_{t=a}^{b} e^{-xp(t)} g(t)\,\mathrm{d}t$ in terms of τ, and then to apply the Theorem of Watson.

We start with the asymptotic expansion of the inverse function $\tau \mapsto t$:

$$t - a \sim \sum_{j=1}^{\infty} c_j \tau^{j/\mu}, \qquad \tau \to 0+0,$$

where the c_j can be computed from the p_k:

$$c_1 = \frac{1}{p_0^{1/\mu}}, \qquad c_2 = -\frac{p_1}{\mu p_0^{1+2/\mu}}, \qquad c_3 = \frac{(\mu+3)p_1^2 - 2\mu p_0 p_2}{2\mu^2 p_0^2 + 3/\mu}, \qquad \dots\ .$$

Next, we transform the integrand:

$$g(t)\,\mathrm{d}t = g(t)\frac{\partial t}{\partial \tau}\,\mathrm{d}\tau = g(t)\frac{1}{p'(t)}\,\mathrm{d}\tau =: h(\tau)\,\mathrm{d}\tau,$$

$$h(\tau) \sim \sum_{j=0}^{\infty} a_j \tau^{\frac{j+\lambda-\mu}{\mu}}, \qquad \tau \to 0+0,$$

where the coefficients a_j depend on the p_k and g_k in a complicated way:

$$a_0 = \frac{g_0}{\mu p_0^{\lambda/\mu}}, \qquad a_1 = \left(\frac{g_1}{\mu} - \frac{(\lambda+1)p_1 g_0}{\mu^2 p_0}\right)\frac{1}{p_0^{(\lambda+1)/\mu}},$$

$$a_2 = \left(\frac{g_2}{\mu} - \frac{(\lambda+2)p_1 g_1}{\mu^2 p_0} + \left((\lambda+\mu+2)p_1^2 - 2\mu p_0 p_2\right)\frac{(\lambda+2)g_0}{2\mu^3 p_0^2}\right)\frac{1}{p_0^{(\lambda+2)/\mu}}.$$

In the case $g(t) \equiv 1$, we have $\lambda = 1$ and $a_j = (j+1)c_{j+1}/\mu$.

Now we are in a position to apply Watson's theorem and obtain:

Corollary 4.5. *Under the above assumptions on p and g, the function f has the following asymptotic expansion for $x \to +\infty$:*

$$f(x) \sim e^{-xp(a)} \sum_{j=0}^{\infty} \Gamma\left(\frac{j+\lambda}{\mu}\right)\frac{a_j}{x^{(j+\lambda)/\mu}}.$$

This result can be used for proving Stirling's asymptotic expansion of the Gamma function a third time.

[1] term-wise differentiation of asymptotic series is, in general, not allowed

4.1.4 The Method of Stationary Phase

We are concerned with an integral

$$I(x) = \int_{t=-\infty}^{\infty} e^{\mathrm{i}xp(t)} g(t)\,\mathrm{d}t,$$

where $x \in \mathbb{R}$ is large, and $p = p(t)$ is real–valued. By a heuristic argumentation, we will find out:

Only the stationary points of the phase, i.e. the points t_0 with $p'(t_0) = 0$, contribute to the principal term of the asymptotic expansion.

For simplicity, we assume:

- the functions p and g are differentiable an infinite number of times,
- there is exactly one point $t_0 \in \mathbb{R}$ with $p'(t_0) = 0$,
- the function $g = g(t)$ and all its derivatives decay for $|t| \to \infty$ sufficiently fast.

The last assumption allows partial integration in the integral I, without introducing boundary terms at $t = \pm\infty$.

Choose a small positive number ε, and define a cut–off function χ_1:

$$\chi_1(t) = \begin{cases} 1 & : -\infty < t \le t_0 - 2\varepsilon, \\ 0 & : t_0 - \varepsilon \le t < \infty, \end{cases}$$

with a smooth transition between $t_0 - 2\varepsilon$ and $t_0 - \varepsilon$. Similarly, we define

$$\chi_3(t) = \begin{cases} 0 & : -\infty < t \le t_0 + \varepsilon, \\ 1 & : t_0 + 2\varepsilon \le t < \infty, \end{cases}$$

also with a smooth transition between $t_0 + \varepsilon$ and $t_0 + 2\varepsilon$. Finally, define

$$\chi_2(t) = 1 - \chi_1(t) - \chi_3(t).$$

We observe that

$$\chi_2(t) = \begin{cases} 0 & : |t - t_0| \ge 2\varepsilon, \\ 1 & : |t - t_0| \le \varepsilon. \end{cases}$$

Then we define

$$I_k(x) = \int_{t=-\infty}^{\infty} e^{\mathrm{i}xp(t)} \chi_k(t) g(t)\,\mathrm{d}t, \qquad k = 1, 2, 3,$$

and have $I(x) = I_1(x) + I_2(x) + I_3(x)$. The term I_1 contains only contributions for t “left from” t_0; and I_3 contains only contributions for t “right from” t_0.

The advantage is that we are allowed to divide by $p'(t)$ for $|t - t_0| \ge \varepsilon$, since $p'(t)$ can not be zero there. Then we are able to show that I_1 and I_3 are not important:

$$\begin{aligned} I_1(x) &= \int_{t=-\infty}^{\infty} e^{\mathrm{i}xp(t)} \chi_1(t) g(t)\,\mathrm{d}t = \int_{t=-\infty}^{\infty} \left(e^{\mathrm{i}xp(t)} p'(t)\right) \left(\frac{\chi_1(t) g(t)}{p'(t)}\right) \mathrm{d}t \\ &= -\frac{1}{\mathrm{i}x} \int_{t=-\infty}^{\infty} \left(e^{\mathrm{i}xp(t)}\right) \left(\frac{\chi_1(t) g(t)}{p'(t)}\right)' \mathrm{d}t; \end{aligned}$$

the last identity follows from partial integration. As a result, we obtain

$$|I_1(x)| \le \frac{C}{x}, \qquad 1 \le x < \infty.$$

We can repeat the partial integration, getting an arbitrary high power of $\frac{1}{x}$:

$$|I_1(x)| \leq \frac{C_N}{x^N}, \qquad 1 \leq x < \infty.$$

The same estimate can be shown for I_3. Since N can be chosen freely, we say that these two sub-integrals do not contribute to the principal term of the asymptotic expansion of $I(x)$ for $x \to +\infty$.

Next, we consider I_2. Supposing that ε is small, we can approximate p and g, by Taylor expansion:

$$\begin{aligned} p(t) &\approx p(t_0) + \frac{1}{2}p''(t_0)(t-t_0)^2, \qquad |t-t_0| \leq 2\varepsilon, \\ g(t) &\approx g(t_0), \qquad |t-t_0| \leq 2\varepsilon. \end{aligned}$$

Then we find

$$\begin{aligned} I_2(x) &\approx \int_{t=t_0-2\varepsilon}^{t_0+2\varepsilon} \exp\left(\mathrm{i}x(p(t_0) + p''(t_0)(t-t_0)^2/2)\right) \chi_2(t) q(t_0)\,\mathrm{d}t \\ &\approx q(t_0) e^{\mathrm{i}xp(t_0)} \int_{t=-\infty}^{\infty} \exp\left(\mathrm{i}x(p''(t_0)(t-t_0)^2/2)\right)\,\mathrm{d}t. \end{aligned}$$

The last integral *does* exist, and it can be evaluated explicitely, because of:

$$\int_{t=-\infty}^{\infty} \exp\left(\pm \mathrm{i}yt^2\right)\,\mathrm{d}t = e^{\pm \mathrm{i}\pi/4} \left(\frac{\pi}{y}\right)^{1/2}, \qquad y > 0.$$

As final result, we get

$$I(x) \approx e^{\pm \mathrm{i}\pi/4} q(t_0) e^{\mathrm{i}xp(t_0)} \left|\frac{2\pi}{xp''(t_0)}\right|^{1/2}, \qquad x \to +\infty,$$

where we chose $+$ or $-$ depending on whether $xp''(t_0) > 0$ or $xp''(t_0) < 0$.

This heuristic argumentation can be made rigorous: define a new integration variable τ by

$$p(t) = p(t_0) + \frac{1}{2}p''(t_0)\tau^2, \qquad |t-t_0| \leq 2\varepsilon,$$

and rewrite the integral I_2 in terms of this new variable τ. You then have

$$\frac{\partial \tau(t)}{\partial t} \approx 1, \qquad |t-t_0| \leq 2\varepsilon.$$

Let the interval (τ_{min}, τ_{max}) correspond to the interval $(t_0 - 2\varepsilon, t_0 + 2\varepsilon)$. Then it follows that

$$\begin{aligned} I_2(x) &= \int_{\tau=\tau_{min}}^{\tau_{max}} \exp\left(\mathrm{i}x(p(t_0) + p''(t_0)\tau^2/2)\right) \chi_2(t(\tau)) q(t(\tau)) \frac{\partial t}{\partial \tau}\,\mathrm{d}\tau \\ &= e^{\mathrm{i}xp(t_0)} \int_{\tau=\tau_{min}}^{\tau_{max}} \exp\left(\mathrm{i}xp''(t_0)\tau^2/2\right) h(\tau)\,\mathrm{d}\tau, \qquad h(\tau) := \chi_2(t(\tau)) q(t(\tau)) \frac{\partial t(\tau)}{\partial \tau}. \end{aligned}$$

And now perform a Taylor expansion of h at $\tau = 0$.

Details can be found in [11].

4.2 Recurrence Formulas

We know several recurrence formulas for special functions:

Chebyshev polynomials: $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, $n \geq 2$,

Legendre polynomials: $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$, $n \geq 2$,

Laguerre polynomials: $nL_n^{(\alpha)}(x) = (2n + \alpha - 1 - x)L_{n-1}^{(\alpha)}(x) - (n + \alpha - 1)L_{n-2}^{(\alpha)}(x)$, $n \geq 2$,

Hermite polynomials: $H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)$, $n \geq 2$,

Bessel functions: $J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x)$.

Our question is the following: given values of these functions for $n = 0$ and $n = 1$, can we calculate the values for large integer n via these recurrence relations ?

4.2.1 An Example

We test the recurrence relation of the Bessel function with the following program (see also [2]):

```
#include<iostream.h>
#include<iomanip.h>
#include<math.h>

int main(){
    double Jn, J0, J1;
    double x;
    int n;

    cout << endl << "x eingeben: ";
    cin >> x;
    cout << endl << "J_0(x) eingeben: ";
    cin >> J0;
    cout << endl << "J_1(x) eingeben: ";
    cin >> J1;
    cout << ::setprecision(16);

    cout << "n = 0" << "               J_n(x) = " << J0 << endl;
    cout << "n = 1" << "               J_n(x) = " << J1 << endl;

    for (n = 2; n < 30; n++) {
        Jn = 2 * (n - 1) * J1 / x - J0;
        cout << "n = " << n << "               J_n(x) = " << Jn << endl;
        J0 = J1;
        J1 = Jn;
    }
}
```

We choose $x = 2.13$ and get the following output (citing [2]):

```
x eingeben: 2.13

J_0(x) eingeben: 0.14960677044884

J_1(x) eingeben: 0.56499698056413
n = 0               J_n(x) = 0.14960677044884
n = 1               J_n(x) = 0.5649969805641299
n = 2               J_n(x) = 0.380906826324991
n = 3               J_n(x) = 0.1503210031447733
n = 4               J_n(x) = 0.04253261915324368
n = 5               J_n(x) = 0.009425923252386066
n = 6               J_n(x) = 0.001720541656080581
n = 7               J_n(x) = 0.0002672691762369261
n = 8               J_n(x) = 3.615715486635142e-05
```

```
n = 9            J_n(x) = 4.333865012661992e-06
n = 10           J_n(x) = 4.670565082569638e-07
n = 11           J_n(x) = 5.164210712170661e-08
n = 12           J_n(x) = 6.633614746019365e-08
n = 13           J_n(x) = 6.958074417255457e-07
n = 14           J_n(x) = 8.427088023837548e-06
n = 15           J_n(x) = 0.000110082814467876
n = 16           J_n(x) = 0.00154203508758005
n = 17           J_n(x) = 0.02305664150598358
n = 18           J_n(x) = 0.3664982518623927
n = 19           J_n(x) = 6.171280009689386
n = 20           J_n(x) = 109.7314549726431
n = 21           J_n(x) = 2054.513320415533
n = 22           J_n(x) = 40401.79880674209
n = 23           J_n(x) = 832536.6357390455
n = 24           J_n(x) = 17939262.63499424
n = 25           J_n(x) = 403432536.8289199
n = 26           J_n(x) = 9452308080.766882
n = 27           J_n(x) = 230357140326.9635
n = 28           J_n(x) = 5830587869222.533
n = 29           J_n(x) = 153062093881486.1
```

This is obvious nonsense, because the following formula follows from the Poisson representation (3.7) of the Bessel function, after some calculation ([1]):

$$J_n(z) = \frac{1}{\pi}\int_{\theta=0}^{\pi} \cos(z\sin\theta - n\theta)\,\mathrm{d}\theta, \qquad n \in \mathbb{N}_+,$$

which yields $|J_n(x)| \leq 1$ for $x \in \mathbb{R}$.

One of the reasons of the instability of this algorithm lies in the factor $2(n-1)/x$ in front of J_n, which becomes large for large n, whereas the final result is expected to be small. A small result can only occur if catastrophic cancellation happens, which makes the algorithm unstable.

4.2.2 Introduction to Recurrence Relations

Example 4.6. *Consider the* FIBONACCI *recurrence:*

$$\begin{cases} f_0, f_1 & : \text{ given, } \in \mathbb{R}, \\ f_n = f_{n-1} + f_{n-2} & : n \geq 2. \end{cases}$$

We obtain the well–known FIBONACCI *numbers for* $f_0 = f_1 = 1$.

It is clear that the sequence $(f_n)_n$ *is uniquely determined upon choosing* f_0, f_1. *Therefore, the set of all possible sequences is a linear space of dimension* 2. *For constructing two linearly independent sequences, we make the ansatz*

$$f_n = c^n, \qquad c \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

and arrive at the equation $c^2 = c + 1$ *with the solutions*

$$c_{1,2} = \frac{1 \pm \sqrt{5}}{2}, \qquad c_1 = 1.618034\ldots, \quad c_2 = -0.618034\ldots.$$

We see that $|c_1| > 1$ *and* $|c_2| < 1$. *The space of all sequences* $(f_n)_n$ *is spanned by an increasing sequence, and a decreasing sequence.*

Definition 4.7. *Let* $(\alpha_n)_n$ *and* $(\beta_n)_n$ *be sequences of given complex numbers. We say that a sequence* $(f_n)_n$ *of complex numbers* satisfies a recurrence relation *if*

$$f_n = \alpha_n f_{n-1} + \beta_n f_{n-2}$$

holds for all $n \geq 2$. *Let* $V_{\alpha,\beta}$ *denote the two-dimensional space of all sequences* $(f_n)_n$ *satisfying this recursion relation.*

If $V_{\alpha,\beta} = \operatorname{span}\{(f_n^+)_n, (f_n^-)_n\}$ *with two linearly independent sequences* $(f_n^+)_n$ *and* $(f_n^-)_n$ *that fulfill*

$$\lim_{n\to\infty} \frac{f_n^-}{f_n^+} = 0,$$

then the sequence $(f_n^+)_n$ *is called* dominant, *and the sequence* $(f_n^-)_n$ *is called* recessive.

A dominant sequence can not be unique, since you can add an arbitrary multiple of the recessive sequence to it, and still get a dominant sequence. However, the recessive sequence is unique (up to a constant factor).

If you start the recurrence with initial values f_0 and f_1 that belong to the recessive sequence, then the recursion should produce that recessive sequence—in theory. However, in practice, the numerical values of f_0 and f_1 will be contaminated with errors that contribute to the dominant solution, which soon will become very big for increasing n.

The evaluation of a recessive sequence is numerically unstable.

In case of the Bessel recurrence, we have a recessive sequence $(J_n(x))_n$, and a dominant sequence $(Y_n(x))_n$ (which fulfills the same recurrence relation). In fact, for fixed $z \in \mathbb{C}$ and $\nu \to +\infty$, we have ([1]) the asymptotic relations

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu, \qquad Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu}.$$

For a general recurrence relation $f_n = \alpha_n f_{n-1} + \beta_n f_{n-2}$, we can define $u_n = \binom{f_{n-1}}{f_n}$, and obtain

$$u_n = A_n u_{n-1}, \qquad A_n = \begin{pmatrix} 0 & 1 \\ \beta_n & \alpha_n \end{pmatrix},$$

or, even more general, $u_n = F(n, u_{n-1})$.

Definition 4.8 ([7]). *Suppose* $F(n,0) = 0$ *for every* n, *and consider the problem*

$$\begin{cases} u_{n_0} & : \text{ given}, \\ u_n = F(n, u_{n-1}) & : n > n_0. \end{cases}$$

We say that the zero sequence $(0)_n$ *is*

a stable solution *if, for every* $\varepsilon > 0$, *there is a* $\delta = \delta(\varepsilon, n_0)$ *such that* $|u_{n_0}| < \delta$ *implies* $|u_n| < \varepsilon$ *for* $n > n_0$,

a uniformly stable solution *if it is a stable solution and* $\delta = \delta(\varepsilon, n_0)$ *can be chosen independently of* n_0,

attractive *if there is a* $\delta = \delta(n_0)$ *such that* $|u_{n_0}| < \delta$ *implies* $\lim_{n\to\infty} u_n = 0$,

asymptotically stable *if it is stable and attractive,*

We cite from [7, Chapter 4]:

Proposition 4.9. *Suppose* $F(n, u_n) = Au_n$, *where the matrix* A *does not depend on* n.

- *The zero solution is asymptotically stable if and only if the eigenvalues of* A *are inside the unit disk, i.e.,* $|\lambda_j| < 1$.
- *The zero solution is stable if and only if the eigenvalues of* A *have modulus less than or equal to one, and those of modulus one are semi-simple (i.e., their algebraic and geometric multiplicities coincide).*

Corollary 4.10. *The zero solution of the Chebyshev recurrence relation is stable, for $-1 \le x \le 1$. The eigenvalues of the recurrence matrix A have modulus one.*

Therefore, we can evaluate $T_n(x)$ by forward recursion, starting from $n = 0, 1$, provided that $-1 \le x \le 1$.

Concerning the recurrence relation of the Legendre polynomials, we are tempted to read it as a perturbed Chebyshev recurrence relation:

$$P_n(x) = \left(2 - \frac{1}{n}\right) xP_{n-1}(x) - \left(1 - \frac{1}{n}\right) P_{n-2}(x), \qquad \alpha_n = \left(2 - \frac{1}{n}\right) x, \quad \beta_n = -\left(1 - \frac{1}{n}\right),$$

and expect a similar behavior as the Chebyshev recurrence relation.

For completeness, we cite [7, Chapter 4] again:

Proposition 4.11. *Consider the recurrences*

$$v_n = Av_{n-1}, \tag{4.1}$$
$$u_n = (A + B_n)u_{n-1}. \tag{4.2}$$

- *If the zero solution to (4.1) is uniformly stable and $\sum_{n=n_0}^{\infty} \|B_n\| < \infty$, then the zero solution to (4.2) is uniformly stable.*
- *If the zero solution to (4.1) is asymptotically stable and $\sum_{n=n_0}^{\infty} \|B_n\| < \infty$, then the zero solution to (4.2) is asymptotically stable.*

The attempt to apply this result to the Legendre recurrence fails, because of $\|B_n\| = \frac{C}{n}$. However, making use of the diagonalizer of A we can show that

$$\|u_n\| \le Cn \|u_1\|, \qquad u_n = \begin{pmatrix} P_{n-1}(x) \\ P_n(x) \end{pmatrix}, \qquad -1 \le x \le 1.$$

The details are left to the reader, as an exercise. In this sense, we can say that the forward recurrence relation of the Legendre polynomials is *almost stable*, provided that $-1 \le x \le 1$.

Unfortunately, the other recurrence relations do not have, in general, stable zero solutions. We cite again from [7, Chapter 3.3]:

Proposition 4.12 (Perron, Poincare). *Suppose* $\lim_{n\to\infty} \alpha_n = \alpha_\infty$*,* $\lim_{n\to\infty} \beta_n = \beta_\infty$*, and $\beta_n \neq 0$, for all n. Let $\lambda_{1,2}$ be the solutions to the* characteristic equation

$$\lambda^2 - \alpha_\infty \lambda - \beta_\infty = 0,$$

and suppose $|\lambda_1| \neq |\lambda_2|$. Then the recurrence relation $f_n = \alpha_n f_{n-1} + \beta_n f_{n-2}$ has two solutions $(f_{n,1})_n$ and $(f_{n,2})_n$ with

$$\lim_{n\to\infty} \frac{f_{n+1,1}}{f_{n,1}} = \lambda_1, \qquad \lim_{n\to\infty} \frac{f_{n+1,2}}{f_{n,2}} = \lambda_2.$$

This applies to the Legendre recursion for $|x| > 1$. In this case, we have $\alpha_\infty = 2x$ and $\beta_\infty = -1$, hence $\lambda_{1,2} = x \pm \sqrt{x^2 - 1}$. We see that the Legendre recurrence relation (for the functions Q_n, which satisfy the same relation) has a dominant and a recessive solution, for $|x| > 1$. The dominant solutions can be evaluated satisfactorily by forward recursion. Unfortunately, the sequence $(Q_n)_n$ is recessive.

For the Laguerre polynomials $L_n^{(\alpha)}$, we get $\lambda_1 = \lambda_2 = 1$, hence no news.

For completeness, we cite one more result from [3]. It includes the previous one; just take $A = B = 0$.

Proposition 4.13 (Perron, Kreuser). *We consider the recurrence $f_n = \alpha_n f_{n-1} + \beta_n f_{n-2}$ under the condition $\beta_n \neq 0$ for all n. Suppose the following asymptotic behavior for α_n and β_n:*

$$\alpha_n \sim an^A, \qquad \beta_n \sim bn^B, \qquad ab \neq 0, \qquad A, B \in \mathbb{R}.$$

If $2A > B$**,** *then there are solution sequences* $(f_{n,1})_n$ *and* $(f_{n,2})_n$ *with*

$$\frac{f_{n+1,1}}{f_{n,1}} \sim an^A, \qquad \frac{f_{n+1,2}}{f_{n,2}} \sim -\frac{b}{a}n^{B-A}, \qquad n \to \infty.$$

If $2A = B$**,** *then define numbers* $\lambda_{1,2}$ *as solutions to* $\lambda^2 - a\lambda - b = 0$*.*

If $|\lambda_1| > |\lambda_2|$**,** *then we have solution sequences* $(f_{n,1})_n$ *and* $(f_{n,2})_n$ *with*

$$\frac{f_{n+1,1}}{f_{n,1}} \sim \lambda_1 n^A, \qquad \frac{f_{n+1,2}}{f_{n,2}} \sim \lambda_2 n^A.$$

If $|\lambda_1| = |\lambda_2|$**,** *then every non-trivial solution sequence* $(f_n)_n$ *satisfies*

$$\limsup_{n\to\infty} \left(\frac{|f_n|}{n!^A}\right)^{1/n} = |\lambda_1|.$$

If $2A < B$**,** *then every non-trivial solution sequence* $(f_n)_n$ *satisfies*

$$\limsup_{n\to\infty} \left(\frac{|f_n|}{n!^{B/2}}\right)^{1/n} = \sqrt{|b|}.$$

We see that, in case $2A < B$, the second solution is recessive, as well as in the first sub-case of the case $2A = B$.

Example 4.14 (Laguerre polynomials). *We have* $a = 2$*,* $A = 0$*,* $b = -1$*,* $B = 0$*, hence* $2A = B$*. It follows that* $|\lambda_1| = |\lambda_2| = 1$*, hence*

$$\limsup_{n\to\infty} \left|L_n^{(\alpha)}(x)\right|^{1/n} = 1.$$

Laguerre polynomials can be evaluated by forward recurrence.

Example 4.15 (Hermite polynomials). *We have* $a = 2x$*,* $A = 0$*,* $b = -2$*,* $B = 1$*, hence* $2A < B$*, from which we obtain*

$$\limsup_{n\to\infty} \left(\frac{|H_n(x)|}{n!^{1/2}}\right)^{1/n} = \sqrt{2}.$$

Hermite polynomials can be evaluated by forward recurrence.

Example 4.16 (Bessel functions). *We have* $a = \frac{2}{a}$*,* $A = 1$*,* $b = -1$*,* $B = 0$*, hence* $2A > B$*, and consequently,*

$$\frac{f_{n+1,1}}{f_{n,1}} \sim \frac{2}{x}n, \qquad \frac{f_{n+1,2}}{f_{n,2}} \sim \frac{x}{2}n^{-1}.$$

The first solution corresponds to the $Y_n(x)$*, the second one to* $J_n(x)$*.*

4.2.3 Backward Recurrence

We wish to compute $J_0(x)$, $J_1(x)$, …, $J_n(x)$, for real x and (possibly) large n. Forward recurrence is not numerically stable, since the sequence $(J_m)_m$ is recessive. But if we go backwards, then $(Y_m)_m$ becomes recessive, and $(J_m)_m$ becomes dominant.

Therefore, our strategy could be as follows:

- Pick a number $M \in \mathbb{N}$, suitably large.
- Put $f_{M+1} = \varepsilon$ and $f_{M+2} = 0$, for some small ε. For instance, $\varepsilon = 10^{-100}$.

- Follow the recurrence formula

$$f_m = \frac{2(m+1)}{x} f_{m+1} - f_{m+2},$$

for $m = M,\, M-1,\, \ldots,\, 1,\, 0$.

- Compute $J_0(x)$ by other means (power series or asymptotic expansion or polynomial approximation or whatever).

- Evaluate a scaling factor $S_0 = \frac{J_0(x)}{f_0}$, and put

$$J_1(x) = S_0 f_1, \ldots, J_n(x) = S_0 f_n.$$

The sequence $(f_m)_m$ is increasing rapidly (for m going downwards), therefore ε should be chosen small, to prevent the arithmetic from overflowing.

The number M should be chosen large, to eliminate the influence of the recessive sequence $(Y_m)_m$, but not too large, for reasons of overflow.

Another possibility of calculating the scaling factor comes from the relation

$$J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x) = 1,$$

or similar relations. The terms $J_{2k}(x)$ decrease exponentially with growing k, provided that $k > |x|$. Then it is no longer necessary to evaluate $J_0(x)$ by other means. Another choice is ([3])

$$J_a(x) + 2\sum_{m=1}^{\infty} (-\mathrm{i})^m \frac{(a+m)\Gamma(2a+m)}{m!\Gamma(1+2a)} J_{a+m}(x) = \frac{(x/2)^a e^{-\mathrm{i}x}}{\Gamma(1+a)},$$

which enables us to compute $J_\nu(x)$ for $\nu \notin \mathbb{N}$.

4.3 Further Methods

A good survey on further methods of evaluating special functions can be found in [5]. See also [4].

Another approach can be found in [9] (and other books by Luke): many special functions can be written in terms of hypergeometric functions ${}_2\mathrm{F}_1(a, b; c; z)$. Given a complex number z, choose a complex number w such that $0 \le z/w \le 1$. Then you have an expansion of the form

$${}_2\mathrm{F}_1(a, b; c; z) = \sum_{n=0}^{\infty} C_n(w) T_n(z/w),$$

where T_n are the Chebyshev polynomials, and C_n are so-called Chebyshev coefficients. In many cases, these coefficients form a sequence $(C_n)_n$ that decreases exponentially fast. Consequently, you need only a few terms in this sum $\sum_{n=0}^{\infty} \ldots$ for the approximation of ${}_2\mathrm{F}_1(a, b; c; z)$. Moreover, these coefficients C_n satisfy a three-term recurrence relation with known coefficients α_n, β_n, which can be solved by backward recurrence. Finally, several scaling conditions of the form $\sum_{m=0}^{\infty} \lambda_m C_m = 1$ are known. Then you can utilize one such scaling condition for the computation of the scaling factor S_0, and another scaling condition for checking the correctness of your result. However, the details are quite involved.

A free source of easily accessible online information on the implementation of algorithms is the `GNU Scientific Library` (GSL).

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