Definable Valuations on Dependent Fields

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Question

When does a dependent (in $\mathcal{L}_{Ring}$) field admit a non-trivial definable valuation (in $\mathcal{L}_{Ring}$, possibly with parameters)?

Definition

Let Γ be an ordered abelian group and let ∞ a symbol such that for all $\gamma \in \Gamma$ $\infty > \gamma$ and $\infty = \infty + \infty = \gamma + \infty = \infty + \gamma$.
A *valuation* v on a field K is a surjective map

$$v : K \twoheadrightarrow \Gamma \cup \{\infty\}$$

such that for all $x, y \in K$

(i) $v(x) = \infty \Rightarrow x = 0$

(ii) $v(xy) = v(x) + v(y)$

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$

We call v trivial if for all $x \neq 0$ $v(x) = 0$.

Definition

We call a subring $\mathcal{O}$ of a field K valuation ring if for every $x \in K \setminus \{0\}$ $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.
We say that $\mathcal{O}$ is non-trivial if $\mathcal{O} \neq K$.

Definition and Lemma

Let v be a valuation on a field K.
Then $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ is a valuation ring on K.

$$\{ \text{ valuations on } K \ \}/\sim \quad \longleftrightarrow \quad \{ \text{ valuation rings on } K \ \}$$

Definition and Lemma

Every valuation ring $\mathcal{O}$ has exactly one maximal ideal $\mathcal{M}$.
We call $\overline{K} := \mathcal{O}/\mathcal{M}$ the residue field of $\mathcal{O}$.

Question

When does a field admit a non-trivial valuation?

Answer

A field admits a non-trivial valuation if and only if it is no algebraic extension of a finite field.

From now on if not stated otherwise no fields are algebraic extension of finite fields.

Theorem (Chevalley)

Let $(K, \mathcal{O})$ be a valued field. Let L/K be an arbitrary field extension.
Then there exists an extension of $\mathcal{O}$ to L i.e. there exists a valuation ring $\mathcal{O}'$ on L such that $\mathcal{O}' \cap K = \mathcal{O}$.

Definition

A valuation (ring) on a field K is called henselian if it extends uniquely to the algebraic closure of K.

Definition

Let $\mathcal{L}_{\text{Ring}} = (0, 1; +, \cdot, -)$ the language of rings.
We call a valuation ring $\mathcal{O}$ on a field K definable if there exists an $\mathcal{L}_{\text{Ring}}(K)$-formula φ in one variable such that $\mathcal{O} = \{x \in K \mid \varphi(x)\}$.

Example

Let $(\mathbb{Q}_p, \mathcal{O}_p)$ be the field of p-adic numbers.
Then

$$\mathcal{O}_p = \left\{x \in \mathbb{Q}_p \mid \exists y\, y^2 - y = px^2\right\}$$

Question

When does a field admit a non-trivial ***definable*** valuation?

henselian **valued** fields *p*-henselian **valued** fields **t-henselian** fields	**dependence** + other **algebraic, combinatorial** and **model theoretic** assumptions
Results of: Koenigsmann and others	

J. Koenigsmann, Definable Valuations, preprint, Delon, Dickmann, Gondard Paris VII, Seminaire Structures algébraiques ordonées (1994)

Definition

A formula $\varphi(x, \underline{y})$ has the *independence property (IP)* in a theory $\mathfrak{T}$ if there exist a model $\mathfrak{M}$ of $\mathfrak{T}$ and

$$\{a_i\}_{i\in\omega} \subseteq M$$

and

$$\{\underline{b}_W\}_{W\subseteq\omega} \subseteq M$$

such that for every $W \subseteq \omega$ and every $i \in \omega$

$$\mathfrak{M} \models \varphi(a_i, \underline{b}_W) \text{ if and only if } i \in W.$$

Definition

A formula is called *dependent* or NIP (not indepence property) (in $\mathfrak{T}$) if it is does not have the independence property (in $\mathfrak{T}$).

Definition

A theory $\mathfrak{T}$ is called *dependent* or NIP if all formulas are dependent in $\mathfrak{T}$.

Definition

A structure $\mathfrak{M}$ is called *dependent* if its theory $\mathrm{Th}(\mathfrak{M})$ is dependent.

The following classes of fields are dependent:

real closed fields stable fields (in particular: algebraically closed fields)	no non-trivial definable valuation

Fact

Let K be a dependent field with $\sqrt{-1} \in K$ such that for all finite extensions L/K and all $p \in \mathbb{N}$ prime $(L^\times : (L^\times)^p) < \infty$. Assume that there exists a finite extension L/K and a $p \in \mathbb{N}$ prime $(L^\times : (L^\times)^p) > 1$. Then K is not real closed and K is not stable.

Conjecture

Let K be a dependent field.
Let $\sqrt{-1} \in K$.
Assume that for all finite extensions L/K and all $p \in \mathbb{N}$ prime $(L^\times : (L^\times)^p) < \infty$.
Further assume there exists a finite extension L/K and a $p \in \mathbb{N}$ prime such that $(L^\times : (L^\times)^p) > 1$.
Then K admits a non-trivial definable valuation.

L		$\mathcal{O}$	non-trivial definable valuation ring
$\vert$	algebraic	$\vert$	
$\vert$	finite	$\vert$	
K		$\mathcal{O} \cap K$	non-trivial definable valuation ring

Fact

Let K be a field. Let L/K be a finite extension.
If $\mathcal{O}$ is a non-trivial definable valuation on L then $\mathcal{O} \cap K$ is a non-trivial definable valuation on K.

Question

How do we find a definable valuation on a field?

Definition

Let $\mathcal{O}$ a valuation ring on a field K with maximal ideal $\mathcal{M}$ and T an additive [multiplicative] subgroup of K.

(a) $\mathcal{O}$ is *compatible* with T if and only if $\mathcal{M} \subseteq T$ [$1 + \mathcal{M} \subseteq T$].

(b) $\mathcal{O}$ is *weakly compatible* with T if and only if $\mathcal{A} \subseteq T$ [$1 + \mathcal{A} \subseteq T$] for some $\mathcal{O}$-ideal $\mathcal{A}$ with $\sqrt{\mathcal{A}} = \mathcal{M}$.

(c) $\mathcal{O}$ is *coarsely compatible* with T if and only if $\mathcal{O}$ is weakly compatible with T and there is no proper coarsening $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ such that $\widetilde{\mathcal{O}}^\times \subseteq T$.

Remark

Let $T \neq K$ [$T \neq K^\times$] and let $\mathcal{O} \neq K$ be weakly compatible with T.

Then there exists a valuation ring $\widetilde{\mathcal{O}}$ which is coarsely compatible with T such that $\mathcal{O} \subseteq \widetilde{\mathcal{O}} \subsetneq K$.

Definition and Lemma

Let $\mathcal{O}_T := \bigcap \{\mathcal{O} \mid \mathcal{O}$ coarsely compatible with $T\}$.
$\mathcal{O}_T$ is a valuation ring on K.

Question

Which subgroups can we choose for T?

T should be a non-trivial, definable, proper subgroup of K.

Definable subgroups of K are:

- The Artin-Schreier group $K^{(p)} := \{x^p - x \mid x \in K\}$ for $p = \mathrm{char}(K)$.
- The group of p-th powers of the units of K $(K^\times)^p$ for any prime p.

Theorem (Kaplan-Scanlon-Wagner)

Let K be an infinite dependent field. Then K is Artin-Schreier closed, e.g. $K^{(p)} = K$ for $p = char(K)$.

Corollary

Let K be an infinite dependent field and $T = K^{(p)}$ for $p = char(K)$.
Then $\mathcal{O}_T$ is trivial.

We will therefore from now on only consider $T = (K^\times)^p$ for p prime.

Question

When is $\mathcal{O}_T$ definable?

Theorem (Koenigsmann)

Let K be a field and T be an additive or multiplicative subgroup of K.
Then $\mathcal{O}_T$ is definable in $\mathcal{L}' := \{0, 1; +, -, \cdot; \underline{T}\}$ in the following cases

	$T \subseteq K$ *additive*	$T \subseteq K^{\times}$ *multiplicative*
group case	*if and only if either* $\mathcal{O}_T$ *is discrete or* $\forall x \in \mathcal{M}_T \; x^{-1}\mathcal{O}_T \subseteq T$	***always***
weak case	*if and only if* $\mathcal{O}_T$ *is discrete*	
residue case	***always***	*if and only if* $\overline{T}$ *is no ordering*

Theorem (Koenigsmann)

Let K be a field let $\sqrt{-1} \in K$. Let $T = (K^{\times})^p$ for some prime p. Then $\mathcal{O}_T$ is definable in $\mathcal{L}_{Ring} := \{0, 1; +, -, \cdot\}$ in the following cases

group case	***always***
weak case	*if and only if $\mathcal{O}_T$ is discrete*
residue case	***always***

Lemma

Let v be a valuation on a field K. Let T be a multiplicative subgroup such that there exists an $n \in \mathbb{N}$ with $(K^{\times})^{n} \subseteq T$ and $\left(n, char\left(\overline{K}\right)\right) = 1$ or $char\left(\overline{K}\right) = 0$ (e.g. $n \in \mathcal{O}^{\times}$)

Then v is compatible with T if and only if it is weakly compatible with T.

Proposition

Let K be a field with $\sqrt{-1} \in K$ and char$(K) > 0$. Let p be prime with char$(K) \neq p$. Let $T := (K^\times)^p$.
Then $\mathcal{O}_T$ is definable.

Proposition

Let K be a field with $\sqrt{-1} \in K$. Let p be prime with $char(K) \neq p$. Let $T := (K^\times)^p$.
Then there exists a definable valuation which induces the same topology as $\mathcal{O}_T$.

Question

When is $\mathcal{O}_T$ non-trivial?

Definition and Lemma

Let $\mathcal{O}_T := \bigcap\{\mathcal{O} \mid \mathcal{O}$ coarsely compatible with $T\}$.
$\mathcal{O}_T$ is a valuation ring on K.

Lemma

If T is proper multiplicative subgroup of $K^\times$ the following are equivalent:

(i) $\mathcal{O}_T$ *is non-trivial*

(ii) *there exists a non-trivial valuation ring $\mathcal{O}$ on K such that $\mathcal{O}$ and T are weakly compatible*

(iii) $\mathcal{B}_T = \{(aT + b) \cap (cT + d) \mid a, b, c, d \in K, a, c \neq 0\}$ *is a basis of a V-topology.*

Definition and Lemma

Let K be a field and $\mathcal{B} \subseteq \mathcal{P}(K)$ such that

(V 1) $\bigcap \mathcal{B} := \bigcap_{U \in \mathcal{B}} U = \{0\}$ and $\{0\} \notin \mathcal{B}$

(V 2) $\forall U, V \in \mathcal{B} \quad \exists W \in \mathcal{B} \quad W \subseteq U \cap V$

(V 3) $\forall U \in \mathcal{B} \quad \exists V \in \mathcal{B} \quad V - V \subseteq U$

(V 4) $\forall U \in \mathcal{B} \quad \forall x, y \in K \quad \exists V \in \mathcal{B} \quad (x + V)(y + V) \subseteq xy + U$

(V 5) $\forall U \in \mathcal{B} \quad \forall x \in K^{\times} \quad \exists V \in \mathcal{B} \quad (x + V)^{-1} \subseteq x^{-1} + U$

(V 6) $\forall U \in \mathcal{B} \quad \exists V \in \mathcal{B} \quad \forall x, y \in K \quad xy \in V \Rightarrow x \in U \vee y \in U$

Then

$$\mathcal{T}_{\mathcal{B}} := \{U \subseteq K \mid \forall x \in U \quad \exists V \in \mathcal{B} \quad x + V \subseteq U\}$$

is a *V-topology* on K.

Fact

*Let K be a field and $\mathcal{T}$ a topology on K.
Then $\mathcal{T}$ is a V-topology if and only if there exists either an archimedean absolute value or a valuation on K whose induced topology coincides with $\mathcal{T}$.*

Lemma (Koenigsmann)

Let $T \subsetneq K^\times$ be a multiplicative subgroup of K and and let $\mathcal{T}_T$ be the topology with basis $\mathcal{B}_T = \{(aT + b) \cap (cT + d) \mid a, b, c, d \in K, a, c \neq 0\}$. Let v be a non-trivial valuation on K.
$\mathcal{T}_v = \mathcal{T}_T$ if and only if T is weakly compatible with some valuation w such that $\mathcal{O}_v \subseteq \mathcal{O}_w \subsetneq K$.

Remark

If $\mathcal{O}_v \subseteq \mathcal{O}_w$ then $\mathcal{T}_v = \mathcal{T}_w$.

If $\mathcal{O}_T$ is non-trivial there exists a non-trivial valuation v which is weakly compatible with T. By the last lemma we have $\mathcal{T}_T = \mathcal{T}_v$ and therefore $\mathcal{T}_T$ is a V-topology.

On the other hand if $\mathcal{T}_T$ is a V-topology then it is induced by a non-trivial absolute value or by a non-trivial valuation. It is possible to show that in our case $\mathcal{T}_T$ is induced by a valuation. Therefore again by the last lemma there exists a non-trivial valuation which is weakly compatible with T. And hence $\mathcal{O}_T$ is non-trivial.