1 Useful English/German Vocabulary

simple group -einfache Gruppe
normal series - Normalreihe
composition series - Kompositionsreihe
refinement - Verfeinerung

2 Isomorphism theorems continued

Theorem 2.1 (Lattice isomorphism theorem)**.** *Let G be a group and let N be a normal subgroup of G. If A is a subgroup of G containing N, let $\overline{A} := A/N$.Let $\pi : G \to G/N$ be the canonical projection.*

The map $A \mapsto \pi(A) = \overline{A}$ is a bijection between the set of subgroups of G containing N and the set of subgroups of G/N.

Moreover, if $A, B \leq G$ with $N \leq A$ and $N \leq B$ then:

1. *$A \leq B$ if and only if $\overline{A} \leq \overline{B}$; and in this case $[B : A] = [\overline{B} : \overline{A}]$*
2. *$A \lhd B$ is and only if $\overline{A} \lhd \overline{B}$; and in this case $B/A \cong \overline{B}/\overline{A}$*
3. *$\overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B}\rangle$*
4. *$\overline{A \cap B} = \overline{A} \cap \overline{B}$*

Proof. UB9 $\square$

Theorem 2.2 (Butterfly Lemma /Zassenhaus Lemma)**.** *Let $a \lhd A$ and $b \lhd B$ be subgroups of a group G. Then*

$$a(A \cap b) \text{ is a normal in } a(A \cap B),$$

$$b(B \cap a) \text{ is normal in } b(B \cap A),$$

$$(A \cap b)(B \cap a) \text{ is normal in } (A \cap B)$$

and

$$\frac{a(A\cap B)}{a(A\cap b)} \cong \frac{(A\cap B)}{(A\cap b)(B\cap a)} \cong \frac{b(B\cap A)}{b(B\cap a)}.$$

Proof. Note first that since $A \le N_G(a)$ and $B \le N_G(b)$, we have that

$$A\cap b \le A\cap B \le N_G(a)$$

and

$$B\cap a \le A\cap B \le N_G(b).$$

Thus $a(A\cap b), a(A\cap B), b(B\cap a)$ and $b(B\cap A)$ are subgroups of G (see lecture 17 corollary 1.10).

We first show that

$$(A\cap b)(B\cap a) \text{ is normal in } (A\cap B).$$

First note that $A\cap b$ and $B\cap a$ are normal in $A\cap B$; if $g\in A\cap B$ and $c\in A\cap b$ then $gcg^{-1}\in b$ since $b \lhd B$ and $gcg^{-1}\in A$ since $g,c\in A$. Thus $(A\cap b)(B\cap a)$ is a subgroup of $A\cap B$. In fact it is a normal subgroup since if $c_1\in A\cap b$, $c_2\in B\cap a$ and $g\in A\cap B$ then $gc_1c_2g^{-1} = gc_1g^{-1}gc_2g^{-1}\in (A\cap b)(B\cap a)$.
If $x\in a(A\cap B)$ then $x=\alpha\gamma$ where $\alpha\in a$ and $\gamma\in A\cap B$. Define

$$f : a(A\cap B) \to \frac{A\cap B}{(A\cap b)(B\cap a)}$$

by

$$x \mapsto \gamma(A\cap b)(B\cap a).$$

The map f is well-defined for if $\alpha\gamma = \alpha'\gamma'$ with $\alpha,\alpha'\in a$ and $\gamma,\gamma'\in A\cap B$ then $\gamma'\gamma^{-1} = (\alpha')^{-1}\alpha \in a\cap A\cap B = a\cap B \le (A\cap b)(B\cap a)$; i.e.

$$\gamma'(A\cap b)(B\cap a) = \gamma(A\cap b)(B\cap a)$$

The map is a homomorphism: if $\alpha, \alpha' \in a$ and $\gamma, \gamma' \in A \cap B$ then $\alpha, \gamma\alpha'\gamma^{-1} \in a$ since $a \lhd A$. So

$$f(\alpha\gamma\alpha'\gamma') = f((\alpha\gamma\alpha'\gamma^{-1})\gamma\gamma') = \gamma\gamma'(A \cap b)(B \cap a)$$

and since $(A \cap b)(B \cap a)$ is normal in $A \cap B$

$$f(\alpha\gamma)f(\alpha'\gamma') = \gamma(A \cap b)(B \cap a)\gamma'(A \cap b)(B \cap a) = \gamma\gamma'(A \cap b)(B \cap a).$$

The map f is surjective by definition.

It remains to find the kernel: if $\alpha \in a$ and $\gamma \in A \cap B$ are such that $f(\alpha\gamma) = 1(A \cap b)(B \cap a)$ then $\gamma \in (A \cap b)(B \cap a) = (B \cap a)(A \cap b)$. Take $x \in (B \cap a)$ and $y \in (A \cap b)$ such that $\gamma = xy$. Then $\alpha\gamma = (\alpha x)\gamma \in a(A \cap b)$.
Conversely, if $\alpha \in a$ and $\gamma \in A \cap B$ with $\alpha\gamma \in a(A \cap b)$ then there exist $t \in a$ and $s \in A \cap b$ such that $\alpha\gamma = ts$. Now $\alpha^{-1}t \in a$ and since $\gamma, s \in B$, $\alpha^{-1}t = \gamma s^{-1} \in B$. Thus $\alpha^{-1}ts = \gamma \in (A \cap b)(B \cap a)$. So $\alpha\gamma \in \ker f$.

So by the first isomorphism theorem, $a(A \cap b)$ is normal in $a(A \cap B)$ and

$$\frac{a(A \cap B)}{a(A \cap b)} \cong \frac{(A \cap B)}{(A \cap b)(B \cap a)}.$$

Exchanging the roles of A and B respectively a and b, we get that $b(B \cap a)$ is normal in $b(B \cap A)$ and

$$\frac{b(A \cap B)}{b(B \cap a)} \cong \frac{(A \cap B)}{(A \cap b)(B \cap a)}.$$

$\square$

3 Jordan-Hölder and Simple and Solvable groups

Definition 3.1. *A group G is simple if $|G| > 1$ and the only normal subgroups of G are 1 and G.*

Remark: A non-trivial abelian group G is simple if and only if its only subgroups are 1 and G (recall: all subgroups of abelian groups are normal).

Thus, if G is simple and abelian then it is generated by every non-identity element of G. So G is cyclic. Recall that if G is infinite and x generates G then x^2 does not generate G (lecture 15 Proposition 5(1)). Thus G is finite. Moreover, if $p \in \mathbb{N}$, a prime, divides $|x|$ then $|x^p| < |x|$ (see lecture 15 Proposition 4(3)) and therefore $x^p = 1$. Thus $|G| = p$. Thus an abelian group is simple if and only if it is finite and of prime order.

Definition 3.2. *Let G be a group. A sequence of subgroups*

$$1 = G_0 \leq G_1 \leq ... \leq G_s = G$$

*is called a **normal series** if G_i is normal in G_{i+1}; we call the quotient groups G_{i+1}/G_i **factor groups** of the series.*

*A normal series is called a **composition series** if each of the quotient groups G_{i+1}/G_i are simple; in this case we call the quotient groups **composition factors** of G (we will see later that the factor groups really do only depend on G).*

A normal series

$$1 = G_0 \lhd G_1 \lhd ... \lhd G_s = G$$

*is a **refinement** of a normal series*

$$1 = H_0 \lhd H_1 \lhd ... \lhd H_r = G$$

if $H_0, ..., H_r$ is a subsequence of $G_0, ..., G_s$.

Example: Since A_4 is index 2 in S_4, A_4 is normal in S_4. You will show on the exercise sheet that the subgroup

$$V := \{(12)(34), (13)(24), (14)(23), e\}$$

is normal in A_4. So

$$\{1\} \lhd V \lhd A_4 \lhd S_4$$

is a normal series for S_4. Its factor groups are $\mathbb{Z}_2$ and $\mathbb{Z}_3$. So it is in fact a composition series.

Definition 3.3. *Two normal series are said to be* ***equivalent*** *if there is a bijection between their factor groups such that corresponding factor groups are isomorphic.*

Example:
Consider the following two composition series of $\mathbb{Z}_30$:

$$\mathbb{Z}_{30} \geq \langle 5 \rangle \geq \langle 10 \rangle \geq \{0\}$$
$$\mathbb{Z}_{30} \geq \langle 3 \rangle \geq \langle 6 \rangle \geq \{0\}$$

The composition factors of the first series are $\mathbb{Z}_{30}/\langle 5 \rangle \cong \mathbb{Z}_5$, $\langle 5 \rangle / \langle 10 \rangle \cong \mathbb{Z}_2$ and $\langle 10 \rangle / \{0\} \cong \mathbb{Z}_3$.
The composition factors of the second series are $\mathbb{Z}_{30}/\langle 3 \rangle \cong \mathbb{Z}_3$, $\langle 3 \rangle / \langle 6 \rangle \cong \mathbb{Z}_2$ and $\langle 6 \rangle / \{0\} \cong \mathbb{Z}_5$.
So the above composition series are equivalent.

Theorem 3.4 (Schreier Refinement Theorem)**.** *Any two normal series of a group G have equivalent refinements.*

Proof. Let
$$1 = G_0 \lhd G_1 \lhd ... \lhd G_s = G$$
and
$$1 = H_0 \lhd H_1 \lhd ... \lhd H_r = G$$
be normal series.
Let $G_{i,j} := G_i(G_{i+1} \cap H_j)$ for $0 \leq j \leq r$. So
$$G_{i,0} = G_i\{1\} = G_i \text{ and } G_{i,r} = G_i(G_{i+1} \cap G) = G_{i+1}.$$
Since $G_i \lhd G_{i+1}$ and $H_j \lhd H_{j+1}$, by Zassenhaus (with $a = G_i, A = G_{i+1}, b = H_j$ and $B = H_{j+1}$),
$$G_{i,j} = G_i(G_{i+1} \cap H_j) \lhd G_i(G_{i+1} \cap H_{j+1}) = G_{i,j+1}.$$
Thus the following series is a refinement of the first normal series:
$$\{1\} = G_{0,0} \lhd G_{0,1} \lhd ... \lhd G_{0,r} = G_{1,0} \lhd G_{1,1} \lhd ... \lhd G_{s-1,r} = G_s = G$$
Let $H_{i,j} := H_i(H_{i+1} \cap G_j)$ for $0 \leq j \leq s$.
Exactly as above,
$$\{1\} = H_{0,0} \lhd H_{0,1} \lhd ... \lhd H_{0,s} = H_{1,0} \lhd H_{1,1} \lhd ... \lhd H_{r-1,s} = H_r = G$$
is a refinement of the second normal series.
It remains now just to note that by the Zassenhaus lemma (with $a = G_i, A = G_{i+1}, b = H_j$ and $B = H_{j+1}$)
$$G_i(G_{i+1} \cap H_{j+1})/G_i(G_{i+1} \cap H_j) \cong H_j(H_{j+1} \cap G_{i+1})/H_j(H_{j+1} \cap G_i);$$
that is
$$G_{i,j+1}/G_{i,j} \cong H_{j,i+1}/H_{j,i}.$$
□

Theorem 3.5 (Jordan-Hölder Theorem). *Let G be a finite group with $G \neq \{1\}$. Then*

1. *G has a composition series and*

2. *all composition series of G are equivalent.*

Proof. (1) Suppose G is not simple. If N is a maximal normal subgroup of G then, by the correspondence theorem, G/N is simple. If G is finite G has a maximal normal subgroup. Thus, by induction on $|G|$, every finite group has a composition series.
(2)Composition series have no refinements by the correspondence theorem; that is, if $G_{i+1} \rhd N \rhd G_i$ then $N/G_i \lhd G_{i+1}/G_i$ and if G_{i+1}/G_i is simple then $N = G_{i+1}$ or $N = G_i$. By the Schreier Refinement Theorem, every two normal series have equivalent refinements. Thus every two composition series of G are equivalent.

□