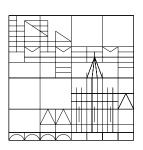
## Universität Konstanz

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# Real Algebraic Geometry II Exercise Sheet 13

The exercise sheet will not be collected. Please come to our office hours if you have any questions.

## Exercise 1 (Multiplicative Lexicographic Decomposition)

Let  $(K, \leq)$  be an ordered field. Assume that the group  $(K^{>0}, \cdot, 1 <)$  is divisible. Let v be the natural valuation on K,  $\overline{K}$  its residue field, G its value group,  $\mathcal{O}$  its valuation ring,  $\mathcal{M}$  the maximal ideal of  $\mathcal{O}$  and  $\mathcal{U} := \mathcal{O} \setminus \mathcal{M}$ .

- (a) Show that there exists a group complement B of  $\mathcal{U}^{>0}$  in  $(K^{>0}, \cdot, 1 <)$  and a group complement B' of  $1 + \mathcal{M}$  in  $(\mathcal{U}^{>0}, \cdot, 1, <)$  such that  $(K^{>0}, \cdot, 1 <) = B \coprod B' \coprod (1 + \mathcal{M}, \cdot, 1, <).$
- (b) Show that every group complement B of  $\mathcal{U}^{>0}$  in  $(K^{>0}, \cdot, 1 <)$  is order isomorphic to G through the isomorphism -v.
- (c) Show that every group complement B' of  $1 + \mathcal{M}$  in  $(\mathcal{U}^{>0}, \cdot, 1, <)$  is order isomorphic to  $(\overline{K}^{>0}, \cdot, 1, <)$ .

**Definition:** Let  $(K, \leq)$  be an ordered field and v its non-trivial natural valuation. Let  $P_K := K^{>0} \setminus R_v$ .

Let  $\sim$  be defined by  $x \sim y$  if and only if  $r \cdot |x| < |y|$  for all  $r \in R_v$  and  $r \cdot |y| < |x|$  for all  $r \in R_v$   $(x, y \in P_K)$ .

 $\sim$  denotes the multiplicative equivalence relation on  $P_K$ .

Denote the class of a by [a] and call it the multiplicative class of a.

## Exercise 2

Let  $(K, \leq)$  be an ordered field. Let v be the non-trivial natural valuation of  $(K, \leq)$  and  $R_v$  its valuation ring. Let w be a valuation on K with valuation ring  $R_w$  such that  $R_w \neq R_v$  is a convex subring. Let G be the value group of v. Let  $G_w := \{v(x) \in G \mid x \in K, w(x) = 0\}$ . Let  $a \in P_K := K^{>0} \setminus R_v$ . Show that the following are equivalent

- (i)  $R_w$  is principal convex generated by a.
- (ii) The class  $[a]^{\cdot}$  is the largest of all classes of  $R_w$ .
- (iii) [a] is a final segment in  $R_w$ .
- (iv)  $G_w$  is principal with smallest archimedean class [v(a)].
- (v)  $v_G(G_w) = \Gamma_w$  is a principal final segment with smallest element  $v_G(v(a))$ . (where  $v_G$  is the natural valuation on G).

#### **Exercise 3**

Let  $\Gamma_1$  and  $\Gamma_2$  be a totally ordered set.

- (a) Show that if  $\Gamma_1 \cong \Gamma_2$ , then  $\Gamma_1^{sf} \cong \Gamma_2^{sf}$ .
- (b) Compute the order type of  $\mathbb{Q}^{fs}$ .