SUMMER SCHOOL GALOIS THEORY AND NUMBER THEORY COURSE IRREDUCIBILITY AND RATIONAL POINTS

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1. Short summary

This course will focus on the geometric aspects of Hilbert's irreducibility theorem and irreducible specializations, namely thin sets of rational points in the sense of Serre and varieties of Hilbert type. It will discuss both some of the basics like the Lang-Weil estimates for rational points over finite fields and weak approximation, interesting applications to number theory and arithmetic geometry, like constructing elliptic curves of high rank, and also recent results on algebraic groups and homogeneous spaces of Hilbert type.

2. Content of the lectures

2.1. **Lecture 1: Basic notions.** Keywords: statement of Hilbert's irreducibility theorem (HIT) and applications, several equivalent definitions of a Hilbertian field, definition of a variety of Hilbert type (HT) in different languages, some permanence principles for Hilbertian fields

2.2. **Lecture 2: Rational points.** Keywords: heights, Mordell conjecture, proof of HIT counting points of bounded height, Riemann Hypothesis for curves and the Lang-Weil estimates, weak weak approximation, proof of HIT counting points over finite fields

2.3. **Lecture 3: Specializing elliptic curves.** Keywords: some basics on elliptic curves, results on finite and infinite rank, question of large rank, proof of Neron's specialization theorem via HIT, construction of elliptic curves of rank at least 9 using the generic cubic

2.4. **Lecture 4: Varieties of Hilbert type.** Keywords: more on varieties of HT, varieties of HT under finite base change, products of varieties of HT with application to algebraic groups ([10]), homogeneous spaces of HT ([11]), specialization of covers of certain varieties that are not of HT ([12])

3. Literature

3.1. **Prerequisites.** The participants should be familiar with basic Galois theory and algebraic geometry, roughly in the scope of

[1] S. Lang, *Algebra*, 3rd edition, Springer 2002. Chapters I-VIII. *or*

[2] J. S. Milne, *Fields and Galois Theory.* Lecture notes available at www.jmilne.org/math/.

and

[3] D. Mumford, *The Red Book of Varieties and Schemes*, Springer 1988. *or*

[4] R. Hartshorne, *Algebraic Geometry*, Springer 1977. Chapters I, II and IV.

Moreover, for lectures 3 and 4, a basic acquaintance with elliptic curves and linear algebraic groups will be useful, see for example

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[5] J. Silverman, *The Arithmetic of Elliptic Curves*, 2nd edition, Springer 2010. Chapters III, VII and VIII.
[6] J. E. Humphreys, *Linear Algebraic Groups*, Springer 1981. Chapter II.

3.2. **Classical material.** The first three lectures will cover mostly classical material that can be found for example in

[7] J.-P. Serre, *Topics in Galois Theory*, Jones and Bartlett 1992.
[8] J.-P. Serre, *Lectures on the Mordell-Weil Theorem*, 3rd edition, Springer 1997.
[9] M. D. Fried and M. Jarden, *Field Arithmetic*, 3rd edition, Springer 2008.

3.3. **Recent research.** The fourth lecture covers recent research, some of it published in

[10] L. Bary-Soroker, A. Fehm and S. Petersen, On varieties of Hilbert type, *Annales de l'Institut Fourier* 64(5), 2014.
[11] M. Borovoi, On homogeneous spaces of Hilbert type, *Int. J. of Number Theory*, 2015.
[12] U. Zannier, Hilbert Irreducibility above algebraic groups, *Duke Math. J.* 153, 2010.

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