Around and about real closed valued fields

Deirdre Haskell
presenting joint work with Clifton Ealy and Jana Maříková
supported by an NSERC discovery grant

McMaster University

Summer school in tame geometry
Konstanz, Germany, July 18–23, 2016
Let $\mathcal{R}$ be a real closed field, consider the field of Laurent series $\mathcal{R}((t))$

$$\mathcal{R}((t)) = \left\{ \sum_{i=N}^{\infty} a_i t^i : a_i \in \mathcal{R}, N \in \mathbb{Z} \right\}.$$ 

$\mathcal{R}((t))$ can be ordered in many different ways; let’s consider $t > 0$, $t < r$ for any positive $r \in \mathcal{R}$. Then

$$t > t^2 > t^3 > \cdots > 0,$$

and in general,

$$\sum_{i=N}^{\infty} a_i t^i < \sum_{i=N}^{\infty} b_i t^i \iff a_N < b_N.$$
A picture
A picture
Another picture
A valuation is a function $v : K \rightarrow \Gamma \cup \{\infty\}$, where $K$ is a field, $\Gamma$ is an ordered abelian group, such that

\begin{align*}
v(xy) &= v(x) + v(y) \\
v(x + y) &\geq \min\{v(x), v(y)\} \\
v(x) = \infty &\iff x = 0
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\]

Valuation ring $\mathcal{O}_K = \{x \in K : v(x) \geq 0\}$
Maximal ideal $m_K = \{x \in K : v(x) > 0\}$
Residue field $k = \mathcal{O}_K/m_K$. 

convexly valued ordered field

In the case of $K = \mathcal{R}((t))$,

$$v\left(\sum_{i=N}^{\infty} a_i t^i\right) = N,$$

$\mathcal{O}_K = \mathcal{R}[[t]]$ and the residue field is $\mathcal{R}$. 

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More generally, a valuation $v$ on a field is convex with respect to an ordering $<$ on the field if for all $0 < x < y$, $v(x) \geq v(y)$. 
Let \( \mathcal{R} \) be an ordered field with a convex valuation.

1) If \( \mathcal{R} \) is real closed then \( \Gamma_{\mathcal{R}} \) is divisible and \( k_{\mathcal{R}} \) is real closed.
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1) If $\mathcal{R}$ is real closed then $\Gamma_{\mathcal{R}}$ is divisible and $k_{\mathcal{R}}$ is real closed.

2) If $\Gamma_{\mathcal{R}}$ is divisible, $k_{\mathcal{R}}$ is real closed and $\mathcal{R}$ is henselian then $\mathcal{R}$ is real closed.

Analogous to algebraically closed valued fields.
Let $K$ be a henselian valued field of characteristic 0. Then $\text{Th}(K)$ in the language of valued fields is determined up to elementary equivalence by $\text{Th}(\Gamma)$ in the language of ordered groups and $\text{Th}(k)$ in the language of fields.
Ax-Kochen, Ersov, late 60’s

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Motto: a valued field is controlled by its value group and residue field.

Motivation: to what extent is this still true when further structure is added? More generally: pursue analogies between algebraically closed valued fields and real closed valued fields.
model theory of pure real closed valued fields

Cherlin-Dickmann 1983

The theory RCVF of real closed fields with a proper convex valuation has quantifier elimination in the language of rings with predicates for the ordering and the valuation ring.
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Mellor 2006
The theory RCVF has elimination of imaginaries in the sorted language $\mathcal{G}$ with sorts for the finitely generated $\mathcal{O}_K$-submodules and their torsors.

Again, analogous to algebraically closed valued fields.
some imaginaries in valued fields

The value group $K^\times/(\mathcal{O}_K \setminus \mathfrak{m}_K)$

$aEb$ if and only if $a/b \in \mathcal{O}_K \setminus \mathfrak{m}_K$ if and only if $v(a) = v(b)$
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The residue field $\mathcal{O}_K/m$

$aEb$ if and only if $a - b \in m$ if and only if $v(a - b) > 0$

The RV sorts $K^\times/(1 + m)$

$aEb$ if and only if $a(1 + m) = b(1 + m)$ if and only if $v(a - b) > v(b)$
Let $L, M, C$ be substructures of $\mathcal{U} \models \text{ACVF}$. Assume that $C$ is maximal and is a substructure of both $L$ and $M$. 

Also assume that $\Gamma_L \cap \Gamma_M = \Gamma_C$ and that $k_L$ is linearly independent from $k_M$ over $k_C$. 

Suppose there is an isomorphism $\sigma : L \to L'$ fixing $C, \Gamma_L, k_L$. Then $\sigma$ can be extended to an isomorphism of the field generated by $L$ and $M$ over $C$, which is the identity on $M$. Equivalently, $tp(M/C) \vdash tp(M/L)$.
Let $L, M, C$ be substructures of $\mathcal{U} \models ACVF$. Assume that $C$ is maximal and is a substructure of both $L$ and $M$. Also assume that $\Gamma_L \cap \Gamma_M = \Gamma_c$ and that $k_L$ is linearly independent from $k_M$ over $k_C$. Suppose there is an isomorphism $\sigma : L \to L'$ fixing $C$, $\Gamma_L$, and $k_L$. Then $\sigma$ can be extended to an isomorphism of the field generated by $L$ and $M$ over $C$, which is the identity on $M$. Equivalently, $\text{tp}(M/C \Gamma_L k_L) \vdash \text{tp}(M/L)$.
H.-Hrushovksi-Macpherson 2008

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Equivalently,

$$tp(M/C\Gamma_L k_L) \models tp(M/L).$$
Theorem (Ealy-H.-Maříková 2016)

The analogous statement with $\mathcal{U} \models \text{RCVF}$.
Define $\sigma : C[L, M] \rightarrow C[L', M]$ by $\sigma(\sum_{i=1}^{n} \ell_i m_i) = \sum_{i=1}^{n} \sigma(\ell_i)m_i$. 

1) Show $\sigma$ preserves the valuation (HHM, also Johnson).

Because $C$ is maximal, WMA that the $m_i$ are a separated basis for the finite-dimensional vector subspace of $M$ that they generate over $C$ such that in addition $v(\sum \ell_i m_i) = \min_{i} \{v(\ell_i) + v(m_i)\}$.

As $\sigma$ fixes $\Gamma_M$ and is an isomorphism on $\Gamma_L$, the result follows.
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Suppose not. Then there is some $a = \sum_{i=1}^{n} \ell_i m_i$ and $m_{n+1} \in M$ with

$$0 < \sigma(a) < m_{n+1} < a$$

and WMA $\{m_i\}$ is a separated basis, and that $a$ is a shortest counterexample.
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and WMA $\{m_i\}$ is a separated basis, and that $a$ is a shortest counterexample. We know that $v(\sigma(a)) = v(a)$, hence also $v(\sigma(m_{n+1})) = v(a)$. 
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For any $x = \sum c_{i}m_{i}$, show that

- it is not the case that $\sigma(a) < x < a$;
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For any $x = \sum c_{i}m_{i}$, show that

- it is not the case that $\sigma(a) < x < a$;
- it is not the case that $v(a - x) > v(a - \sigma(a))$,

hence there is a closest (in the valuation sense) such element $x$ to $a$. 
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hence there is a closest (in the valuation sense) such element $x$ to $a$.
Now construct a pseudo-Cauchy sequence to find a closer such element $x$ to $a$, giving a contradiction.
Theorem (Ealy-Haskell-Maříková 2016)

Assume $L, M, C$ are elementary substructures of $\mathcal{U} \models \text{RCVF}$, with $C$ a common substructure of $L$ and $M$, $C$ maximal, $\Gamma_L \subseteq \Gamma_M$. Assume that $\text{kInt}^L_{C\Gamma_L}$ is algebraically independent from $\text{kInt}^M_{C\Gamma_L}$ over $C\Gamma_L$. 
the $k$-internal sorts

Given parameter set $A$, $kInt_A$ is the collection of sets, definable over $A$, that are \textit{internal} to the residue field.
That is, $E \subseteq \text{dcl}(k \cup a)$, where $a$ is a finite tuple from $A$
Ex: $E = a(1 + m)$. 

the \( k \)-internal sorts

Given parameter set \( A \), \( k\text{Int}_A \) is the collection of sets, definable over \( A \), that are *internal* to the residue field.

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Ex: \( E = a(1 + m) \).

Given fields \( C \subseteq L \), \( S \subseteq \Gamma(L) \)

\[
\text{kInt}^L_{CS} = \text{kInt}_{CS} \cap \text{acl}(Ck_L\{\text{RV}_\gamma(L)\}_{\gamma \in S})
\]
a stronger theorem

**Theorem (Ealy-Haskell-Maříková 2016)**

Assume $L, M, C$ are elementary substructures of $\mathcal{U} \models \text{RCVF}$, with $C$ a common substructure of $L$ and $M$, $C$ maximal, $\Gamma_L \subseteq \Gamma_M$.

Assume that $\text{kInt}^L_{C\Gamma_L}$ is algebraically independent from $\text{kInt}^M_{C\Gamma_L}$ over $C\Gamma_L$.

Let $\sigma : L \to L'$ be an ordered valued field isomorphism fixing $\text{kInt}^M_{C\Gamma_L}$.

Then $\sigma$ extends to an ordered valued field isomorphism from $C(L, M)$ to $C(L'M)$ fixing $M$. 
Define $\sigma$ as before.
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Show that $\sigma$ preserves the valuation: perturb the valuation to $\nu'$ with $\Gamma_{\nu'}(L) \cap \Gamma_{\nu'}(M) = \Gamma_{\nu'}(C)$. Note that $\nu'$ is no longer convex with respect to the ordering, so apply pure valued field version of previous theorem and deduce that $\sigma$ preserves $\nu'$ and hence (by construction) also preserves $\nu$. 
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Show that $\sigma$ preserves the ordering: if not, then there is a change in an order relation on balls defined over $M$, which is a formula in the type of $L$ over $\text{kInt}^M_{C \Gamma_L}$. 
perturbing the valuation

Choose \(a_1, \ldots, a_r\) from \(L\) and \(e_1, \ldots, e_r\) from \(M\) such that \(v(a_i) = v(e_i)\) and \(\{a_i\}\) is a \(\mathbb{Q}\)-basis for \(\Gamma_L\) over \(\Gamma_C\).

Choose \(b_1, \ldots b_s\) from \(L\) such that \(\{\text{res}(b_i)\}\) is a transcendence basis for \(k_L\) over \(k_C\).

The assumption that \(k\text{Int}^L_{C\Gamma_L}\) is algebraically independent from \(k\text{Int}^M_{C\Gamma_L}\) over \(C\Gamma_L\) is equivalent to saying the elements

\[
\text{res}(a_1/e_1), \ldots \text{res}(a_r/e_r), \text{res}(b_1), \ldots, \text{res}(b_s)
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are algebraically independent over \(k_M\).
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Choose \( a_1, \ldots, a_r \) from \( L \) and \( e_1, \ldots, e_r \) from \( M \) such that \( v(a_i) = v(e_i) \) and \( \{a_i\} \) is a \( \mathbb{Q} \)-basis for \( \Gamma_L \) over \( \Gamma_C \).

Choose \( b_1, \ldots, b_s \) from \( L \) such that \( \{\text{res}(b_i)\} \) is a transcendence basis for \( k_L \) over \( k_C \).

The assumption that \( k\text{Int}_{C\Gamma_L}^L \) is algebraically independent from \( k\text{Int}_{C\Gamma_L}^M \) over \( C\Gamma_L \) is equivalent to saying the elements

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are algebraically independent over \( k_M \).

For each \( 0 \leq j \leq r - 1 \) choose a place

\[
p^{(j)} : \text{dcl}(k_M, \text{res}(b_1), \ldots, \text{res}(b_s), \text{res}(a_1/e_1), \ldots, \text{res}(a_{j+1}/e_{j+1})) \to \\
\text{dcl}(k_M, \text{res}(b_1), \ldots, \text{res}(b_s), \text{res}(a_1/e_1), \ldots, \text{res}(a_j/e_j))
\]

Let \( p_{v'} : \text{dcl}(C(L, M)) \to \text{dcl}(k_M, k_L) \) be the composition. Let \( v' \) be a valuation associated to the place \( p_{v'} \).

Then \( \Gamma_{v'}(L) \cap \Gamma_{v'}(M) = \Gamma_{v'}(C) \) and \( k_{v'}(L), k_{v'}(M) \) are linearly disjoint over \( k_{v'}(C) \).
$\sigma$ preserves the ordering

Suppose not. Let $a = \sum_{i=1}^{n} \ell_i m_i > 0$ be a minimal counterexample. As before, WMA that the $m_i$ form a separated basis for the space that they generate over $C$ with respect to $v'$. From the construction of $v'$, in fact, the basis is also separated over $v$ and over $L$. Hence we may assume that $v(\ell_i m_i) = 0$, so $v(m_i) \in \Gamma_L$. 
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Thus $a > 0$ implies that $a + \sum_{i=1}^n \ell_i d_i > 0$ for any $d_i$ with $v(d_i) > v(m_i) = -v(\ell_i)$. In other words, the formula

$$x_1 B^{op}_{v(m_1)}(m_1) + \cdots + x_n B^{op}_{v(m_n)}(m_n) > 0$$

is a formula in the type of $L$ over $k\text{Int}^M_{C\Gamma_L}$. Since we assumed $\sigma$ preserves this type, $\sigma(a) > 0$. 
As in previous theorems, let $C, M$ be substructures of $\mathcal{U} \models \text{RCVF}$, $A = \text{dcl}(Ce)$, where $e$ is a tuple of imaginaries in $\mathcal{G}$.

- Suppose $\Gamma_A \cap \Gamma_M = \Gamma_C$ and $k_A$ and $k_M$ are linearly independent over $k_C$. Then
  $$\text{tp}(A/Ck_M\Gamma_M) \vdash \text{tp}(A/M).$$

- Suppose $k\text{Int}^M_{C\Gamma_A}$ is independent from $k\text{Int}^A_{C\Gamma_A}$. Then
  $$\text{tp}(A/CT_Ak\text{Int}^M_{C\Gamma_L}) \vdash \text{tp}(A/M).$$
As in previous theorems, let $C, M$ be substructures of $\mathcal{U} \models \text{RCVF}$, $A = \text{dcl}(Ce)$, where $e$ is a tuple of imaginaries in $\mathcal{G}$.

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- Suppose $k\text{Int}^M_{C\Gamma_A}$ is independent from $k\text{Int}^A_{C\Gamma_A}$. Then
  \[ \text{tp}(A/CT_A k\text{Int}^M_{C\Gamma_L}) \vdash \text{tp}(A/M). \]

Proof: Find a resolution of $A$ in the field sort with same value group and residue field. Then apply previous theorems.
further directions

Extend to a general $T$-convex theory.

How do functions behave on the interaction of $L$ and $M$?

Resolutions still exist for the geometric sorts (provided the underlying o-minimal theory is power bounded). Are new sorts required to eliminate imaginaries?