Fachbereich
Mathematik und Statistik

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Lineare Algebra I
Wiederholungsblatt

Solution W1

See your lecture notes.

Solution W2

(a) TRUE Suppose $v, w \in V$. Then $f(v) = f(w)$ if and only if $f(v-w) = 0$, since f is linear. If $\ker f = 0$ then $v - w = 0$, so $v = w$. Hence f is injective. Conversely, if f is injective then $f(v) = 0$ implies $v = 0$. So $\ker f = 0$.

(b) FALSE For any scalar product $\langle \cdot, \cdot \rangle$, $\langle v, v \rangle$ takes only non-negative values.

(c) TRUE Every basis for U can be extended to a basis for V.

(d) FALSE For instance, a 1×1 matrix can not be multiplied by a 2×2 matrix.

(e) TRUE If f were linear, then for all $v \in V$, $f(0) = f(0.v) = 0.f(v) = 0$.

(f) FALSE Take $V = \mathbb{R}^2$ as an $\mathbb{R}$-vector space and $v_1 = (0,0)$, $v_2 = (1,0)$. Then v_2 is not a linear combination of v_1.

(g) TRUE Since $\{v_1, ..., v_n, v_{n+1}\}$ has $n+1$ elements and $\dim V = n$, there exists $0 < i \leq n+1$ such that v_i is a linear combination of elements in $\{v_1, ..., v_{n+1}\} \backslash \{v_i\}$. The set $\{v_1, ..., v_{n+1}\} \backslash \{v_i\}$ spans V. Since $\dim V = n$, $\{v_1, ..., v_{n+1}\} \backslash \{v_i\}$ is a basis for V. Therefore $\{v_1, ..., v_{n+1}\} \backslash \{v_i\}$ is linearly independent.

(h) TRUE Suppose V, U, W are K-vector spaces and $f : V \to U$, $g : U \to W$ are linear maps. Let $u, v \in V$ and $\lambda \in K$. Then

$$(g \circ f)(u + \lambda.v) = g(f(u + \lambda v)) = g(f(u) + \lambda.f(v)) = g(f(u)) + \lambda.g(f(v)) = (g \circ f)(u) + \lambda.(g \circ f)(v).$$

So $g \circ f$ is linear.

(i) TRUE See lecture notes.

(j) TRUE See lecture notes.

(k) TRUE Check group axioms.

(l) FALSE Let $V = \mathbb{R}$ as an $\mathbb{R}$-vector space. No basis contains both 1 and 2 since they are linearly dependent but $1 \neq 2$.

(m) TRUE Any set containing the zero vector in linearly dependent since $1.0 = 0$.

(n) FALSE Let $V = \mathbb{R}$ as an $\mathbb{R}$-vector space. The vectors $1, 2, 3$ span (generate) V but V has dimension 1.

(o) FALSE For any vector space V and any non-zero $w \in V$ the function $f(v) = w - v$ is not linear since $f(0) = w \neq 0$.

(p) TRUE Since x, y, z are linearly dependent, there exists $a, b, c \in K$ not all zero such that $a.x + b.y + c.z = 0$. Since x, y are linearly independent, $c \neq 0$. Therefore $z = -a/c.x - b/c.y$.

(q) FALSE Let $V = \mathbb{R}^2$ and $x = (1,0), y = (0,1), z = (0,1)$. Then x, y are linearly independent, x, y, z are linearly dependent and x is not in the span of y, z.

(r) TRUE Suppose x, y are linearly dependent. Then there exists $a, b \in \mathbb{K}$ (with at least one of a, b non-zero) such that $ax + by = 0$. If $a \neq 0$ then $x = -b/ay$. If $b \neq 0$ then $y = -a/bx$.

(s) TRUE For any such f, $\dim \ker f + \dim \mathrm{im} f = \dim V$. So $\dim f \leq \dim V$.

(t) TRUE

(u) FALSE Let $V = \mathbb{R}^2$ and U be the space spanned by $(1,1)$. Then $(1,0), (0,1)$ is a basis for V and neither vector is contained in U.

(v) TRUE Suppose $v \in V$. Then $f(v) \in \mathrm{im} f$. So $f(v) \in \ker f$. So $f^2(v) = f(f(v)) = 0$. Therefore $f^2 = 0$.

(w) TRUE See Bemerkung 2.1.11.

(x) TRUE Suppose that $v \in V$ and v is non-zero. Then $\langle v, v \rangle > 0$. Let $\langle v, v \rangle = t$. For any strictly positive $s \in \mathbb{R}$, s/t has a square root in $\mathbb{R}$. We have that $\langle \sqrt{s/t}v, \sqrt{s/t}v \rangle = s$ and $\langle 0, 0 \rangle = 0$. So $\langle v, v \rangle$ takes all positive values in $\mathbb{R}$.

Solution W3

(i) The subset U is not a subspace, since U does not contain the zero vector.

(ii) The subset U is a subspace. For all $x \in \mathbb{R}$, $x^2 \geq 0$. Therefore $x_1^2 \leq 0$ implies $x_1 = 0$. Let $x, y \in U$ and $r \in \mathbb{R}$. Then

$$x = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix}$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$. So

$$x + ry = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + r \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 + ry_2 \\ x_3 + ry_3 \end{pmatrix} \in U.$$

Therefore U is a subspace of $\mathbb{R}^3$.

A basis for U is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The dimension is 2.

(iii) The subset U is a subspace. Let $x, y \in U$ and $r \in \mathbb{R}$. There exists $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{pmatrix}.$$

So

$$x + ry = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix} + r \begin{pmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + ry_1 \\ x_2 + ry_2 \\ x_1 + x_2 + r(y_1 + y_2) \end{pmatrix} \in U$$

since $(x_1 + ry_1) + x_2 + ry_2 = x_1 + x_2 + r(y_1 + y_2)$.

A basis for U is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(iv) The subset U is not a subspace, since $(1, 0, 0)$ and $(0, 1, 0)$ are in U but $(1, 1, 0)$ is not.

(v) The subset U is not a subspace, since $(2, 0, 0)$ is in U but $(4, 0, 0)$ is not.

(vi) This question was modified so that $\alpha(x_1, x_2, x_3) = (0, x_1 - 2x_2, x_3 - x_2, x_2 - 4x_1)$. The kernel of a linear map is always a subspace. A vector (x_1, x_2, x_3) is in the kernel of α if and only if $x_1 = 2x_2$, $x_3 = x_2$ and $x_2 = 4x_1$. It is easy to see that this implies that $x_1 = x_2 = x_3 = 0$. Therefore the kernel is the zero subspace.

Solution W4

(a) It is equivalent to show that if $u, \alpha u, \alpha^2 u, ..., \alpha^{n-1} u$ are linearly dependent then $\alpha^{n-1} = 0$. Suppose that $b_i \in \mathbb{K}$ are such that $\sum_{i=0}^{n-1} b_i \alpha^i u = 0$ and not all b_i are zero. Let j be least such that $b_j \neq 0$. By applying α^{n-1+j} to $\sum_{i=0}^{n-1} b_i \alpha^i u$, we get $b_j \alpha^{n-1} u = 0$. Hence $\alpha^{n-1} u = 0$, since $b_j \neq 0$.

(b) Let $u_1, ..., u_{n+1}$ be a basis for V. The map $\alpha : V \to V$, defined by $\alpha(\sum_{i=1}^{n+1} b_i u_i) = \sum_{i=1}^{n} b_i u_{i+1}$ is a linear map and satisfies the required properties.

Solution W5

(a) There are exactly 3 possibilities for the value of ab. If $ab = a$ then $b = e$. If $ab = b$ then $a = e$. So $ab = e$.

(b) If $a^2 = a$ then $a = e$. If $a^2 = e$ then $aab = b$. So $a = b$. Thus $a^2 = b$. Similarly $b^2 = a$. Since right inverses are left inverses, $ba = e$.

$$\begin{array}{c||c|c|c} & e & a & b \\ \hline\hline e & e & a & b \\ \hline a & a & b & e \\ \hline b & b & e & a \end{array}.$$

Solution W6

Since $\mathbb{R}$ is a field, commutativity and associativity of addition and multiplication are inherited by $\mathbb{Q}[\sqrt{3}]$ from $\mathbb{R}$. Distributivity is also inherited from $\mathbb{R}$. The element $0 + 0\sqrt{3}$ is the zero element and the element $1 + 0\sqrt{3}$ is the multiplicative identity (to prove this, either do a simple computation or argue that these elements inherit their required properties from $\mathbb{R}$). If $a, b \in \mathbb{Q}$ then $a + b\sqrt{3}$ has additive inverse $(-a) + (-b)\sqrt{3}$.

It remains to show that every non-zero element in $\mathbb{Q}[\sqrt{3}]$ has a multiplicative inverse in $\mathbb{Q}[\sqrt{3}]$. In the following proof we will assume that $\sqrt{3} \notin \mathbb{Q}$. (incase you have not seen a proof of this, one is included at the end of this solution)

First note that, for all $a, b \in \mathbb{Q}$, $a + b\sqrt{3} = 0$ if and only if $a = b = 0$. This is because if $b \neq 0$ and $a + b\sqrt{3} = 0$ then $-a/b = \sqrt{3}$, contradicting the fact that $\sqrt{3} \notin \mathbb{Q}$. Therefore $b = 0$, so $a = 0$.

Suppose that $a + b\sqrt{3} \neq 0$ and $a, b \in \mathbb{Q}$. Then $a + (-b)\sqrt{3} \neq 0$. We are now ready to show that the multiplicative inverse (in $\mathbb{R}$) of $a + b\sqrt{3}$ is in $\mathbb{Q}[\sqrt{3}]$. Consider

$$\frac{1}{a + b\sqrt{3}} = \frac{a - b\sqrt{3}}{(a + b\sqrt{3})(a - b\sqrt{3})} = \frac{a}{a^2 - 3b^2} + \frac{-b}{a^2 - 3b^2}\sqrt{3}.$$

Proof that $\sqrt{3}$ is irrational:

Claim: Let $a \in \mathbb{N}$. Then 3 divides a^2 implies 3 divides a.

It is equivalent to show that a not divisible by 3 implies a^2 is not divisible by 3.

Suppose 3 does not divide a. Then $a = 3n+1$ or $a = 3n+2$ for some $n \in \mathbb{N}$. If $a = 3n+1$ then $a^2 = 3(3n^2 + 2n) + 1$, which is not divisible by 3. If $a = 3n + 2$ then $a^2 = 3(3n^2 + 4n + 1) + 1$ which is not divisible by 3. So we have proved the claim.

We are now ready to prove that $\sqrt{3}$ is irrational. Suppose, for a contradiction that $\sqrt{3} = a/b$ for integers a and b. Since $\sqrt{3}$ is positive in $\mathbb{R}$, we may assume that a and b are positive. By dividing through by the highest common factor of a and b, we may assume that a and b have highest common factor 1. Squaring both side of $\sqrt{3} = a/b$ we get that $3b^2 = a^2$. Therefore 3 divides a^2 and hence, using the claim, 3 divides a. Let $a_1 \in \mathbb{N}$ be such that $3a_1 = a$. Then $3b^2 = 9a_1^2$. Thus $b^2 = 3a_1^2$. So 3 divides b^2 and hence 3 divides b. This contradicts our assumption that a and b have highest common factor 1. Therefore $\sqrt{3}$ is irrational.

Solution W7
Let
$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
and let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$. Then

$$u_1 = e_2 + e_3, u_2 = e_1 + e_3, u_3 = e_1 + e_2.$$

So the matrix
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
takes a vector written in terms of u_1, u_2, u_3 and writes it in terms of e_1, e_2, e_3.
Now let
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
and let e_1, e_2, e_3, e_4 be the standard basis for $\mathbb{R}^4$. Then

$$e_1 = v_1, e_2 = -v_1 + v_2, e_3 = -v_2 + v_3, e_4 = -v_3 + v_4.$$

So the matrix
$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
takes a vector written in terms of e_1, e_2, e_3, e_4 and writes it in terms of v_1, v_2, v_3, v_4.
Thus α in terms of (u_1, u_2, u_3) and (v_1, v_2, v_3, v_4) has matrix representation

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 & -1 \\ -2 & -2 & 0 \\ 5 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix}.$$

Solution W8
We have that $V_1 \cap V_2 \subset V_1 \subset V_1 + V_2$. Therefore

$$\dim V_1 \cap V_2 \leq \dim V_1 \leq \dim V_1 + V_2 = \dim V_1 \cap V_2 + 1.$$

So either $\dim V_1 = \dim V_1 \cap V_2$, in which case $V_1 = V_1 \cap V_2$ or $\dim V_1 = \dim V_1 + V_2$, so $V_1 = V_1 + V_2$.
If $V_1 = V_1 \cap V_2$ then $V_1 \subset V_2$. So $V_2 = V_1 + V_2$. Conversely, if $V_1 = V_1 + V_2$ then $V_1 \supset V_2$. So $V_2 = V_1 \cap V_2$.

Solution W9

(a) First note that, if $\langle x, y\rangle = 0$ then $\langle y, x\rangle = \overline{\langle x, y\rangle} = 0$. Suppose $\langle x, y\rangle = 0$. Then

$$||x+y||^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle =$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle y, y\rangle = ||x||^2 + ||y||^2.$$

(b) From the computation above, we see that $||x+y||^2 = ||x||^2 + ||y||^2$ if and only if $\langle x, y\rangle + \langle y, x\rangle = 0$. If $K = \mathbb{R}$ then $\langle x, y\rangle = \langle y, x\rangle$. So $\langle x, y\rangle + \langle y, x\rangle = 0$ if and only if $2\langle x, y\rangle = 0$ if and only if $\langle x, y\rangle = 0$. So, when $\mathbb{K} = \mathbb{R}$, the converse holds.

When $\mathbb{K} = \mathbb{C}$ the converse does not hold. Let $V = \mathbb{C}$, $x = 1, y = i$ and $\langle \cdot, \cdot\rangle$ the standard scalar product on $\mathbb{C}$. Then $\langle x, y\rangle = -i$ and $\langle y, x\rangle = i$. So $\langle x, y\rangle + \langle y, x\rangle = 0$ but $\langle x, y\rangle \neq 0$.

(c) Suppose that $||x+y|| = ||x|| + ||y||$. Then

$$||x+y||^2 = \langle x+y, x+y\rangle = (||x|| + ||y||)^2 = \langle x, x\rangle + 2||x||||y|| + \langle y, y\rangle.$$

Since $\mathbb{K} = \mathbb{R}$,

$$||x+y||^2 = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle.$$

Therefore $2||x||||y|| = 2\langle x, y\rangle$.

Now take $s, t \geq 0$, then

$$||sx+ty||^2 = s^2||x||^2 + 2st\langle x, y\rangle + t^2||y||^2 = ||x||^2 + 2st||x||||y|| + ||y||^2 = (s||x|| + t||y||)^2.$$

Therefore $s||x|| + t||y|| = ||sx+ty||$,

The converse direction is obviously true.