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Real Algebraic Geometry II
Ordered Sets and Ordinal Numbers

1 Notation. Let A and B be sets. We denote by $A \sqcup B$ the disjoint union of A and B.

2 Definition. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two ordered sets.

(a) We define the sum of ordered sets

$$(A, \leq_A) + (B, \leq_B) = A + B := (A \sqcup B, \leq_+)$$

such that for any $x_1, x_2 \in A \sqcup B$

$$x_1 \leq_+ x_2 \Leftrightarrow \begin{cases} x_1, x_2 \in A \text{ and } x_1 \leq_A x_2 & \text{or} \\ x_1, x_2 \in B \text{ and } x_1 \leq_B x_2 & \text{or} \\ x_1 \in A \text{ and } x_2 \in B & . \end{cases}$$

(b) We define the product of ordered sets

$$(A, \leq_A) \,.\, (B, \leq_B) = A.B := (A \times B, \leq_{\text{rlex}})$$

such that $\leq_{\text{rlex}}$ is the reverse lexicographic order, i.e. for any $(x_1, y_1), (x_2, y_2) \in A \times B$

$$(x_1, y_1) \leq_{\text{rlex}} +(x_2, y_2) \Leftrightarrow \begin{cases} y_1 <_A y_2 & \text{or} \\ y_1 = y_2 \text{ and } x_1 \leq_B x_2 & . \end{cases}$$

3 Definition. A set A is called transitive if any element of A is also a subset of A.

4 Definition. A set α is called an ordinal if it is transitive and if $(\alpha, \in)$ is a well-ordered set.

5 Definition. Let $(A,<)$ be a well-ordered set. The order type of $(A,<)$, denoted $\mathrm{ot}(A)$, is defined as the unique ordinal to which $(A,<)$ is isomorphic.

6 Definition. Let α and β be two ordinal numbers. Let $n \in \mathbb{N}$.

(a) We define the sum of ordinals as $\alpha + \beta := \mathrm{ot}\,(\alpha + \beta)$.

(b) We define $\omega.n := \underbrace{\omega + \cdots + \omega}_{n-\text{times}}$.

(c) We define the product of ordinals $\alpha.\beta := \mathrm{ot}(\alpha.\beta)$.

(d) We define $\omega^n := \underbrace{\omega. \cdots .\omega}_{n-\text{times}}$.