DECIDABILITY FOR THEORIES OF MODULES OVER VALUATION DOMAINS

LORNA GREGORY

Abstract. Extending work of Puninski, Puninskaya and Toffalori in [PPT07], we show that if V is an effectively given valuation domain then the theory of all V-modules is decidable if and only if there exists an algorithm which, given $a, b \in V$, answers whether $a \in \operatorname{rad}(bV)$. This was conjectured in [PPT07] for valuation domains with dense value group where it was proved for valuation domains with dense archimedean value group. The only ingredient missing from [PPT07] to extend the result to valuation domains with dense value group or infinite residue field is an algorithm which decides inclusion for finite unions of Ziegler open sets. We go on to give an example of a valuation domain with infinite Krull dimension which has decidable theory of modules with respect to one effective presentation and undecidable theory of modules with respect to another. We show that for this to occur infinite Krull dimension is necessary.

1. Introduction

Throughout this paper all rings have 1 and all modules are unital.

In [PPT07] Puninski, Puninskaya and Toffalori conjectured that the theory of modules of an effectively given valuation domain V with dense value group is decidable if and only if there is an algorithm which, given $a, b \in V$, answers whether there exists an $n \in \mathbb{N}$ such that $a^n \in bV$, that is answers whether $a \in \operatorname{rad}(bV)$. We show that this conjecture is unconditionally true, i.e. without any restriction on the value group of V (theorem 7.1). This is the main result of our paper.

For valuation domains with non-archimedean dense value groups or infinite residue fields the only ingredient missing from the proof in [PPT07] is an algorithm which decides whether inclusions hold for finite unions of Ziegler basic open sets. We explicitly describe such an algorithm in section 4.

On the other hand, when V has non-dense value group and finite residue field, the number of indecomposable pure-injective modules with finite but not equal to 1 Baur-Monk invariant sentences increases significantly. The proof in [PPT07] for a valuation domain with dense value group and finite residue field uses the fact that for each Baur-Monk invariant $|\varphi/\psi|$ there are only finitely many indecomposable pure-injective modules (up to

Date: April 18, 2013.
Author supported by an EPSRC Doctoral training Grant.

isomorphism) with $|\varphi/\psi|$ finite and non-zero. For valuation domains with non-dense value group this is no longer true. Luckily, this problem is still not too combinatorially difficult (see section 6).

In section 5 we discuss duality for the Ziegler spectrum of a valuation domain. Prest [Pre88, Chapter 8] defined the dual $D\varphi$ of a pp-formula φ. This map induces a lattice anti-isomorphism between the lattice of left and right pp-formulae of a ring such that $D^2\varphi = \varphi$. Herzog in [Her93] extended this notion to a lattice isomorphism from the lattice of open sets of the left Ziegler spectrum of a ring to the lattice of open sets of the right Ziegler spectrum of a ring. It is not known in general whether this map is induced by a homeomorphism. If there is such a homeomorphism, we call it a duality homeomorphism. Note that for a commutative ring R this will in general be a non-trivial automorphism of Zg_R. We give an explicit duality homeomorphism for the Ziegler spectrum of a valuation domain. We use Herzog's results to show that if $D : \mathrm{Zg}_R \to_R \mathrm{Zg}$ is such a duality homeomorphism then for all pairs of pp-formulae φ/ψ, $|\varphi(N)/\psi(N)| = |D\psi(DN)/D\varphi(DN)|$. This is used in section 6 to reduce the number of indecomposable pure-injective modules N for which we need to explicitly calculate $|\varphi(N)/\psi(N)|$.

In the final section we give an example of a valuation domain V with infinite Krull dimension which has undecidable theory of modules with respect to one effective presentation and decidable theory of modules with respect to another. We do this by constructing a recursive totally ordered abelian group in which the relation

$$\alpha \gg \beta \text{ if and only if } \forall n\ |\alpha| \geq n|\beta|$$

is not recursive. We note that if V is an effectively given valuation domain with finite Krull dimension then its theory of modules is decidable.

Throughout this paper we will use a naive notion of cardinality. That is, if X, Y are sets then $|X| = |Y|$ means, if either X or Y is of finite cardinality then their cardinality is equal.

2. Background

For general background on model theory of modules see [Pre88].

Let R be a ring. Let $\mathcal{L}_R := (0, +, (r)_{r\in R})$ be the language of (right) R-modules. A (right) **pp-n-formula** is a formula of the form

$$\exists \overline{y}\ (\overline{y}, \overline{x})A = 0$$

where l, n, m are natural numbers, A is an $(l+n)\times m$ matrix with entries from R, and $\overline{y}$ is an l-tuple of variables and $\overline{x}$ is an n-tuple of variables.

The solution set $\varphi(M)$ of a pp-n-formula φ in an R-module M is a subgroup of M^n. If we cosmetically weaken our definition of a pp-n-formula to include all formulae (in n-variable) in the language of (right) R-modules, $\mathcal{L}_R$, which are equivalent over the theory of R-modules, T_R, to a pp-n-formula then the T_R-equivalence classes of pp-n-formulae become a lattice under implication with the join of two formulae φ, ψ given by

$$(\varphi + \psi)(\overline{x}) := \exists \overline{y}, \overline{z}(\overline{x} = \overline{y} + \overline{z} \wedge \varphi(\overline{y}) \wedge \psi(\overline{z}))$$

and the meet given by $\varphi \wedge \psi$.
Let φ, ψ be pp-1-formulae and $n \in \mathbb{N}$. There is a sentence, $|\varphi/\psi| \geq n$ in the language of (right) R-modules, which expresses in every R-module M that the quotient $\varphi(M)/\varphi \wedge \psi(M)$ has at least n elements. Such sentences will be referred to as **invariant sentences**. We will write $|\varphi/\psi| = n$ for the sentence $|\varphi/\psi| \geq n \wedge \neg(|\varphi/\psi| \geq n+1)$. For an R-module M, we will write $|\varphi(M)/\psi(M)| \geq n$ instead of $M \models |\varphi/\psi| \geq n$. We will also write $|\varphi(M)/\psi(M)| = \infty$ to mean that $|\varphi(M)/\psi(M)| \geq n$ for all $n \in \mathbb{N}$. This final statement is of course not necessarily expressed by a first order sentence in the language of R-modules.

Theorem 2.1 (Baur-Monk Theorem)**.** [Pre88] *Let R be a ring. Every sentence $\chi \in \mathcal{L}_R$ is equivalent to a boolean combination of invariant sentences.*

The above theorem together with the fact that the theory of modules of a recursively given ring R is recursively axiomatised means that, in order to show that the theory of R-modules is recursive, it is enough to show that there is an algorithm which given a boolean combination of invariant sentences χ answers whether there is an R-module in which χ is true.
A **pp-type** is a set of pp-formulae. If M is an R-module and $a \in M$ then the set of pp-formulae satisfied by a in M is called the pp-type of a. We say a pp-type is **complete** if it is the pp-type of an element of a module or equivalently if it is closed under implications (with respect to the theory of all R-modules) and conjunctions.
A **pure-embedding** between two modules is an embedding which preserves and reflects the solution sets of pp-formulae. We say a module N is **pure-injective** if for every pure-embedding $g : N \to M$, the image of N in M is a direct summand of M; equivalently, it is injective with respect to pure-embeddings. For every R-module M, there exists a pure-injective module $PE(M)$ such that M is a pure-submodule of $PE(M)$ and for all pure-injectives M' and all pure-embeddings $f : M \hookrightarrow M'$ there is an extension of f embedding $PE(M)$ purely into M'. Moreover, $PE(M)$ is unique up to isomorphism over M. We call $PE(M)$ the **pure-injective hull** of M. All modules are elementary equivalent to their pure-injective hull [Pre88, Theorem 4.21]. Every module is elementary equivalent to a direct sum of indecomposable pure-injective modules [Pre88, Corollary 4.36]. Combining this fact with the Baur-Monk theorem and that the solution sets of pp-formulae commute with direct sums we get that any sentence χ in the language of R-modules is true in some module if and only if it is true in some finite direct sum of indecomposable pure-injective modules.
The (right) **Ziegler spectrum** of a ring R, denoted Zg_R, is a topological space whose points are isomorphism classes of indecomposable pure-injective

(right) modules and which has a basis of open sets given by:

$$(\varphi/\psi) = \{M \mid \varphi(M) \supsetneq \psi(M) \cap \varphi(M)\}$$

where φ, ψ range over (right) pp-1-formulae. The left Ziegler spectrum ${}_R\mathrm{Zg}$ of a ring is defined analogously.
A commutative integral domain V is called a **valuation domain** if the lattice of ideals of V is a chain. This implies that a subset I of V is an ideal of V if and only if for all $r \in V$ and $a \in I$, $ar \in I$. Note that all finitely generated ideals are principal. Throughout we will assume that V is not a field (the theory of K-vector spaces for a recursively given field K is decidable). Unless otherwise stated, V will always denote a valuation domain and $\mathfrak{m}$ will denote its unique maximal ideal. The field $V/\mathfrak{m}$ is called the **residue field** of V.
The reader should note that the value group of V is dense if and only if the maximal ideal of V is not principal.

3. Decidability and modules

Let R be a ring. The theory of R-modules, T_R, is **decidable** if there is an algorithm which, given a sentence χ in $\mathcal{L}_R$, answers whether $\chi_R \in T_R$ or not. Since algorithms and their formalisms (Turing machines, partial recursive functions etc) are usually expected to take natural numbers as input and output natural numbers, in order to talk (formally) about decidability of T_R we must have some way of converting ring elements into natural numbers. So we assume that our algorithms are implemented with respect to a functions $\pi : R \to \mathbb{N}$. Of course this means that R must be countable.
We now discuss conditions we must impose on π in order to have any hope of T_R being decidable. For more details see [Pre88, Chapter 17]. Firstly, for all $r_1, r_2 \in R$, $r_1 = r_2$ if and only if $T_R \models \forall x(xr_1 = xr_2)$. So we must be able to decide equality of elements and therefore, may as well assume that π is a bijection. For similar reasons, we must assume that given $a, b \in R$ we can compute $a + b$ and $a \cdot b$. Thus we assume that $+$ and $\cdot$ induce recursive functions on $\mathbb{N}$ via π. Finally, for all $r \in R$, r is a unit in R if and only if $T_R \models \forall x(xr = 0 \to x = 0)$. Thus we must be able to compute whether an element is a unit or not. Thus, we assume that the image of the units of R under π is a recursive subset of $\mathbb{N}$.
Note that for a valuation domain V the set of units of V is exactly the complement of $\mathfrak{m}$. Thus we get the following definition (which is obviously equivalent to the definition given in [PPT07]).

Definition 3.1. *An effectively given valuation domain is a (countable) valuation domain V together with a bijection $\pi : \mathbb{N} \to V$ such that the pre-image of the maximal ideal of V under π is a recursive set and addition and multiplication induce recursive functions on $\mathbb{N}$ via π. We call the map π an effective presentation of V.*

Note that this implies that there is an algorithm (with respect to π) which given $a, b \in V$ either computes c such that $a = bc$ or decides that such a c does not exist [PPT07, Remark 3.2] and that there is an algorithm (with respect to π) which given a unit $a \in V$ outputs a^{-1} [PPT07, Remark 3.1]. We will work with an informal notion of algorithm, in the knowledge that, given the time and inclination, we could rewrite all proofs in terms of recursive functions.

The following lemma is the easy direction of our main theorem and occurs as lemma 9.1 of [PPT07] with the restriction that R is a valuation domain. This restriction is unnecessary, so we include a proof.

Lemma 3.2. *Let R be a countable commutative ring together with a bijection $\pi : R \to \mathbb{N}$ with decidable theory of modules (with respect to π). Then there is an algorithm which, given $a, b \in V$ decides whether $a \in \mathrm{rad}(bR)$.*

Proof. Claim:

$$T_R \models \exists x(x \neq 0 \wedge xb = 0) \to \exists y(y \neq 0 \wedge xa = 0)$$

if and only if

$$a \in \mathrm{rad}(bR).$$

First suppose that $a \in \mathrm{rad}(bR)$. There exists an $n \in \mathbb{N}$ such that $a^n \in bV$. Suppose N is an R-module and $x \in N$ such that $x \neq 0$ and $xb = 0$. Then $xa^n = 0$. Take m least such that $xa^m = 0$, then $(xa^{m-1})a = 0$ and $xa^{m-1} \neq 0$.

Now suppose that

$$T_R \models \exists x(x \neq 0 \wedge xb = 0) \to \exists y(y \neq 0 \wedge xa = 0).$$

Let $\mathfrak{p} \lhd R$ be a prime ideal such that $b \in \mathfrak{p}$. Then $1 + \mathfrak{p} \in R/\mathfrak{p}$ is annihilated by b and non-zero. Hence there exists $y \in V \backslash \mathfrak{p}$ such that $ya \in \mathfrak{p}$. Therefore $a \in \mathfrak{p}$. Thus $a \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ containing b. Hence $a \in \mathrm{rad}(bV)$ since $\mathrm{rad}(bV)$ is the intersection of all prime ideals containing b. $\square$

4. Algorithms and the Ziegler spectrum

In this section we show that if V is an effectively given valuation domain with an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$ then there exists an algorithm which given $n \in \mathbb{N}$, a pp-pair φ/ψ and n pp-pairs ϑ_i/ξ_i, answers whether

$$\left(\frac{\varphi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right).$$

For any $n \in \mathbb{N}$, pp-1-formulae φ, ψ and pp-1-formulae ϑ_i, ξ_i for $1 \leq i \leq n$, $T_R \models \neg\left(\left|\frac{\varphi}{\psi}\right| > 1 \wedge \bigwedge_{i=1}^{n} \left|\frac{\vartheta_i}{\xi_i}\right| = 1\right)$ is equivalent to $\left(\frac{\varphi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right)$. Hence, decidability of T_R implies we can effectively decide whether $\left(\frac{\varphi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right)$.

We start by recalling some facts about Ziegler spectra of valuation domains.

Lemma 4.1. [PPT07, Lemma 3.3] *Let V be an effectively given valuation domain. There exists an algorithm which, given a pp-1-formula φ, produces a formula of the form $\sum_{i=1}^{n}(xa_i = 0 \wedge b_i|x)$ equivalent to φ.*

Lemma 4.2. [PPT07, Corollary 3.4] *Let V be an effectively given valuation domain. There exists an algorithm which, given a pp-1-formula φ, produces a formula of the form $\bigwedge_{i=1}^{n}(xa_i = 0 + b_i|x)$ equivalent to φ.*

Lemma 4.3. [Pun99][PPT07, Corollary 4.3] *The collection of open sets*

$$\mathcal{W}_{a,b,g,h} := \left(\frac{(xag = 0) \wedge (b|x)}{(xa = 0) + (bh|x)}\right)$$

for non-zero $a, b \in V$ and $g, h \in \mathfrak{m}$ form a basis for Zg_V.
Moreover, if V is effectively given then there exists an algorithm which, given φ/ψ a pp-pair, returns the symbol $\emptyset$ if $\left(\frac{\varphi}{\psi}\right)$ is empty and otherwise returns $n \in \mathbb{N}$, $a_i, b_i \in V\backslash\{0\}$ and $g_i, h_i \in \mathfrak{m}$ such that

$$\left(\frac{\varphi}{\psi}\right) = \bigcup_{i=1}^{n} \mathcal{W}_{a_i,b_i,g_i,h_i}.$$

A **pair** over a valuation domain is a pair of proper ideals $\langle I, J\rangle$. To each pair over V, we can associate a pp-type

$$p\langle I, J\rangle = \{xb = 0 \mid b \in I\} \cup \{a|x \mid a \notin J\}.$$

Recall that every complete pp-type is realised in a (unique up to isomorphism) minimal pure-injective module, denoted $N(p)$. We say a complete pp-type is **indecomposable** if $N(p)$ is indecomposable. We say that $\langle I, J\rangle \sim \langle K, L\rangle$ if there exists non-zero $a, b \in R$ such that at least one of the following holds:

(1) $Ia = K$ and $J = La$ or
(2) $I = Ka$ and $Ja = L$.

Lemma 4.4. [Pun99] *Every pp-type $p\langle I, J\rangle$ has a unique extension to a complete indecomposable pp-type and every indecomposable pp-type arises in this way. We write $N(I, J)$ for the unique (up to isomorphism) indecomposable pure-injective realising $\langle I, J\rangle$. Moreover, for two pairs $\langle I, J\rangle$ and $\langle K, L\rangle$ over V, $N(I, J)$ is isomorphic to $N(K, L)$ if and only if $\langle I, J\rangle \sim \langle K, L\rangle$.*

From now on we will write (I, J) for both the equivalence class of $\langle I, J\rangle$ and the corresponding isomorphism class of indecomposable pure-injective modules. We will refer to (I, J) as a point or a point in Zg_V. So, $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if there exists a pair $\langle K, L\rangle$ such that $\langle K, L\rangle \sim \langle I, J\rangle$ and $a \notin K, b \notin L, ag \in K$ and $bh \in L$. We will write $N(I, J)$ only when we want to emphasise that points in the Ziegler spectrum are modules.
Let R be a commutative ring, $I \lhd R$ and $a \notin I$. Define

$$(I : a) := \{r \in V \mid ar \in I\}.$$

It is easy to see that for $I, J \lhd V$ proper ideals of a valuation domain and $a \notin J$, we have that:

$$Ia = J \text{ if and only if } I = (J : a). \tag{1}$$

We can now reformulate $\sim$ in terms of ideal quotients (this follows directly from (1)):
Let $\langle I, J\rangle$ and $\langle K, L\rangle$ be pairs over V. We have that $\langle I, J\rangle \sim \langle K, L\rangle$ if and only if at least one of the following holds:

(i) there exists $a \notin K$ such that $I = (K : a)$ and $J = La$
(ii) there exists $a \notin J$ such that $I = Ka$ and $J = (L : a)$.

Using the above observation we can now reformulate what it means for a point in Zg_V to be contained in a basic open set:

Lemma 4.5. *Let $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. A point $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if one of the following holds:*

(i) there exists $r \notin I$ such that $a \notin (I : r), b \notin Jr, ag \in (I : r)$ and $bh \in Jr$;
(ii) there exists $s \notin J$ such that $a \notin Is, b \notin (J : s), ag \in Is$ and $bh \in (J : s)$.

The lemma below shows that in fact the open sets of the form $\mathcal{W}_{1,\lambda,g,h}$ are a basis for Zg_V.

Lemma 4.6. *Let $a, b \in V\backslash\{0\}$, $g, h \in \mathfrak{m}$ and (I, J) a point in Zg_V. The following statements are equivalent:*

(i) $(I, J) \in \mathcal{W}_{a,b,g,h}$,
(ii) $(I, J) \in \mathcal{W}_{1,ab,g,h}$,
(iii) $(I, J) \in \mathcal{W}_{ab,1,g,h}$.

For a proper ideal $I \lhd V$, let $I^\#$ be the prime ideal $\cup_{a\notin I}(I : a)$. Note that for all proper ideals $I, J \lhd V$, $a \in V\backslash\{0\}$ and $b \notin I$, we have $(Ia)^\# = I^\#$, $(I : b)^\# = I^\#$ and $(IJ)^\# = I^\# \cap J^\#$ (see [FS01, Lemma 4.6] for a proof). If $\mathfrak{p}$ is a prime ideal then $\mathfrak{p}^\# = \mathfrak{p}$.
We will use the following simple remark without comment.

Remark 4.7. *Let $I \lhd V$ be a non-zero proper ideal of V. The following are equivalent:*

$$(a)\ r \notin I \qquad (b)\ r\mathfrak{m} \supseteq I \qquad (c)\ rI^\# \supseteq I.$$

Theorem 4.8. [Gre13, Theorem 4.3] *Let $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Let (I, J) be a point in Zg_V. The following are equivalent:*

(i) $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$
(ii) $\lambda gh \in IJ$, $g \in I^\#$, $h \in J^\#$ and $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

The condition $g \in I^\#$ simply means that there is some non-zero element $a \in N(I, J)$ such that $ag = 0$. Similarly the condition $h \in J^\#$ means that there is some $a \in N(I, J)$ which is not divisible by h. The condition $(I, J) \in \mathcal{W}_{1,\lambda,0,0} = \mathcal{W}_{\lambda,1,0,0}$ means exactly that $\lambda \notin \mathrm{ann}_V N(I, J)$.
Note that $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ always implies $\lambda \notin IJ$ [Gre13, Lemma 4.2] but the converse is not always true. This motivates the following definition.

Definition 4.9. *We say a point (I, J) in Zg_V is* ***normal*** *if for all $\lambda \notin IJ$, $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Otherwise we say (I, J) is* ***abnormal****.*

In terms of modules, $N(I, J)$ is abnormal if and only if $\mathrm{ann}_V N(I, J) \supsetneq IJ$.

Lemma 4.10. [Gre13, Lemma 4.5] *Let (I, J) be a point in Zg_V such that $I^\# \neq J^\#$. Then for all $\lambda \in V\backslash\{0\}$, $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin IJ$. That is, if $I^\# \neq J^\#$ then (I, J) is normal.*

Lemma 4.11. [Gre13, Lemma 4.9] *Let (I, J) be an abnormal point with $I^\# = J^\# = \mathfrak{p}$. Then $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda\mathfrak{p} \supsetneq IJ$, $\lambda gh \in IJ$, $g \in I^\#$ and $h \in J^\#$.*

Thus, up to topological indistinguishability, a point (I, J) is completely determined by IJ, $I^\#$, $J^\#$ and whether or not it is abnormal.
The following proposition determines all abnormal points up to topological indistinguishability.

Proposition 4.12. [Gre13, Proposition 4.10] *Let $\mathfrak{p} \lhd V$ be a prime ideal.*

(i) If $\mathfrak{p}^2 \neq \mathfrak{p}$ and $a \in V\backslash\{0\}$ then the point $(\mathfrak{p}, a\mathfrak{p})$ is an abnormal.
(ii) For all non-zero $a \in \mathfrak{p}$ there is an abnormal point (I, J) such that $IJ = a\mathfrak{p}$ and $I^\# = J^\# = \mathfrak{p}$.
(iii) Let (I, J) be an abnormal point with $I^\# = J^\# = \mathfrak{p}$. There exists non-zero $a \in \mathfrak{p}$ such that $IJ = a\mathfrak{p}$.

Lemma 4.13. *Let $\mathfrak{p} \lhd V$ be such that $\mathfrak{p}^2 = \mathfrak{p}$. Then, for all $a \in V\backslash\{0\}$, the point $(\mathfrak{p}, a\mathfrak{p})$ is normal.*

Proof. Let $\lambda \in V\backslash\{0\}$. Then $(\mathfrak{p}, a\mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if there exists $t \notin \mathfrak{p}$ such that $\lambda \notin at\mathfrak{p}$. Since $t \notin \mathfrak{p}$, $at\mathfrak{p} = a\mathfrak{p}$. Thus $(\mathfrak{p}, a\mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin a\mathfrak{p} = a\mathfrak{p}^2$. So, $(\mathfrak{p}, a\mathfrak{p})$ is normal. $\square$

We are now ready to start to construct an algorithm which, given $n \in \mathbb{N}$, $\lambda, \mu_1, ...\mu_n \in V\backslash\{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{m}$, answers whether

$$\mathcal{W}_{1,\lambda,g,h} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}.$$

We start by showing that it is enough to check the inclusion on finitely many subspaces of the form

$$X_{\mathfrak{p},\mathfrak{q}} := \{(I, J) \in \mathrm{Zg}_V \mid I^\# = \mathfrak{p} \text{ and } J^\# = \mathfrak{q}\}$$

where $\mathfrak{p}, \mathfrak{q} \lhd V$ are prime ideals. Moreover, we show that we can compute, given $\lambda, \mu_1, ...\mu_n \in V\backslash\{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n$, a finite set of elements $c_1, ..., c_m \in \mathfrak{m}$ such that it is enough to check the inclusion for the subspaces $X_{\mathfrak{p},\mathfrak{q}}$ where $\mathfrak{p} = \mathrm{rad}(c_iV)$ and $\mathfrak{q} = \mathrm{rad}(c_jV)$.

Definition 4.14. *Let $t \in \mathfrak{m}$. Denote by $\mathfrak{p}_t$ the smallest prime ideal containing t.*

Note that, for any $t \in \mathfrak{m}$, the ideal $\mathfrak{p}_t$ exists since the ideals of a valuation domain are totally ordered and note that $\mathfrak{p}_t$ is exactly the radical of tV.

Definition 4.15. *Suppose $x, y \in V$. We define $< x, y >$ as*

$$< x, y > := \begin{cases} y/x & \text{if } x|y, \\ x/y & \text{otherwise} \end{cases}$$

If V is effectively given then this function is computable.
We have split the proof of the following proposition into two lemmas the first dealing with normal points and the second with abnormal points.

Proposition 4.16. *Let $n \in \mathbb{N}$, $\lambda \in V\backslash\{0\}$, $g, h \in \mathfrak{m}$ and for each natural number $1 \leq i \leq n$ let $\mu_i \in V\backslash\{0\}$, $a_i, b_i \in \mathfrak{m}$. The following are equivalent:*

(1)

$$\mathcal{W}_{1,\lambda,g,h} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$$

(2) For all $\mathfrak{p} = \mathrm{rad}(tV)$ and $\mathfrak{q} = \mathrm{rad}(sV)$

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$$

where $s, t \in < T, T > \cap \mathfrak{m}$ and

$$T := \{\mu_i a_i b_i \ , \ \mu_i \mid 1 \leq i \leq n\} \cup \{1 \ , \ \lambda \ , \ g \ , \ h \ , \ \lambda gh\}.$$

Lemma 4.17. *Let n be a natural number, $\lambda \in V\backslash\{0\}$, $g, h \in \mathfrak{m}$ and for each natural number $1 \leq i \leq n$ let $\mu_i \in V\backslash\{0\}$, $a_i, b_i \in \mathfrak{m}$. If there exists (I, J) a normal point such that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ and $(I, J) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$ then there exists a point $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$ and $(K, L) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$ such that $K^{\#} = \mathfrak{p}_r$ $L^{\#} = \mathfrak{p}_s$ where $r =< x, y > \in \mathfrak{m}$ and $s =< u, w > \in \mathfrak{m}$ and x, y, u, w are taken from the set*

$$\{\mu_i a_i b_i \ , \ \mu_i \mid 1 \leq i \leq n\} \cup \{1 \ , \ \lambda \ , \ g \ , \ h \ , \ \lambda gh\}.$$

Proof. By definition, a normal point (I, J) is such that $(I, J) \notin \mathcal{W}_{1,\mu,a,b}$ if and only if either $\mu \in IJ$, $\mu ab \notin IJ$, $a \notin I^{\#}$ or $b \notin J^{\#}$. Therefore, if $(I, J) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$ then for each $1 \leq i \leq n$, either $\mu_i \in IJ$, $\mu_i a_i b_i \notin IJ$, $a_i \notin I^{\#}$ or $b_i \notin J^{\#}$.
Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ is normal and $(I, J) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$.
Let $p_1 \in \{\lambda, \mu_i a_i b_i \mid \mu_i a_i b_i \notin IJ\}$ be such that λ divides p_1 and if $\mu_i a_i b_i \notin IJ$ then $\mu_i a_i b_i$ divides p_1. Note that for any ideal $K \lhd V$, $p_1 \notin K$ implies $\lambda \notin K$ and if $\mu_i a_i b_i \notin IJ$ then $\mu_i a_i b_i \notin K$.
Let $p_2 \in \{\lambda gh, \mu_i \mid \mu_i \in IJ\}$ be such that p_2 divides λgh and if $\mu_i \in IJ$ then p_2 divides μ_i. Note that for any ideal $K \lhd V$, $p_2 \in K$ implies $\lambda gh \in K$ and if $\mu_i \in IJ$ then $\mu_i \in K$.
Since $p_1 \notin IJ$ and $p_2 \in IJ$, $p_2 = p_1 t$ for some $t \in V$ and $t \in (IJ)^{\#} = I^{\#} \cap J^{\#}$ by definition of $(IJ)^{\#}$.
Note that if $a_i \notin I^{\#}$ then $a_i \notin \mathfrak{p}_g \cup \mathfrak{p}_t$ since $\mathfrak{p}_g \cup \mathfrak{p}_t \subseteq I^{\#}$ and if $b_i \notin J^{\#}$ then $b_i \notin \mathfrak{p}_h \cup \mathfrak{p}_t$ since $\mathfrak{p}_h \cup \mathfrak{p}_t \subseteq J^{\#}$.

We now split the proof into two cases.

Case 1: $\mathfrak{p}_g \cup \mathfrak{p}_t \neq \mathfrak{p}_h \cup \mathfrak{p}_t$, or $\mathfrak{p}_g \cup \mathfrak{p}_t = \mathfrak{p}_h \cup \mathfrak{p}_t$ and $(\mathfrak{p}_g \cup \mathfrak{p}_t)^2 = \mathfrak{p}_g \cup \mathfrak{p}_t$
The point $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t))$ is a normal point (see lemmas 4.13 and 4.10) and

$$(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t) = (\mathfrak{p}_g \cup \mathfrak{p}_t) \cap (\mathfrak{p}_h \cup \mathfrak{p}_t).$$

So $t \in (\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$.
The point $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \in \mathcal{W}_{1,\lambda,g,h}$ since $g \in \mathfrak{p}_g \cup \mathfrak{p}_t$; $h \in \mathfrak{p}_h \cup \mathfrak{p}_t$; $p_1 \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$ implies $\lambda \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$ and $p_2 = p_1 t \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$ implies $\lambda gh \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$.
It remains to show $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $1 \leq i \leq n$.
As remarked above, if $a_i \notin I^\#$ then $a_i \notin (\mathfrak{p}_g \cup \mathfrak{p}_t)$ and if $b_i \notin J^\#$ then $b_i \notin (\mathfrak{p}_h \cup \mathfrak{p}_t)$. Since $p_1 \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$, if $\mu_i a_i b_i \notin IJ$ then $\mu_i a_i b_i \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$. Since $p_2 \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$, if $\mu_i \in IJ$ then $\mu_i \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$. Therefore, since $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t))$ is a normal point, for all $1 \leq i \leq n$, $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$.

Case 2: $\mathfrak{p} := \mathfrak{p}_g \cup \mathfrak{p}_t = \mathfrak{p}_h \cup \mathfrak{p}_t$ and $\mathfrak{p}^2 \neq \mathfrak{p}$
Since $\mathfrak{p} \neq \mathfrak{p}^2$, if $K \lhd V$ is such that $K^\# = \mathfrak{p}$ then $K = a\mathfrak{p}$ for some $a \in V\backslash\{0\}$. So, by proposition 4.12 (i), any point (K, L) with $K^\# = L^\# = \mathfrak{p}$ is necessarily abnormal.
First suppose that $\lambda gh \in p_1\mathfrak{p}^2$. Since $p_1 \notin p_1\mathfrak{p}$, we have $\lambda \notin p_1\mathfrak{p}$. So $\lambda\mathfrak{p} \supseteq p_1\mathfrak{p} \supsetneq p_1\mathfrak{p}^2$. By definition of $\mathfrak{p}$, $g, h \in \mathfrak{p}$. By lemma 4.11, $(\mathfrak{p}, p_1\mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$.
As in the first case, if $a_i \notin I^\#$ then $a_i \notin \mathfrak{p}$ and if $b_i \notin J^\#$ then $b_i \notin \mathfrak{p}$. If $\mu_i \in IJ$, then, since $p_2 \in p_1\mathfrak{p}$, $\mu_i \in p_1\mathfrak{p}$ and hence $p_1\mathfrak{p}^2 \supseteq \mu_i\mathfrak{p}$. If $\mu_i a_i b_i \notin IJ$, then, since $p_1 \notin p_1\mathfrak{p}$, $\mu_i a_i b_i \notin p_1\mathfrak{p}$ and hence $\mu_i a_i b_i \notin p_1\mathfrak{p}^2$. So, for all $1 \leq i \leq n$, either $a_i \notin \mathfrak{p}$, $b_i \notin \mathfrak{p}$, $\mu_i a_i b_i \notin p_1\mathfrak{p}^2$ or $p_1\mathfrak{p}^2 \supseteq \mu_i\mathfrak{p}$. Thus, by lemma 4.11, for all $1 \leq i \leq n$, $(\mathfrak{p}, p_1\mathfrak{p}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$.
Now suppose that $\lambda gh \notin p_1\mathfrak{p}^2$. Since $h \in \mathfrak{p}$, $\lambda g \notin p_1\mathfrak{p}$. Thus $\mathfrak{p} \supseteq \lambda\mathfrak{p} \supsetneq p_1\mathfrak{p}$. Therefore $p_1 \in \mathfrak{p}$.
Therefore, by proposition 4.12 (iii), there exists an abnormal point (K, L) with $K^\# = L^\# = \mathfrak{p}$ and $KL = p_1\mathfrak{p}$.
Since $p_2 \in p_1\mathfrak{p}$, $\lambda gh \in p_1\mathfrak{p}$. We have already noted that $\lambda\mathfrak{p} \supsetneq p_1\mathfrak{p}$. So, since $g, h \in \mathfrak{p}$, lemma 4.11 implies that $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$.
Since $p_2 \in p_1\mathfrak{p}$, if $\mu_i \in IJ$ then $\mu_i \in p_1\mathfrak{p}$. Since $p_1 \notin p_1\mathfrak{p}$, if $\mu_i a_i b_i \notin IJ$, then $\mu_i a_i b_i \notin KL$. So, for all $1 \leq i \leq n$, either $a_i \notin \mathfrak{p}$, $b_i \notin \mathfrak{p}$, $\mu_i a_i b_i \notin p_1\mathfrak{p}$ or $\mu_i \in p_1\mathfrak{p}$. Thus, by lemma 4.11 for all $1 \leq i \leq n$, $(K, L) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$.
Finally note that $\mathfrak{p}_t \cup \mathfrak{p}_g = \mathfrak{p}_r$ and $\mathfrak{p}_t \cup \mathfrak{p}_h = \mathfrak{p}_s$ for some $r =< x, y >$ and $s =< u, v >$ where x, y, v, u are taken from the set:

$$\{1\ ,\ \lambda\ ,\ g\ ,\ h\} \cup \{\mu_i\ ,\ \mu_i a_i b_i \mid 1 \leq i \leq n\}\,. \qquad \square$$

Lemma 4.18. *Let $n \in \mathbb{N}$, $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$ and for each natural number $1 \leq i \leq n$ let $\mu_i \in V\backslash\{0\}$ and $a_i, b_i \in \mathfrak{m}$. If there exists (I, J) an abnormal point such that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ and $(I, J) \notin \bigcup_{i=1}^n \mathcal{W}_{1,\mu_i,a_i,b_i}$ then there exists a point $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$ and $(K, L) \notin \bigcup_{i=1}^n \mathcal{W}_{1,\mu_i,a_i,b_i}$ such that*

$K^{\#} = \mathfrak{p}_r$ $L^{\#} = \mathfrak{p}_s$ where $r =< x, y >$ and $s =< u, w >$ and x, y, u, w are taken from the set

$$\{\mu_i a_i b_i \ , \ \mu_i \mid 1 \leq i \leq n\} \cup \{1 \ , \ \lambda \ , \ g \ , \ h \ , \ \lambda gh\}\,.$$

Proof. First note that since (I, J) is abnormal, by lemma 4.10, $I^{\#} = J^{\#}$. Let $\mathfrak{p} = I^{\#}$.
We now choose $\mu, d \in V$ as follows:
Suppose there exists $1 \leq i \leq n$ such that $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$. Let $\mu = \mu_i$ for some $1 \leq i \leq n$ such that $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$ and μ divides μ_i for all $1 \leq i \leq n$ such $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$.
It is easy to check that if $a, b \in V\backslash\{0\}$ and $a|b$ then $\mathcal{W}_{1,b,0,0} \subseteq \mathcal{W}_{1,a,0,0}$. So, for any pair $(K, L) \notin \mathcal{W}_{1,\mu,0,0}$, if $1 \leq i \leq n$ is such that $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$ then $(K, L) \notin \mathcal{W}_{1,\mu_i,0,0}$. If for all $1 \leq i \leq n$, $(I, J) \in \mathcal{W}_{1,\mu_i,0,0}$, let $\mu = 0$.
Suppose there exists $1 \leq i \leq n$ such that $\mu_i a_i b_i \notin IJ$. Let $d = \mu_i a_i b_i$ for some $1 \leq i \leq n$ such that $\mu_i a_i b_i \notin IJ$ and $\mu_i a_i b_i$ divides d for all $\mu_i a_i b_i \notin IJ$. Note, this means for any ideal K, if $d \notin K$ and $1 \leq i \leq n$ is such that $\mu_i a_i b_i \notin IJ$ then $\mu_i a_i b_i \notin K$. If for all $1 \leq i \leq n$, $\mu_i a_i b_i \in IJ$, let $d = 1$.
If $\mu \in IJ$ then set $p_1 = \lambda$ if $d|\lambda$ and $p_1 := d$ otherwise, set $p_2 := \lambda gh$ if $\lambda gh|\mu$ and $p := \mu$ otherwise. Then proceed as in the proof of lemma 4.17.
Otherwise, $\mu \notin IJ$ and $(I, J) \notin \mathcal{W}_{1,\mu,0,0}$. Thus $\mu\mathfrak{p} \supseteq IJ$ and by lemma 4.11 $IJ \supseteq \mu\mathfrak{p}$. Thus $\lambda\mathfrak{p} \supsetneq \mu\mathfrak{p} = IJ \supseteq \lambda ghV$ and $\mu \neq 0$. Note that $\mu \in \mathfrak{p}$ since $\mathfrak{p} \supseteq \lambda\mathfrak{p} \supsetneq \mu\mathfrak{p}$.
We now choose $t \in V$ and $\gamma \in V$ as follows:
Let $t \in V$ be such that $\mu = \lambda t$ and $\gamma \in V$ such that $\lambda gh = \mu\gamma$. Let $\mathfrak{q} := \mathfrak{p}_t \cup \mathfrak{p}_\gamma \cup \mathfrak{p}_g \cup \mathfrak{p}_h$. Note that $t, \gamma, g, h \in \mathfrak{p}$. So $\mathfrak{p} \supseteq \mathfrak{q}$. By proposition 4.12 and since $\mu \in \mathfrak{q}$, there exists an abnormal point (K, L) such that $KL = \mu\mathfrak{q}$. Since $\mu \in \lambda\mathfrak{q}$, $\lambda\mathfrak{q} \supsetneq \mu\mathfrak{q}$. Further $\lambda gh \in \mu\mathfrak{q}$, $g \in \mathfrak{q}$ and $h \in \mathfrak{q}$. Thus $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$.
If $a_i \notin \mathfrak{p}$ then $a_i \notin \mathfrak{q}$ and if $b_i \notin \mathfrak{p}$ then $b_i \notin \mathfrak{q}$. Since $d \notin IJ = \mu\mathfrak{p}$, we have that $d \notin \mu\mathfrak{q}$. Thus, if $\mu_i a_i b_i \notin IJ$ then $\mu_i a_i b_i \notin \mu\mathfrak{q}$. Finally $(K, L) \notin \mathcal{W}_{1,\mu,0,0}$. So, if $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$ then $(K, L) \notin \mathcal{W}_{1,\mu_i,0,0}$. Thus $(K, L) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $1 \leq i \leq n$. $\square$

We now reinterpret the inclusion

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$$

in terms of inclusions of intervals in the following order.

Definition 4.19. *Let $a, b \in V$ and $\mathfrak{p} \lhd V$ be prime. We write*

$$a <_{\mathfrak{p}} b \text{ if and only if } b \in a\mathfrak{p},$$
$$a =_{\mathfrak{p}} b \text{ if and only if } a\mathfrak{p} = b\mathfrak{p} \text{ and}$$
$$a \leq_{\mathfrak{p}} b \text{ if and only if } b\mathfrak{p} \subseteq a\mathfrak{p}.$$

Remark 4.20. *(i) If $X_{\mathfrak{p},\mathfrak{p}}$ contains normal points then V together with the order $<_{\mathfrak{p}}$ is dense.*

(ii) If $I \lhd V$ and $I^{\#} = \mathfrak{p}$ then $t \notin I$ and $s \in I$ implies $t <_{\mathfrak{p}} s$.

(iii) Let $(I, J) \in X_{\mathfrak{p},\mathfrak{p}}$ be abnormal. Let $a \in \mathfrak{p}$ be such that $IJ = a\mathfrak{p}$. Let $g, h \in \mathfrak{p}$ Then $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda <_{\mathfrak{p}} a <_{\mathfrak{p}} \lambda gh$.

Proof. (i) Since $X_{\mathfrak{p},\mathfrak{p}}$ contains a normal point $\mathfrak{p}^2 = \mathfrak{p}$ (See proposition 4.12 (i) and note that if $(I^{\#})^2 \neq I^{\#}$ then $I = aI^{\#}$ for some $a \in V$). Suppose $a <_{\mathfrak{p}} b$. Then $b \in a\mathfrak{p}$. Let $\gamma_1, \gamma_2 \in \mathfrak{p}$ such that $b = a\gamma_1\gamma_2$. Then $b \in a\gamma_1\mathfrak{p}$ and $a\gamma_1 \in a\mathfrak{p}$. So $a <_{\mathfrak{p}} a\gamma_1 <_{\mathfrak{p}} b$.
(ii) Suppose $I^{\#} = \mathfrak{p}$, $t \notin I$ and $s \in I$. Let $r \in V$ be such that $tr = s$. By definition of $I^{\#}$, $r \in I^{\#}$. Thus $s \in t\mathfrak{p}$. So $t <_{\mathfrak{p}} s$.
(iii) Suppose $(I, J) \in X_{\mathfrak{p},\mathfrak{p}}$ is abnormal. Then, by proposition 4.12 (iii) $IJ = a\mathfrak{p}$ for some $a \in \mathfrak{p}$. So by lemma 4.11 $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ means exactly that $\lambda gh \in IJ$ and $\lambda\mathfrak{p} \supsetneq IJ$. Thus $\lambda gh >_{\mathfrak{p}} a$ and $a >_{\mathfrak{p}} \lambda$. □

Definition 4.21. *Let $\mathfrak{p} \lhd V$ be prime, $t \in V$ and $s \in \mathfrak{p}$. We define*

$$(t, st)_{\mathfrak{p}} := \{r \in V \mid t <_{\mathfrak{p}} r <_{\mathfrak{p}} st\}, \text{ and}$$
$$[t, st)_{\mathfrak{p}} := \{r \in V \mid t \leq_{\mathfrak{p}} r <_{\mathfrak{p}} st\}$$

Proposition 4.22. *Let V be an effectively given valuation domain. Suppose $\mathfrak{p}, \mathfrak{q} \lhd V$ are prime ideals and that $\mathfrak{p} \neq \mathfrak{q}$. Suppose $\lambda, \mu_1, ...\mu_n \in V\backslash\{0\}$, $g, a_1, ..., a_n \in \mathfrak{p}$ and $h, b_1, ..., b_n \in \mathfrak{q}$. Then*

$$[\lambda, \lambda gh)_{\mathfrak{q}\cap\mathfrak{p}} \subseteq \cup_{i=1}^{n}[\mu_i, \mu_i a_i b_i)_{\mathfrak{q}\cap\mathfrak{p}}$$

if and only if

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}.$$

Proof. Because $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $(J, I) \in \mathcal{W}_{1,\lambda,h,g}$, we may assume without loss of generality that $\mathfrak{p} \subsetneq \mathfrak{q}$.
Note that, by lemma 4.10 all $(I, J) \in X_{\mathfrak{p},\mathfrak{q}}$ are abnormal since $\mathfrak{p} \neq \mathfrak{q}$.
Suppose

$$[\lambda, \lambda gh)_{\mathfrak{p}} \subseteq \cup_{i=1}^{n}[\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}.$$

We may assume that $\cup_{i=1}^{n}[\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$ is an irredundant union.
By reordering, we may assume,

$$\mu_i a_i b_i <_{\mathfrak{p}} \mu_{i+1} a_{i+1} b_{i+1}$$

for $1 \leq i < n$.
From the irredundancy of $\cup_{i=1}^{n}[\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$ and the reordering, we get that $\mu_i <_{\mathfrak{p}} \mu_{i+1}$, $\mu_1 \leq_{\mathfrak{p}} \lambda$ and $\lambda gh \leq_{\mathfrak{p}} \mu_n a_n b_n$.
Take $(I, J) \in \mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}}$. So $\lambda \notin IJ$ and $\lambda gh \in IJ$. We now need to show that there exists $1 \leq k \leq n$ such that $\mu_k \notin IJ$ and $\mu_k a_k b_k \in IJ$.
Since $\mu_1 \leq_{\mathfrak{p}} \lambda$ and $\lambda gh \leq_{\mathfrak{p}} \mu_n a_n b_n$, $\mu_1 \notin IJ$ and $\mu_n a_n b_n \in IJ$.

Let k be least such that $\mu_k a_k b_k \in IJ$. Then either $k=1$ or $\mu_{k-1}a_{k-1}b_{k-1} \notin IJ$. If $k=1$, then, since $\mu_1 \notin IJ$, $(I,J) \in \mathcal{W}_{1,\mu_1,a_1,b_1}$. If $\mu_k \notin IJ$ then $(I,J) \in \mathcal{W}_{1,\mu_k,a_k,b_k}$.
Suppose for a contradiction that $\mu_k \in IJ$ and $k > 1$. Then $\lambda <_{\mathfrak{p}} \mu_k$, $\mu_{k-1}a_{k-1}b_{k-1} <_{\mathfrak{p}} \mu_k$ and $\mu_{k-1}a_{k-1}b_{k-1} <_{\mathfrak{p}} \lambda gh$, since $\lambda \notin IJ$, $\mu_k \in IJ$, $\mu_{k-1}a_{k-1}b_{k-1} \notin IJ$ and $\lambda gh \in IJ$.
Thus there exists $d \in V$ such that $\lambda \leq_{\mathfrak{p}} d <_{\mathfrak{p}} \lambda gh$ and $\mu_{k-1}a_{k-1}b_{k-1} \leq_{\mathfrak{p}} d <_{\mathfrak{p}} \mu_k$. Since $d <_{\mathfrak{p}} \mu_k$, $d <_{\mathfrak{p}} \mu_i$ for all $i \geq k$. Since $\mu_{k-1}a_{k-1}b_{k-1} \leq_{\mathfrak{p}} d$, $\mu_i a_i b_i \leq_{\mathfrak{p}} d$ for all $i \leq k-1$. So $d \notin (\mu_i, \mu_i a_i b_i]$ for all $1 \leq i \leq n$. But, since $\lambda \leq_{\mathfrak{p}} d <_{\mathfrak{p}} \lambda gh$, $d \in [\lambda, \lambda gh)$. This contradicts our assumption. Thus $\mu_k \notin IJ$. So $(I,J) \in \mathcal{W}_{1,\mu_k,a_k,b_k}$.
Now suppose that

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}.$$

Suppose $d \in [\lambda, \lambda gh)$. Then $\lambda \notin d\mathfrak{p}$ and $\lambda gh \in d\mathfrak{p}$. Note that $d\mathfrak{p}\mathfrak{q} = d\mathfrak{p}$. The point $(d\mathfrak{p}, \mathfrak{q})$ is normal (lemma 4.10), since $(d\mathfrak{p})^{\#} = \mathfrak{p} \neq \mathfrak{q}$. Thus, by theorem 4.8, $(d\mathfrak{p}, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}}$. Thus $(d\mathfrak{p}, \mathfrak{q}) \in \mathcal{W}_{1,\mu_k,a_k,b_k} \cap X_{\mathfrak{p},\mathfrak{q}}$ for some $1 \leq k \leq n$. So, $\mu_k \notin d\mathfrak{p}$ and $\mu_k a_k b_k \in d\mathfrak{p}$. Thus $\mu_k \leq_{\mathfrak{p}} d$ and $d <_{\mathfrak{p}} \mu_k a_k b_k$. So $d \in \cup_{i=1}^{n} [\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$. □

Corollary 4.23. *Let V be an effectively given valuation domain. Suppose $\mathfrak{p}, \mathfrak{q} \lhd V$ are prime ideals such that $\mathfrak{p} \neq \mathfrak{q}$. Suppose there is an algorithm that given $a \in V$, answers whether $a \in \mathfrak{p}$ and an algorithm that given $a \in V$, answers whether $a \in \mathfrak{q}$. Then for any natural number n there is an algorithm that given $\lambda, \mu_1, ...\mu_n \in V \backslash \{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{m}$, answers whether*

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}.$$

Proof. If $g \notin \mathfrak{p}$ or $h \notin \mathfrak{q}$ then $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} = \emptyset$. So $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$.
Suppose $g \in \mathfrak{p}$ and $h \in \mathfrak{q}$. Then $(\mathfrak{p}, \lambda\mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ since $g \in \mathfrak{p}$, $\lambda \notin \lambda\mathfrak{q}$ and $\lambda h \in \lambda\mathfrak{q}$. If, for all $1 \leq i \leq n$, either $a_i \notin \mathfrak{p}$ or $b_i \notin \mathfrak{q}$ then $\bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}} = \emptyset$. Hence $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \nsubseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$.
Now suppose $g \in \mathfrak{p}$ and $h \in \mathfrak{q}$ and there exists $1 \leq i \leq n$ such that $a_i \in \mathfrak{p}$ and $b_i \in \mathfrak{q}$. Let $\mathcal{J}$ be the set of all $1 \leq i \leq n$ such that $a_i \in \mathfrak{p}$ and $b_i \in \mathfrak{q}$. Then $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$ if and only if $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i \in \mathcal{J}} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$.
By proposition 4.22, $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i \in \mathcal{J}} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$ if and only if $[\lambda, \lambda gh)_{\mathfrak{p}\cap\mathfrak{q}} \subseteq \bigcup_{i \in \mathcal{J}} [\mu_i, \mu_i a_i b_i)_{\mathfrak{p}\cap\mathfrak{q}}$.
The existence of an algorithm which, given $a \in V$, answers whether $a \in \mathfrak{p} \cap \mathfrak{q}$ means, since V is effectively given, there exists an algorithm which, given $a, b \in V$, answers whether $a \in b(\mathfrak{p} \cap \mathfrak{q})$. Therefore, there is an algorithm which given $\lambda, \mu_1, ..., \mu_k \in V \backslash \{0\}$ and $g, h, a_1, ..., a_k, b_1, ..., b_k \in \mathfrak{p} \cap \mathfrak{q}$, answers whether $[\lambda, \lambda gh)_{\mathfrak{p}\cap\mathfrak{q}} \subseteq \bigcup_{i \in \mathcal{J}} [\mu_i, \mu_i a_i b_i)_{\mathfrak{p}\cap\mathfrak{q}}$. □

Proposition 4.24. *Suppose $\mathfrak{p} \lhd V$ is prime, $n \in \mathbb{N}$, $\lambda, \mu_1, ..., \mu_n \in V \backslash \{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{p}$. Then the following are equivalent:*

(1)
$$(\lambda, \lambda gh)_{\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$$

(2)
$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}.$$

Proof. (1)⇒(2) Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}}$ is normal. Suppose, for a contradiction, that $(I, J) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $1 \leq i \leq n$.
Let $S := \{1 \leq i \leq n \mid \mu_i \in IJ\}$. Let $T := \{1 \leq i \leq n \mid \mu_i a_i b_i \notin IJ\}$. Thus, since $(I, J) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $1 \leq i \leq n$, we have that either $\mu_i \in IJ$ or $\mu_i a_i b_i \notin IJ$ for all $1 \leq i \leq n$. So $S \cup T = \{1, 2, ..., n\}$.
First we show that neither S nor T is empty. Suppose S is empty. Then $\mu_i a_i b_i \notin IJ$ for all $1 \leq i \leq n$ because $S \cup T = \{1, 2, ..., n\}$. Since $\lambda gh \in IJ$, by remark 4.20 (ii), $\mu_i a_i b_i <_{\mathfrak{p}} \lambda gh$ for all $1 \leq i \leq n$. This contradicts (1). Suppose T is empty. Then $\mu_i \in IJ$ for all $1 \leq i \leq n$. Since $\lambda \notin IJ$, by remark 4.20 (ii), $\lambda <_{\mathfrak{p}} \mu_i$ for all $1 \leq i \leq n$. This contradicts (1).
Take z_1 maximal with respect to the $<_{\mathfrak{p}}$ order such that $z_1 = \mu_i a_i b_i$ for some $i \in T$. Take z_2 minimal with respect to the $<_{\mathfrak{p}}$ order such that $z_2 = \mu_i$ for some $i \in S$. Thus $z_1 \notin IJ$ and $z_2 \in IJ$. So $z_1 <_{\mathfrak{p}} z_2$. Since $\lambda \notin IJ$, $\lambda <_{\mathfrak{p}} z_2$. Since $\lambda gh \in IJ$, $z_1 < \lambda gh$.
By remark 4.20 (i), there exists $d \in (z_1, z_2) \cap (\lambda, \lambda gh)$. So, using (1), $d \in (\mu_i, \mu_i a_i b_i)$ for some $1 \leq i \leq n$. But then $z_1 <_{\mathfrak{p}} d <_{\mathfrak{p}} \mu_i a_i b_i$ and $\mu_i <_{\mathfrak{p}} d <_{\mathfrak{p}} z_2$. Thus $\mu_i \notin IJ$ and $\mu_i a_i b_i \in IJ$.
Thus (2) holds when restricted to normal points. That (2) holds for abnormal points too follows straightforwardly from remark 4.20 (iii).
(2)⇒(1) This follows directly from remark 4.20 (iii). □

Corollary 4.25. *Let V be an effectively given valuation domain. Suppose $\mathfrak{p} \lhd V$ is a prime ideal. Suppose there is an algorithm that given $a \in V$, answers whether $a \in \mathfrak{p}$. Then for any natural number n there is an algorithm that given $\lambda, \mu_1, ...\mu_n \in V \backslash \{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{m}$, answers whether*

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}.$$

Proof. Almost exactly as in corollary 4.23. □

Lemma 4.26. *Let $n \in \mathbb{N}$. Let V be an effectively given valuation domain such that there exists an algorithm which, given $a, b \in V$, answers whether $a \in \text{rad}(bV)$. Then there exists an algorithm which, given $a, b, \alpha_i, \beta_i \in V \backslash \{0\}$ and $g, h, \gamma_i, \delta_i \in \mathfrak{m}$ for each $1 \leq i \leq n$, answers whether*

$$\mathcal{W}_{a,b,g,h} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{\alpha_i,\beta_i,\gamma_i,\delta_i}.$$

Proof. First note for any $a, b \in V \backslash \{0\}$ and $g, h \in \mathfrak{m}$, $\mathcal{W}_{a,b,g,h} = \mathcal{W}_{1,ab,g,h}$. Suppose $n \in \mathbb{N}$, $\lambda, \mu_i \notin V\backslash\{0\}$ and $g, h, a_i, b_i \in \mathfrak{m}$. Let $T = \{< u, v >\in \mathfrak{m} \mid u, v \in \{1 , \lambda , g , h , \mu_i a_i b_i , \mu_i \mid 1 \leq i \leq n\}\}$. Note that T is a finite set and there is an algorithm which, given λ, g, h and μ_i, a_i, b_i for $1 \leq i \leq n$, computes T since the function $< , >$ and multiplication of ring elements is recursive.
Then in order to check whether

$$\mathcal{W}_{1,\lambda,g,h} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$$

by lemma 4.17 and lemma 4.18 it is enough to check

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$$

for $\mathfrak{p} = \mathrm{rad}(tV)$ and $\mathfrak{q} = \mathrm{rad}(sV)$ for each $t, s \in T$. Note that $\mathfrak{p} \subsetneq \mathfrak{q}$ if and only if $s \notin \mathrm{rad}(tV)$.
By corollary 4.23 and corollary 4.25 there exists an algorithm determining the truth of the above statement. □

Theorem 4.27. *Let V be an effectively given valuation domain with an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$. Let $n \in \mathbb{N}$. Then there is an algorithm which, given φ/ψ a pp-pair and ϑ_i/ξ_i a pp-pair for each $1 \leq i \leq n$, answers whether:*

$$\left(\frac{\varphi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right).$$

Proof. By lemma 4.3, given a pp-pair φ/ψ we can effectively check whether $\left(\frac{\varphi}{\psi}\right)$ is non-empty.

Again using lemma 4.3, given a pp-pair φ/ψ, if $\left(\frac{\varphi}{\psi}\right)$ is non-empty we can effectively find $a_j, b_j \in V\backslash\{0\}$ and $g_j, h_j \in \mathfrak{m}$ such that:

$$\left(\frac{\varphi}{\psi}\right) = \bigcup_{j} \mathcal{W}_{a_j,b_j,g_j,h_j}$$

and for each i, if $\left(\frac{\vartheta_i}{\xi_i}\right)$ is non-empty we can effectively find $\alpha_{i,k}, \beta_{i,k} \in V\backslash\{0\}$ and $\gamma_{i,k}, \delta_{i,k} \in \mathfrak{m}$ such that:

$$\left(\frac{\vartheta_i}{\xi_i}\right) = \bigcup_{i,k} \mathcal{W}_{\alpha_{i,k},\beta_{i,k},\gamma_{i,k},\delta_{i,k}}$$

Therefore it is enough to check for each j whether:

$$\mathcal{W}_{a_j,b_j,g_j,h_j} \subseteq \bigcup_{i,k} \mathcal{W}_{\alpha_{i,k},\beta_{i,k},\gamma_{i,k},\delta_{i,k}}.$$

By lemma 4.26 there exists an algorithm which determines the truth of the above statement. □

5. Duality

In this section we will discuss the duality map for the Ziegler spectrum of valuation domains. The results in this section are used in section 6. It is unnecessary to invoke duality in the sense that the results of this paper may be obtained by more elementary methods. These elementary methods involve calculating the size of pp-quotients in certain uniserial modules (see [Gre11]). Considering the duality map means that we only have to do two thirds of these computations.
A duality between the lattice of right pp-n-formulae and the lattice of left pp-n-formulae was first introduced by Prest [Pre88, Section 8.4] and then extended by Herzog [Her93] to give an isomorphism between the lattice of open set of the left Ziegler spectrum of a ring and the lattice of open sets of the right Ziegler spectrum of a ring.

Definition 5.1. *Let φ be a pp-n-formula in the language of right R-modules of the form $\exists \bar{y}(\bar{x},\bar{y})H = 0$ where $\bar{x}$ is a tuple of n variable, $\bar{y}$ is a tuple of l variables, $H = (H'\ H'')^T$ and H' (respectively H'') is a $n \times m$ (respectively $l \times m$) matrix with entries in R. Then $\mathrm{D}\varphi$ is the pp-n-formula in the language of left R-modules $\exists \bar{z} \begin{pmatrix} I & H' \\ 0 & H'' \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = 0$.*
Similarly, let φ be a pp-n-formula in the language of left R-modules of the form $\exists \bar{y} H \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = 0$ where $\bar{x}$ is a tuple of n variable, $\bar{y}$ is a tuple of l variables, $H = (H'\ H'')$ and H' (respectively H'') is a $m \times n$ (respectively $m \times l$) matrix with entries in R. Then $\mathrm{D}\varphi$ is the pp-n-formula in the language of right R-modules $\exists \bar{z}(\bar{x},\bar{z}) \begin{pmatrix} I & 0 \\ H' & H'' \end{pmatrix} = 0$.

Note that the pp-formula $a|x$ for $a \in R$ is mapped by D to a formula equivalent with respect to T_R to $xa = 0$ and the pp-formula $xa = 0$ for $a \in R$ is mapped by D to a formula equivalent with respect to T_R to $a|x$.

Theorem 5.2. [Pre88, Chapter 8] *The map $\varphi \to D\varphi$ induces an anti-isomorphism between the lattice of right pp-n-formulae and the lattice of left pp-n-formulae. In particular, if φ, ψ are pp-n-formulae then $D(\varphi + \psi)$ is equivalent to $D\varphi \wedge D\psi$ and $D(\varphi \wedge \psi)$ is equivalent to $D\varphi + D\psi$.*

This gives rise "at the level of open sets" to a homeomorphism from the left Ziegler spectrum of R to the right Ziegler spectrum of R. To be precise:

Theorem 5.3. [Her93] *The map D given on basic opens sets by*

$$\left(\frac{\varphi}{\psi}\right) \mapsto \left(\frac{D\psi}{D\varphi}\right)$$

is an idempotent lattice isomorphism from the lattice of open sets of Zg_R *to the lattice of open sets of* ${}_R\mathrm{Zg}$.

It is unknown whether this lattice isomorphism always comes from a homeomorphism or even if this map always comes from a homeomorphism between Zg_R and ${}_R\mathrm{Zg}$ after identifying topologically indistinguishable points in both spaces.
For a commutative ring R we identify the left and right Ziegler spectra.
In the case of valuation domains we are in the lucky position of having a very canonical homeomorphism which give rise to this map.

Proposition 5.4. *The map* $t : \mathrm{Zg}_R \to \mathrm{Zg}_R : N(I,J) \mapsto N(J,I)$ *is a well-defined homeomorphism. Moreover,* t *induces the lattice isomorphism* D *given in theorem 5.3.*

Proof. First we note that t is well-defined since $\langle I,J\rangle \sim \langle K,L\rangle$ if and only if $\langle J,I\rangle \sim \langle L,K\rangle$.
<u>Claim</u>: For any $a,b \in V\backslash\{0\}$, $g,h \in \mathfrak{m}$ and pair of ideals (I,J), $(I,J) \in \mathcal{W}_{a,b,g,h}$ if and only if $(J,I) \in \mathcal{W}_{b,a,h,g}$.
Suppose $(I,J) \in \mathcal{W}_{a,b,g,h}$ then there exists (K,L) such that $(I,J) \sim (K,L)$ and $a \notin K$, $ag \in K$, $b \notin L$ and $bh \in L$. Therefore $(L,K) \in \mathcal{W}_{b,a,h,g}$ and $(J,I) \sim (L,K)$ so $(J,I) \in \mathcal{W}_{b,a,h,g}$. The reverse direction is by symmetry.
Therefore t is a homeomorphism and

$$N(I,J) \in \left(\frac{xag = 0 \wedge b|x}{xa = 0 + bh|x}\right) \text{ if and only if } N(J,I) \in \left(\frac{xbh = 0 \wedge a|x}{xb = 0 + ag|x}\right).$$

Since t is a homeomorphism, it induces an automorphism t^{latt} on the lattice of open sets of Zg_V. From the fact that D and t are equal on a basis of the lattice of open sets of Zg_V (by a basis of a lattice L we simply mean a subset B of L such that every element of L can be written as a supremum of elements in B) we get that t^{latt} and D are the same automorphism. $\square$

We call a homeomorphism from Zg_R to ${}_R\mathrm{Zg}$ which gives rise to the lattice isomorphism in 5.3 a duality homeomorphism for Ziegler spectra.
The following result is essentially due to Herzog [Her93] (although it is not explicitly stated).

Theorem 5.5. *If* $D : \mathrm{Zg}_R \to_R \mathrm{Zg}$ *is a duality homeomorphism for Ziegler spectra,* φ/ψ *is a pp-pair and* N *is a pure-injective indecomposable (right)* R*-module then*

$$\left|\frac{\varphi(N)}{\psi(N)}\right| = \left|\frac{D\psi(DN)}{D\varphi(DN)}\right|.$$

Proof. If $|\varphi(N)/\psi(N)|$ and $|D\psi(DN)/D\varphi(DN)|$ are always either 1 or infinite for all pp-1-formulae φ,ψ then the statement is true by definition.
Suppose $|\varphi(N)/\psi(N)|$ is finite but not equal to one for some pp-pair φ/ψ. Then there exists a pp-pair σ/τ which is N-minimal i.e. $\sigma(N) \supsetneq \tau(N)$ and for all pp-1-formulae θ, $\sigma(N) \supseteq \theta(N) \supseteq \tau(N)$ implies either $\sigma(N) = \theta(N)$

or $\theta(N) = \tau(N)$. Then N is reflexive in the sense of Herzog [Her93, page 51], that is, there exists a pp-pair χ/τ such that for all indecomposable pure-injective modules U in the closure of N (with respect to the Ziegler topology), χ/τ is either U-minimal or $\chi(U) = \tau(U)$. So, now by [Her93, Theorem 6.6] and the modularity of the lattice of pp-formulae,

$$\left|\frac{\varphi(N)}{\varphi \wedge \psi(N)}\right| = \left|\frac{(\varphi+\psi)(N)}{\psi(N)}\right| = \left|\frac{D\psi(DN)}{(D\varphi \wedge D\psi)(DN)}\right|.$$

□

Putting this together with proposition 5.4 we get that:

Proposition 5.6. *Let V be a valuation domain. For all proper ideal $I, J \lhd V$ and all pp-pairs φ/ψ we have that*

$$|\varphi(N(I,J))/\psi(N(I,J))| = |D\psi(N(J,I))/D\varphi(N(J,I))|.$$

6. Finite invariants

We start by recalling some useful results from the model theory of modules over valuation domains.

A module M is called uniserial if its lattice of submodules form a chain. Clearly every submodule and quotient module of a uniserial module is also uniserial. Less obviously we have the following theorem due to Ziegler.

Theorem 6.1. [Zie84] *Every indecomposable pure-injective module over a valuation domain is the pure-injective hull of a uniserial module and the pure-injective hull of a uniserial module is indecomposable.*

Despite pure-injective modules over valuation domains not in general being uniserial, they are uniserial as modules over their endomorphism ring (see [Pun01]) and thus we get the following theorem and corollary:

Theorem 6.2. [Pun01, Corollary 11.5] *If M is an indecomposable pure-injective module over a valuation domain then for any two pp-formulae $\varphi(x), \psi(x)$ either $\varphi(M) \subseteq \psi(M)$ or $\psi(M) \subseteq \varphi(M)$.*

Corollary 6.3. *Let N be an indecomposable pure-injective module over a valuation domain V. If $\varphi := \sum_i \varphi_i$ and $\varphi := \wedge_j \psi_j$ where φ_i and ψ_j are pp-formulae then $|\varphi(N)/\psi(N)| = \max_{i,j} |\varphi_i(N)/\psi_j(N)|$.*

Bearing in mind that we have effective procedures for rewriting pp-formulae in the form $\sum_{i=1}^n (xa_i = 0 \wedge b_i|x)$ (Lemma 4.1) and $\bigwedge_{i=1}^n (xa_i = 0 + b_i|x)$ (Lemma 4.2). This means it is enough to consider invariant sentences of the form $\left|\frac{(xag=0)\wedge(b|x)}{(xa=0)+(bh|x)}\right| \geq m$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$.

If V is a valuation domain with infinite residue field then the only finite V-module is the zero module. Since [PPT07] already dealt with finite invariant sentences for valuation domains with dense value groups we won't include results for this case. Thus in this section we will focus on valuation domains with non-dense value group and finite field residue field.

Let R be a commutative ring. For every indecomposable pure-injective module N the set of $r \in R$ whose action on N is not bijective is a prime ideal of R (see for instance [Zie84, Theorem 5.4]). We call this prime ideal the attached prime of N.
For a valuation domain V the attached prime of $N(I, J)$ is $I^{\#} \cup J^{\#}$. This follows easily from lemma 4.4, the reformulation of the equivalence relation $\sim$ just after lemma 4.4 and the definition of $I^{\#}$ and $J^{\#}$.

Lemma 6.4. *Let V be a valuation domain with finite residue field. Let φ, ψ be pp-1-formulae and let $I, J \lhd V$. If $\left|\frac{\varphi(N(I,J))}{\psi(N(I,J))}\right|$ is finite and not equal to 1 then either $I^{\#} = \mathfrak{m}$ or $J^{\#} = \mathfrak{m}$.*

Proof. Suppose $\left|\frac{\varphi(N)}{\psi(N)}\right|$ is finite and not equal to 1. There exists a pp-1-formula ψ' such that $\varphi(N) \supsetneq \psi'(N) \supseteq \psi(N)$ and φ/ψ' is an N-minimal pair. Since $\left|\frac{\varphi(N)}{\psi(N)}\right|$ is finite and not equal to 1, $\left|\frac{\varphi(N)}{\psi'(N)}\right|$ is finite and not equal to 1. Suppose N has attached prime $\mathfrak{p}$ not equal to $\mathfrak{m}$. Then, for all $r \in \mathfrak{p}$ and all non-zero $x \in N$, xr has strictly greater pp-type than x by [Pre88, Chapter 4 section 4.4]. Hence if $x \in \varphi(N)$ then $xr \in \psi'(N)$. Therefore $\frac{\varphi(N)}{\psi'(N)}$ is an $V/\mathfrak{p}$-module. All $r \notin \mathfrak{p}$ act as automorphisms on N. Hence $\frac{\varphi(N)}{\psi'(N)}$ is a $V_{\mathfrak{p}}/\mathfrak{p}$-module (i.e. vector space) and therefore infinite or the zero module since $V/\mathfrak{p}$ is of infinite size.
Therefore, if $\left|\frac{\varphi(N)}{\psi(N)}\right|$ is finite and not equal to 1 then its attached prime is $\mathfrak{m}$. Thus, $I^{\#} \cup J^{\#} = \mathfrak{m}$. Therefore either $I^{\#} = \mathfrak{m}$ or $J^{\#} = \mathfrak{m}$. $\square$

For a valuation domain V with dense value group and finite residue field the situation is significantly simpler. If $\varphi(N(I, J))/\psi(N(I, J))$ is non-zero and finite for some pp-pair φ/ψ then either $I = a\mathfrak{m}$ and $J = b\mathfrak{m}$ for some non-zero $a, b \in V$ or $I = aV$ and $J = bV$ for some non-zero $a, b \in \mathfrak{m}$ (see [PPT07, Section 7]).
A valuation domain having non-dense value group exactly means that its maximal ideal is principal. It is easy to derive from this that $I^{\#} = \mathfrak{m}$ if and only if I is principal. Thus for all $I \lhd V$ with $I^{\#} = \mathfrak{m}$ there exists $a \in \mathfrak{m}$ such that $(I : a) = \mathfrak{m}$. Thus we need only consider finite invariant sentences for indecomposable pure-injective modules of the form $N(I, \mathfrak{m})$, $N(\mathfrak{m}, J)$ and $N(\mathfrak{m}, xV)$ where $x \in \mathfrak{m}\backslash\{0\}$, $I^{\#} \subsetneq \mathfrak{m}$ and $J^{\#} \subsetneq \mathfrak{m}$.

Lemma 6.5. *Let V be a valuation domain with residue field consisting of q elements. Then any finite non-zero module is of size q^n for some $n \in \mathbb{N}$.*

Proof. Suppose M is a finite non-zero V-module. Let

$$M = M_k \supsetneq \ldots \supsetneq M_2 \supsetneq M_1 \supsetneq 0 = M_0$$

be a chain of submodules of M such that each quotient M_{i+1}/M_i is cyclic. Every finite cyclic (non-zero) module is isomorphic to $V/\mathfrak{m}^w$ for some $w \in \mathbb{N}$ and $V/\mathfrak{m}^w$ has q^w elements. $\square$

Note that the above lemma implies that for any pp-pair φ/ψ and any V-module M, $\left|\frac{\varphi(M)}{\psi(M)}\right| = q^n$ for some $n \in \mathbb{N}_0$ or $\left|\frac{\varphi(M)}{\psi(M)}\right|$ is infinite.

Lemma 6.6. *Let V be a valuation domain with non-dense value group and finite residue field. Let φ be the pp-1-formula $(xag = 0 \wedge b|x)$ and let ψ be the pp-1-formula $(xa = 0 + bh|x)$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. If $x \in \mathfrak{m}$ is such that $N(\mathfrak{m}, xV) \in \left(\frac{\varphi}{\psi}\right)$ then*

$$\left|\frac{\varphi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| = \ min \ \left\{\left|\frac{V}{gV}\right| \ , \ \left|\frac{V}{hV}\right| \ , \ \left|\frac{xV}{abghV}\right| \ , \ \left|\frac{abV}{xV}\right|\right\}.$$

Proof. The type $p(xV, \mathfrak{m})$ is realised by 1 in the module V/xV. Since V/xV is uniserial, $N(xV, \mathfrak{m})$ is isomorphic to the pure-injective hull of V/xV. Thus V/xV and $N(xV, \mathfrak{m})$ are elementary equivalent. So we need only calculate the size of $\frac{\varphi(V/xV)}{\psi(V/xV)}$.
Note that, by proposition 4.12 (i) the point $(xV, \mathfrak{m})$ is an abnormal point since $\mathfrak{m}$ is principally generated and thus $xV = t\mathfrak{m}$ for some $t \in V\backslash\{0\}$ and $\mathfrak{m}^2 \neq \mathfrak{m}$. Note that $ab\mathfrak{m} \supsetneq x\mathfrak{m}$ if and only if $ab \notin xV$. So, by lemma 4.11 the condition that $N(\mathfrak{m}, xV) \in \left(\frac{\varphi}{\psi}\right)$ means that $ab \notin xV$ and $abgh \in x\mathfrak{m}$. Thus $bV \supseteq (xV : a)$ and $bhV \subseteq (xV : ag)$.
The solution set to the formula $xag = 0$ in V/xV is $(xV : ag)/xV$. The solution set to the formula $b|x$ in V/xV is bV/xV. The solution set to the formula $xa = 0$ in V/xV is $(xV : a)/xV$. The solution set to the formula $bh|x$ in V/xV is $(bhV + xV)/xV$. Thus

$$\left|\frac{\varphi(V/xV)}{\psi(V/xV)}\right| = \min\left\{\left|\frac{(xV : ag)}{(xV : a)}\right| \ , \left|\frac{(xV : ag)}{bhV}\right| \ , \left|\frac{bV}{(xV : a)}\right| \ , \left|\frac{bV}{bhV}\right|\right\}.$$

Since V is a domain $\frac{(xV:ag)}{(xV:a)} \cong V/gV$, $\frac{(xV:ag)}{bhV} \cong xV/abghV$, $\frac{bV}{(xV:a)} \cong abV/xV$ and $\frac{bV}{bhV} \cong V/hV$. □

Proposition 6.7. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Let φ be the pp-formula $(xag = 0 \wedge b|x)$ and let ψ be the pp-formula $(xa = 0 + bh|x)$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Suppose $I \lhd V$ is a proper ideal such that $I^\# \subsetneq \mathfrak{m}$. Then*

$$\left|\frac{\varphi(N(I, \mathfrak{m}))}{\psi(N(I, \mathfrak{m}))}\right| = \begin{cases} 1, & \text{if } ab \in I \text{ or } abgh \notin I \text{ or } g \notin I^\#; \\ q^v, & ab \notin I,\ abgh \in I,\ g \in I^\# \text{ and } hV = \mathfrak{m}^v; \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. By lemma 6.5, if $\left|\frac{\varphi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right|$ is finite then it is either of size 1 or q^v for some $v \in \mathbb{N}$.

From theorem 4.8 and lemma 4.10 we have that $N(I,\mathfrak{m}) \in \left(\frac{\varphi}{\psi}\right)$ if and only if $ab \notin I$, $abgh \in I$ and $g \in I^{\#}$. So $\left|\frac{\varphi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = 1$ if and only if $ab \in I$ or $abgh \notin I$ or $g \notin I^{\#}$.
We now assume that $ab \notin I$, $abgh \in I$ and $g \in I^{\#}$.
Note that the pp-type $p(I,\mathfrak{m})$ is realised by $1+I$ in the uniserial module V/I. The pure-injective hulls of uniserial modules are indecomposable 6.1 and thus the pure-injective hull of V/I is isomorphic to $N(I,\mathfrak{m})$. Every module is elementary equivalent to its pure-injective hull. Hence

$$|\varphi(V/I)/\psi(V/I)| = |\varphi(N(I,\mathfrak{m}))/\psi(N(I,\mathfrak{m}))|\,.$$

The pp-subgroup defined by $(xa = 0 + bh|x)$ in V/I is

$$\frac{(I:a)+bhV}{I}.$$

Note that $bV \supsetneq I$ since $ab \notin I$. The pp-subgroup defined by $(xag = 0 \wedge b|x)$ in V/I is

$$\frac{(I:ag)\cap bV}{I}.$$

Thus the pp-quotient defined by φ/ψ in V/I is

$$\frac{(I:ag)\cap bV}{(I:a)+bhV}.$$

Since V/I is uniserial,

$$\left|\frac{(I:ag)\cap bV}{(I:a)+bhV}\right| = \min\left\{\left|\frac{(I:ag)}{(I:a)}\right| \ ,\ \left|\frac{(I:ag)}{bhV}\right| \ ,\ \left|\frac{bV}{(I:a)}\right| \ ,\ \left|\frac{bV}{bhV}\right|\right\}.$$

Thus

$$\left|\frac{(I:ag)\cap bV}{(I:a)+bhV}\right| = \min\left\{\left|\frac{I}{Ig}\right| \ ,\ \left|\frac{I}{abghV}\right| \ ,\ \left|\frac{abV}{I}\right| \ ,\ \left|\frac{V}{hV}\right|\right\}.$$

Note that any finite non-zero uniserial module is cyclic and further isomorphic to $V/\mathfrak{m}^n$ for some $n \in \mathbb{N}$. Thus, since I is not principally generated, the first three quotients are infinite. Thus

$$\left|\frac{(I:ag)\cap bV}{(I:a)+bhV}\right| = \left|\frac{V}{hV}\right| = q^v$$

if and only if $hV = \mathfrak{m}^v$. □

Using section 7 we get the dual statement as a corollary. This statement could alternatively be proved by elementary but tedious calculations (see [Gre11]). This corollary will not be used later but we include it to show explicitly how the duality works.

Corollary 6.8. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Let $v \in \mathbb{N}\backslash\{0\}$, let φ be the pp-formula $(xag = 0 \wedge b|x)$ and let ψ be the pp-formula $(xa = 0 + bh|x)$*

where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Suppose $J \lhd V$ is a proper ideal such that $J^{\#} \subsetneq \mathfrak{m}$. Then

$$\left|\frac{\varphi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = \begin{cases} 1, & \text{if } ab \in I \text{ or } abgh \notin I \text{ or } g \notin I^{\#}; \\ q^v, & ab \notin J,\ abgh \in J,\ h \in J^{\#} \text{ and } gV = \mathfrak{m}^v; \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. By proposition 5.6

$$\left|\frac{\varphi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = \left|\frac{D\psi(N(J,\mathfrak{m}))}{D\varphi(N(J,\mathfrak{m}))}\right|.$$

Note that $D\varphi$ is $(ag|x + xb = 0)$ and $D\psi$ is $a|x \wedge xbh = 0$. Thus, proposition 6.7 gives the required statement. □

By a **boolean combination of conditions on an ideal** we mean a boolean combination Δ of conditions of the form $r \in I$ and $s \in I^{\#}$ where $r, s \in V$. We will say that an ideal $J \lhd V$ satisfies Δ if when we replace the symbol I by J the statement is true. We will write $\perp$ for the condition on an ideal which is false for all ideals. In what follows, when V is an effectively given valuation domain with non-dense value group, k will denote a fixed generator for the maximal ideal of V.

Proposition 6.9. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and φ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal, such that for all $I \lhd V$, I satisfies Δ if and only if $I^{\#} \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\varphi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| \geq q^v.$$

Proof. We start with the special case where φ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$ for some $\alpha, \beta, \gamma, \delta \in V$.

First note that if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$ then for all V-modules M, $\left|\frac{\varphi(M)}{\psi(M)}\right| = 1$. We can effectively check if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$. In this situation let $\Delta = \perp$.

Otherwise let $a = \gamma$, $b = \beta$, $g = \alpha/\gamma$ and $h = \delta/\beta$.

By proposition 6.7, if $I^{\#} \subsetneq \mathfrak{m}$, the following statements are equivalent:

(1) $\left|\frac{\varphi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v$.

(2) $abgh \in I$, $ab \notin I$, $g \in I^{\#}$ and $h = \mathfrak{m}^v$.

The condition $hV = \mathfrak{m}^v$ is equivalent to k^v divides h and k^{v+1} does not divide h. This can be effectively checked. So, if $\mathfrak{m}^v \neq hV$, let $\Delta = \perp$. If $k^v V = hV$, let Δ be

$$(abgh \in I) \wedge (ab \notin I) \wedge (g \in I^{\#}) \wedge (k \notin I^{\#}).$$

Now suppose that φ and ψ are arbitrary pp-1-formulae. By lemmas 4.1 and 4.2 we can effectively rewrite φ as $\sum_{i=1}^{n} \varphi_i$ where φ_i is $(xa_i = 0 \wedge b_i|x)$and

ψ as $\bigwedge_{j=1}^m \psi_j$ where ψ_j is $(xc_j = 0 + d_j|x)$. Then by corollary 6.3, for any pure-injective module N

$$\left|\frac{\varphi(N)}{\psi(N)}\right| = \max_{i,j}\left\{\left|\frac{\varphi_i(N)}{\psi_j(N)}\right|\right\}.$$

We can now use the above special case to effectively produce an appropriate boolean combination of conditions on an ideal. □

Since $|\varphi/\psi|$ can only take values of the form q^v for some $v \in \mathbb{N}_0$ this means:

Corollary 6.10. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}_0$ and φ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal, such that for all $I \lhd V$, I satisfies Δ if and only if $I^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\varphi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v.$$

Taking the dual of a pp-formula is clearly effective. Thus we may now use section 5 to get the dual statements as corollaries.

Corollary 6.11. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and φ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal, such that for all $J \lhd V$, J satisfies Δ if and only if $J^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\varphi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| \geq q^v.$$

Corollary 6.12. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}_0$ and φ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal, such that for all $J \lhd V$, J satisfies Δ if and only if $J^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\varphi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = q^v.$$

Proposition 6.13. *Let V be an effectively given valuation domain with an algorithm which, given $a, b \in V$, answers whether $a \in \operatorname{rad}(bV)$. There exists an algorithm which, given a boolean combination of conditions on an ideal Δ, answers whether there is an ideal $J \lhd V$ satisfying Δ.*

Proof. In order to show that we can effectively decide whether there exists an ideal $J \lhd V$ satisfying a given boolean combination of conditions on an ideal, it is enough to show that we can effectively decide whether there exists an ideal $J \lhd V$ satisfying a condition of the following form:

$$(*) \qquad \left(\bigwedge_{g=1}^{k} r_g \in J\right) \wedge \left(\bigwedge_{h=1}^{l} s_h \notin J\right) \wedge \left(\bigwedge_{i=1}^{m} t_i \in J^\#\right) \wedge \left(\bigwedge_{j=1}^{n} u_j \notin J^\#\right).$$

where $k, l, m, n \in \mathbb{N}$ and $r_g, s_h, t_i, u_j \in V$ for $1 \leq g \leq k$, $1 \leq h \leq l$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
Since V is a valuation domain there exists $1 \leq g \leq k$ such that r_g generates the ideal $r_1V + ... + r_kV$, let $r = r_g$. There exists $1 \leq i \leq m$ such that t_i generates the ideal $t_1V + ... + t_mV$, let $t = t_i$. There exists $1 \leq h \leq l$ such that s_h generates $\cap_{h=1}^{l} s_hV$, $s = s_h$. There exists $1 \leq j \leq n$ such that u_j generates $\cap_{j=1}^{n} u_jV$, let $u = u_j$. It is clear that such r, s, t and u can be found effectively.
Note that $J \lhd V$ satisfies $(*)$ if and only if $r \in J$, $s \notin J$, $t \in J^{\#}$ and $u \notin J^{\#}$.

Claim: For any $r, s, t, u \in V$, there exists $J \lhd V$ such that $r \in J$, $s \notin J$, $t \in J^{\#}$ and $u \notin J^{\#}$ if and only if s divides r, $u \notin \text{rad}(tV)$ and $u \notin \text{rad}((r/s)V)$.

Suppose $J \lhd V$ and $r \in J$, $s \notin J$, $t \in J^{\#}$ and $u \notin J^{\#}$. Since $J^{\#}$ is prime and $t \in J^{\#}$, $\text{rad}(tV) \subseteq J^{\#}$. Therefore $u \notin \text{rad}(tV)$. Clearly s divides r. Let $\gamma = r/s$. Then $s \notin J$ and $\gamma s \in J$ so $\gamma \in J^{\#}$. Therefore $\text{rad}(\gamma V) \subseteq J^{\#}$ so $u \notin \text{rad}(\gamma V)$.
Suppose s divides r, $u \notin \text{rad}(tV)$ and $u \notin \text{rad}(r/sV)$. Let $\gamma = r/s$ and $J = s(\text{rad}(tV) \cup \text{rad}(\gamma V))$. Then $J^{\#} = \text{rad}(tV) \cup \text{rad}(\gamma V)$ so $t \in J^{\#}$ and $u \notin J^{\#}$. Clearly $s \notin J$ and $\gamma \in \text{rad}(\gamma V)$ so $r = s\gamma \in J$. $\square$

By a **boolean combination of conditions on an element** we mean a boolean combination Δ of conditions of the form $x \in rV$ where $r \in V$. We will say that an element $w \in V$ satisfies Δ if when we replace the symbol x by w the statement is true. We will write $\perp$ for the condition on an element which is false for all elements.

Lemma 6.14. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and φ, ψ pp-1-formulae, produces Δ, a boolean combination of conditions on an element, such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left| \frac{\varphi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))} \right| \geq q^v.$$

Proof. We start with the special case where φ is $x\alpha = 0 \wedge \beta | x$ and ψ is $x\gamma = 0 + \delta | x$ for some $\alpha, \beta, \gamma, \delta \in V$.
As in proposition 6.9 if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$ then for all V-modules M, $\left| \frac{\varphi(M)}{\psi(M)} \right| = 1$. We can effectively check if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$. In this situation let $\Delta = \perp$.
Otherwise let $a = \gamma$, $b = \beta$, $g = \alpha/\gamma$ and $h = \delta/\beta$.
For $x \in \mathfrak{m}$, $N(\mathfrak{m}, xV)$ is an abnormal point since $\mathfrak{m}^2 \neq \mathfrak{m}$(see proposition 4.12 (i)). Thus $N(\mathfrak{m}, xV) \in (\varphi/\psi)$ is equivalent to $ab\mathfrak{m} \supsetneq x\mathfrak{m}$ and $abgh \in x\mathfrak{m}$ since $g, h \in \mathfrak{m}$. Note that since $\mathfrak{m}$ is finitely generated, $ab\mathfrak{m} \supsetneq x\mathfrak{m}$ if and only if $ab \notin xV$.

By lemma 6.6, if $N(\mathfrak{m}, xV) \in (\varphi/\psi)$ then

$$\left|\frac{\varphi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| \geq q^v$$

if and only if

$$|V/gV| \geq q^v \ , \ |V/hV| \geq q^v \ , \ |xV/abghV| \geq q^v \ , \ |abV/xV| \geq q^v.$$

Note that if $c, d \in V$ with $d \in cV$ then $|cV/dV| \geq q^v$ if and only if $d \in c\mathfrak{m}^v$. If $g \notin \mathfrak{m}^v$ or $h \notin \mathfrak{m}^v$ then let $\Delta = \perp$ (note that this can be effectively checked). Otherwise, let $r = g/k^v$ (we can effectively calculate r). Note that the condition $x \notin abrhkV$ is the same as $abrh \in xV$, which is the same as $abgh \in xk^vV$.
Let Δ be

$$x \in abk^vV \wedge x \notin abrhkV.$$

For arbitrary pp-formulae use lemmas 4.1, 4.2 and corollary 6.3 as in proposition 6.9. □

Corollary 6.15. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}_0$ and φ, ψ pp-1-formulae, produces Δ, a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left|\frac{\varphi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| = q^v.$$

Lemma 6.16. *Let V be an effectively given valuation domain. There exists an algorithm which, given Δ a boolean combination of conditions on an element, answers whether there exists $x \in V$ satisfying Δ.*

Proof. In order to show that we can effectively decide whether there exists $x \in V$ satisfying a given boolean combination of conditions on an element, it is enough to show that we can effectively decide whether there exists $x \in V$ satisfying a condition of the form:

$$\Delta = \bigwedge_{i=1}^{n} (x \in r_iV) \wedge \bigwedge_{j=1}^{m} (x \notin s_jV)$$

where $n, m \in \mathbb{N}$ and $r_i, s_j \in V$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Since V is a valuation domain there exists $1 \leq i \leq n$ such that $r_iV = \cap_{i=1}^{n} r_iV$, let $r = r_i$ and note that we can effectively find such an i. Again, since V is a valuation domain there exists $1 \leq j \leq m$ such that $s_jV = \cup_{j=1}^{m} s_jV$, let $s = s_j$ and note we that we can effectively find such a j.
There exists x satisfying Δ if and only if there exists $x \in V$ such that $x \in rV$ and $x \notin sV$ if and only if $sV \subsetneq rV$ if and only if $s \in r\mathfrak{m}$. Given any $r, s \in V$ we can effectively answer whether $s \in r\mathfrak{m}$. □

7. Main theorem

Theorem 7.1. *Let V be an effectively given valuation domain. The following are equivalent:*

(i) The theory of V-modules, T_V, is decidable.
(ii) There exists an algorithm which, given $a, b \in V$, answers whether $a \in \operatorname{rad}(bV)$.

Proof. For the cases where V has infinite residue field or dense value group we refer the reader to the proofs of Theorem 6.2 and Theorem 8.2 of [PPT07] where the only missing ingredient for valuation domains with non-archimedean value groups is an algorithm for answering whether one Ziegler basic open set is contained in a finite union of others (we produced such an algorithm in section 4).

Let V be an effectively given valuation domain with finite residue field and non-dense value group such that there is an algorithm which, given $a, b \in V$ answers whether $a \in \operatorname{rad}(bV)$. First note that since V is effectively given, T_V is recursively axiomatised. Hence we have an algorithm which produces a list of all sentences true in all V-modules. Since the theory T_V is not complete, in order to show that T_V is decidable, we need to effectively produce a list of sentences which are false in some V-module. Equivalently, we need to effectively produce a list of sentences which are true in at least one V-module.

By the Baur-Monk theorem, every sentence is equivalent to a boolean combination of invariant sentences. Since T_V is recursively axiomatised, given a sentence χ we can effectively find a boolean combination of invariant sentences equivalent to χ.

Thus, it is enough to show that there is an algorithm which given a conjunction of invariant sentences and negations of invariant sentences χ, answers whether there exists a module M satisfying χ. Suppose χ is a conjunction of the following sentences:

$$(1)\ \left|\frac{\varphi_i^1}{\psi_i^1}\right| = q^{v_i} \qquad (2)\ \left|\frac{\varphi_j^2}{\psi_j^2}\right| \geq q^{w_j} \qquad (3)\ \left|\frac{\varphi_k^3}{\psi_k^3}\right| = 1$$

where $l, m, n \in \mathbb{N}$ and for all $1 \leq i \leq l$, $1 \leq j \leq m$, $1 \leq k \leq n$, φ_i^1, ψ_i^1, φ_j^2, ψ_j^2, φ_k^3, ψ_k^3 are pp-1-formulae and $v_i, w_j \in \mathbb{N}$.

It is enough to consider sentences of this form as any finite V-module is either the zero module or has q^v elements for some strictly positive $v \in \mathbb{N}$, by lemma 6.5.

If τ is a conjunction of invariant sentences like those in (1), (2) and (3) then we call $\sum_{i=1}^l v_i$ the exponent of the statement.

We proceed by induction on $\sum_{i=1}^l v_i$.

First consider the situation when $\sum_{i=1}^l v_i = 0$, that is, (1) is empty. Suppose there exists a module M satisfying χ. We may assume $M = \bigoplus_{\mu \in \mathcal{M}} N_\mu$, for some finite indexing set $\mathcal{M}$. Therefore, for each $1 \leq j \leq m$, there is $\mu \in \mathcal{M}$

such that

$$\left|\frac{\varphi_j^2(N_\mu)}{\psi_j^2(N_\mu)}\right| > 1$$

and for all $\mu \in \mathcal{M}$ and all $1 \leq k \leq n$,

$$\left|\frac{\varphi_k^3(N_\mu)}{\psi_k^3(N_\mu)}\right| = 1.$$

Hence, for each $1 \leq j \leq m$, there exists N_μ such that $N_\mu \in \left(\frac{\varphi_j^2}{\psi_j^2}\right)$ and $N_\mu \notin \left(\frac{\varphi_k^3}{\psi_k^3}\right)$ for all $1 \leq k \leq n$. For each $1 \leq j \leq m$, let N_j be such a module. Then there exists $t \in \mathbb{N}$ such that $\left(\bigoplus_{j=1}^m N_j\right)^t$ satisfies (2) and (3).
Hence, there exists a module M satisfying (2) and (3) if and only if for all $1 \leq j \leq m$

$$\left(\frac{\varphi_j^2}{\psi_j^2}\right) \not\subseteq \bigcup_{k=1}^n \left(\frac{\varphi_k^3}{\psi_k^3}\right).$$

Theorem 4.27 asserts that there exists an algorithm to check this, so we are done.
Now suppose $L = \sum_{i=1}^l v_i > 0$, so (1) is not empty and that for any conjunction Θ of invariant sentences and negations of invariant sentences with exponent strictly smaller that L, there is an algorithm which answers whether there exists a module M satisfying Θ.
Suppose there exists M satisfying χ. We may assume $M = \bigoplus_{\mu \in \mathcal{M}} N_\mu$ where $\mathcal{M}$ is a finite indexing set and each N_μ is an indecomposable pure-injective module. Hence there exists $\mu \in \mathcal{M}$ such that

$$q \leq \left|\frac{\varphi_1^1(N_\mu)}{\psi_1^1(N_\mu)}\right| \leq q^{v_1}$$

and for all $\mu \in \mathcal{M}$, for all $1 \leq i \leq l$ and for all $1 \leq k \leq n$

$$\left|\frac{\varphi_i^1(N_\mu)}{\psi_i^1(N_\mu)}\right| \leq q^{v_i} \text{ and } \left|\frac{\varphi_k^3(N_\mu)}{\psi_k^3(N_\mu)}\right| = 1.$$

Let $\mathcal{U}$ be the set of functions $u : \{1, ..., l+m\} \to \mathbb{N} \cup \{\infty\}$. Let $\mathcal{U}^*$ be the subset of $\mathcal{U}$ consisting of functions $u \in \mathcal{U}$ such that $1 \leq u(1) \leq v_1$, for all $2 \leq i \leq l$, $0 \leq u(i) \leq v_i$ and for all $1 \leq j \leq m$, either $0 \leq u(l+j) < w_j$ or $u(l+j) = \infty$. Note that $\mathcal{U}^*$ is a finite set.
We now show that for each $u \in \mathcal{U}^*$ we can effectively answer whether there exists an indecomposable pure-injective V-module satisfying the following sentences:

(i) $\left|\frac{\varphi_i^1}{\psi_i^1}\right| = q^{u(i)}$.

(ii) If $u(j+l) \neq \infty$, $\left|\frac{\varphi_j^2}{\psi_j^2}\right| = q^{u(j+l)}$. Otherwise $\left|\frac{\varphi_j^2}{\psi_j^2}\right| \geq q^{w_j}$.

(iii) $\left|\frac{\varphi_k^3}{\psi_k^3}\right| = 1$.

Since $1 \leq u(1)$, by lemma 6.4 if $I, J \lhd V$ are such that $N(I,J)$ satisfies (i), (ii) and (iii) then either $I^{\#} = \mathfrak{m}$ or $J^{\#} = \mathfrak{m}$. So, if $N(I,J)$ satisfies (i), (ii) and (iii), then we may assume either $I = \mathfrak{m}$ and $J = xV$ for some $x \in \mathfrak{m}$, $I = \mathfrak{m}$ and $J^{\#} \subsetneq \mathfrak{m}$ or $J = \mathfrak{m}$ and $I^{\#} \subsetneq \mathfrak{m}$.
Therefore it is enough to show how to answer the following 3 questions effectively:

Question 1: Does there exist $x \in \mathfrak{m}$ such that $N(\mathfrak{m}, xV)$ satisfies (i),(ii) and (iii)?
By corollary 6.15, given any sentence $\left|\frac{\varphi}{\psi}\right| = q^v$ where φ, ψ are pp-1-formulae and $v \in \mathbb{N}_0$ we can effectively produce Ω a boolean combination of conditions on an element such that $x \in V$ satisfies Ω if and only if $x \in \mathfrak{m}$ and $\left|\frac{\varphi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = q^v$. By lemma 6.14, given any sentence $\left|\frac{\varphi}{\psi}\right| \geq q^v$ where φ, ψ are pp-1-formulae and $v \in \mathbb{N}$ we can effectively produce Ω a boolean combination of conditions on an element such that $x \in V$ satisfies Ω if and only if $x \in \mathfrak{m}$ and $\left|\frac{\varphi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$.
Hence we can effectively produce a boolean combination of conditions Θ on an element $x \in V$ such that x satisfies Θ if and only if $x \in \mathfrak{m}$ and $N(\mathfrak{m}, xV)$ satisfies (i), (ii) and (iii).
By lemma 6.16, we can effectively decide whether there exists $x \in V$ satisfying Θ.

Question 2: Does there exist $I \lhd V$ such that $I^{\#} \subsetneq \mathfrak{m}$ and $N(I, \mathfrak{m})$ satisfies (i), (ii) and (iii)?
Use corollary 6.10 and proposition 6.9 to produce Θ a boolean condition on an ideal such that $I \lhd V$ satisfies Θ if and only if $I^{\#} \subsetneq \mathfrak{m}$ and $N(I, \mathfrak{m})$ satisfies (i), (ii) and (iii). By proposition 6.13, we can effectively decide whether there exists $I \lhd V$ satisfying Θ.

Question 3: Does there exist $J \lhd V$ such that $J^{\#} \subsetneq \mathfrak{m}$ and $N(\mathfrak{m}, J)$ satisfies (i), (ii) and (iii)?
Same as question 2 replacing corollary 6.10 by corollary 6.12 and proposition 6.9 by corollary 6.11.
Let $\mathcal{U}^{**}$ be the set of $u \in \mathcal{U}^*$ such that an indecomposable pure-injective N exists satisfying (i),(ii) and (iii). If $\mathcal{U}^{**}$ is empty then there does not exist a module M satisfying (1), (2) and (3).
For each $u \in \mathcal{U}^{**}$ we effectively produce a new list of sentences $(1)^u$, $(2)^u$ and $(3)^u$. For each u start with $(1)^u$ and $(2)^u$ empty, and $(3)^u$ containing all sentences in (3).
For each $1 \leq i \leq l$, if $u(i) < v_i$, add the sentence $\left|\frac{\varphi_i^1}{\psi_i^1}\right| = q^{v_i - u(i)}$ to $(1)^u$. If $u(i) = v_i$, add the sentence $\left|\frac{\varphi_i^1}{\psi_i^1}\right| = 1$ to $(3)^u$. For each $1 \leq j \leq m$, if $u(l+j) < w_j$, add the sentence $\left|\frac{\varphi_j^2}{\psi_j^2}\right| \geq q^{w_j - u(l+j)}$ to $(2)^u$.

Now there exists a module M satisfying (1), (2) and (3) if and only if there exists a module M' satisfying $(1)^u$, $(2)^u$ and $(3)^u$ for some $u \in \mathcal{U}^{**}$.
Note that for each $u \in \mathcal{U}^{**}$ the exponent of the conjunction of conditions in $(1)^u$ is strictly smaller than $L = \sum_{i=1}^{l} v_i$. Hence by the induction hypothesis, for each $u \in \mathcal{U}^{**}$ there is an algorithm which answers whether there exists a module satisfying $(1)^u$, $(2)^u$ and $(3)^u$.
The other direction is lemma 3.2. □

8. An effectively given valuation domain with undecidable theory of modules

In this section we *sketch* how to construct a valuation domain with infinite Krull dimension which has decidable theory of modules with respect one effective presentation and undecidable theory of modules with respect to another. We do this by constructing a recursively presented totally ordered abelian group Γ (which is classically isomorphic to $\oplus_\omega \mathbb{Z}$) such that the relation $\ll$ on Γ, given by $a \ll b$ if and only if $n|a| < |b|$ for all $n \in \mathbb{N}$, codes up the halting problem. We then construct an effectively given valuation domain V out of fractions of polynomials with exponents in Γ such that the $\ll$ relation on Γ becomes the radical relation on V.
We show that valuation domains with finite Krull dimension have decidable theory of modules with respect to any effective presentation.

Group construction: Let Γ be the abelian group generated by the set $\{N_i | i \in \mathbb{N}\} \cup \{\epsilon_i | i \in \mathbb{N}\}$ with the relation $\epsilon_i = nN_i$ holding if and only if program i halts after exactly n steps. Note that for $n_i, m_j \in \mathbb{Z}$

$$\sum_{i=1}^{t} n_i N_i + \sum_{j=1}^{t} m_j \epsilon_j = 0$$

if and only if

$$n_i N_i + m_i \epsilon_i = 0$$

for $1 \leq i \leq t$ if and only if $-n_i/m_i \in \mathbb{N}$ and

$$-n_i/m_i N_i = \epsilon_i$$

if and only if the ith program halts after exactly $-n_i/m_i$ steps. So we can compute equality of elements in our group.
We now put an order on Γ. Set $0 < nN_i < N_j$ for all $n \in \mathbb{N}$ and $i < j$. Set $n\epsilon_i < N_j$ for all $n \in \mathbb{N}$ and all $i < j$. Set $nN_i < \epsilon_i$ if the ith program takes more that n steps to halt. Note that

$$\sum_{i=1}^{t} n_i N_i + \sum_{i=1}^{t} m_i \epsilon_i > 0$$

if and only if there exists a $1 \leq j \leq t$ such that for all $i > j$

$$n_i N_i + m_i \epsilon_i = 0 \quad \text{and} \quad n_j N_j + m_j \epsilon_j > 0.$$

Thus there is a recursive presentation of Γ as a totally ordered abelian group such that the sets $\{N_i \mid i \in \mathbb{N}\}$ and $\{\epsilon_i \mid i \in \mathbb{N}\}$ are recursive but of course the $\ll$ relation is not, as if it were then we could decide the halting problem. Let this recursive presentation be given by a bijective map $\lambda : \mathbb{N} \to \Gamma$. Note that this group is classically isomorphic to $\oplus_\omega \mathbb{Z}$ lexicographically ordered.

Valuation domain construction: Let F be any recursive field. Let $\pi : \mathbb{N} \to F(\Gamma)$ be a recursive presentation of the field of fractions of the group ring $F\Gamma$ coded up by pairs in $F\Gamma$ such that the map from Γ to $F(\Gamma)$ defined by $g \mapsto t^g$ induce a recursive function from $\mathbb{N}$ to $\mathbb{N}$ via π and λ and such that the map $v : F(\Gamma) \mapsto \Gamma \cup \{\infty\}$ given by

$$(\sum_{g\in\Gamma} \mu_g t^g, \sum_{g\in\Gamma} \lambda_g t^g) \mapsto \min\{g \in \Gamma \mid \mu_g \neq 0\} - \min\{g \in \Gamma \mid \lambda_g \neq 0\}$$

if $\sum_{g\in\Gamma} \mu_g t^g \neq 0$ and

$$(\sum_{g\in\Gamma} \mu_g t^g, \sum_{g\in\Gamma} \lambda_g t^g) \mapsto \infty$$

otherwise, also induces a recursive function from $\mathbb{N}$ to $\mathbb{N}$ via π and λ.
Note that v defines a valuation on the field $F(\Gamma)$ and is recursive. Thus v defines a valuation domain V as a recursive subset (via π) of $F(\Gamma)$ and we have a recursive section from $\Gamma_{\geq 0}$ to V. This of course means we may now define a bijective map $i : \mathbb{N} \to V$ such that $\pi^{-1} \circ l \circ i$ is a recursive map from $\mathbb{N}$ to $\mathbb{N}$ where l is the inclusion of V in $F(\Gamma)$. The valuation map from V to $\Gamma_{\geq 0}$ induces a recursive function from $\mathbb{N}$ to $\mathbb{N}$ via i and λ and has a recursive section (via i and λ) from $\Gamma_{\geq 0}$ into V.
Suppose $g, h \in \Gamma_{\geq 0}$. Then $ng < h$ for all $n \in \mathbb{N}$ if and only if $(t^g)^n \notin t^h V$ for all $n \in \mathbb{N}$, which is if and only if $t^g \notin \mathrm{rad}(t^h V)$. Thus the radical relation on V can not possibly be recursive as then the $\ll$ relation on Γ would be recursive.
If on the other hand we were to take a recursive presentation of $\oplus_\omega \mathbb{Z}$ in which the $\ll$ relation is recursive and construct a valuation domain as above, then we get a valuation domain with decidable theory of modules.
The same construction would still work if we replace $\oplus_\omega \mathbb{Z}$ lexicographically ordered by $\oplus_\omega \mathbb{Q}$ lexicographically ordered. Thus non-density of the value group is not important.

Proposition 8.1. *Let V be an effectively given valuation domain with finite Krull dimension. Then the theory of V modules is decidable.*

Proof. Suppose V has prime ideals

$$\mathfrak{p}_m := \mathfrak{m} \supsetneq ... \supsetneq \mathfrak{p}_2 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0 := 0.$$

For $0 \leq i \leq m$ fix b_i such that $\mathrm{rad}(b_i V) = \mathfrak{p}_i$. We describe an algorithm which given $a \in V$ outputs $0 \leq i \leq m$ such that $\mathrm{rad}(aV) = \mathrm{rad}(b_i V)$ and $m+1$ if $a \notin \mathfrak{m}$. We can effectively decide whether $a \in \mathfrak{m}$ and if not output $m+1$. If $a = 0$ then output 0. Now assume that $a \in \mathfrak{m}$ is

non-zero and find $0 \leq i \leq m$ such that $a \in b_{i+1}V$ and $a \notin b_iV$. Such an i exists since a is non-zero and we can do this effectively since V is effectively given. Thus $\mathrm{rad}(aV) = \mathrm{rad}(b_iV)$ or $\mathrm{rad}(aV) = \mathrm{rad}(b_{i+1}V)$. Now $a \in \mathrm{rad}(b_iV)$ if and only if there exists an $n \in \mathbb{N}$ such that $a^n \in b_iV$ and $b_{i+1} \in \mathrm{rad}(aV)$ if and only if there exists an $n \in \mathbb{N}$ such that $b_{i+1}^n \in aV$. Exactly one of these two possibilities must occur. Thus, in order to check whether $\mathrm{rad}(aV) = \mathrm{rad}(b_iV)$ or $\mathrm{rad}(aV) = \mathrm{rad}(b_{i+1}V)$ we must for each $n \in \mathbb{N}$ ask whether $b_{i+1}^n \in aV$ or $a^n \in b_iV$.
Now if we are given $a, c \in V$ we may effectively find $0 \leq i, j \leq m$ such that $\mathrm{rad}(aV) = \mathrm{rad}(b_iV)$ and $\mathrm{rad}(cV) = \mathrm{rad}(b_jV)$. If $i \leq j$ then $a \in \mathrm{rad}(cV)$. $\square$

Acknowledgements The results in this paper were part of my PhD thesis under the supervision of Mike Prest at The University of Manchester. I am very grateful for Mike's support and patient supervision. I would like to thank Merlin Carl for helping construct the recursive group in section 8. I would like to thank my thesis examiners, Dugald Macpherson and Alex Wilkie, for their comments and suggestions.

References

[FS01] László Fuchs and Luigi Salce. *Modules over non-Noetherian domains*, volume 84 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[Gre11] Lorna Gregory. *Ziegler Spectra of Valuation Rings*. Phd thesis, University of Manchester, 2011.

[Gre13] Lorna Gregory. Sobriety for the Ziegler spectrum of a Prüfer domain. *J. Pure Appl. Algebra. http://dx.doi.org/10.1016/j.jpaa.2013.01.007*, 2013.

[Her93] Ivo Herzog. Elementary duality of modules. *Trans. Amer. Math. Soc.*, 340(1):37–69, 1993.

[PPT07] Gennadi Puninski, Vera Puninskaya, and Carlo Toffalori. Decidability of the theory of modules over commutative valuation domains. *Ann. Pure Appl. Logic*, 145(3):258–275, 2007.

[Pre88] Mike Prest. *Model theory and modules*, volume 130 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[Pun99] Gennadi Puninski. Cantor-Bendixson rank of the Ziegler spectrum over a commutative valuation domain. *J. Symbolic Logic*, 64(4):1512–1518, 1999.

[Pun01] Gennadi Puninski. *Serial rings*. Kluwer Academic Publishers, Dordrecht, 2001.

[Zie84] Martin Ziegler. Model theory of modules. *Ann. Pure Appl. Logic*, 26(2):149–213, 1984.

Fachbereich Mathematik und Statistik, Universität Konstanz, D-78457 Konstanz, Deutschland
E-mail address: Lorna.Gregory@uni-konstanz.de