SOBRIETY FOR THE ZIEGLER SPECTRUM OF A PRÜFER DOMAIN

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Abstract. We show that Ziegler Spectra of Prüfer domains are sober.

Contents

1. Introduction

The (right) Ziegler spectrum, Zg_R, of a ring R is a topological space attached to the (right) module category of R. The points of Zg_R are isomorphism classes of indecomposable (right) pure-injective modules and the closed sets correspond to complete theories of modules closed under arbitrary direct sums. This space was first introduced by Ziegler in [Zie84].

In [Her93], Herzog showed that every irreducible closed subset of Zg_R with a countable neighbourhood basis of open sets is the closure of a point. Thus he showed that, when R is countable, Zg_R is sober. We say a topological space is sober if every irreducible closed set C has a generic point x, that is C is the closure of x.

In this paper, we prove that Zg_R is sober when R is a Prüfer domain with no restriction on the cardinality of R. A commutative domain is called a Prüfer domain if its localisations at all maximal ideals are valuation domains.

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The reason for interest in whether Zg_R is sober or not is twofold. Firstly, if viewed from the point of view of the functor category $(R\text{-mod}, \mathrm{Ab})$, its construction looks very similar to that of the Hochster dual Spec^*R of the prime spectrum of a (commutative) ring R. That is, the topological space with points given by isomorphism classes of indecomposable injective modules and basis of open sets of the form

$$(M) := \{E \in \mathrm{inj}_R \mid \mathrm{Hom}_R(M, E) \neq 0\}$$

where M ranges over finitely presented modules is homeomorphic to Spec^*R after identifying topologically indistinguishable points. The Ziegler spectrum of a ring R can be constructed as the topological space with set of point the indecomposable injective functors in $(R\text{-mod}, \mathrm{Ab})$ and basis of open sets given by the sets of indecomposable injective functors F with $\mathrm{Hom}(M, F) \neq 0$ where M is a finitely presented functor in $(R\text{-mod}, \mathrm{Ab})$. For more details of this point of view see [Pre09, §13] and [Her97].

Secondly, Herzog [Her93] used Prest's notion of duality for pp-formulae to show that if both the right and left Ziegler spectra of a ring R are sober then they are homeomorphic after identifying topologically indistinguishable points. Of course, in the commutative case, the left and right Ziegler spectra are already homeomorphic but for most commutative rings it would give rise to a non-trivial automorphism of Zg_R after identifying topologically indistinguishable points.

In order to prove that Zg_R is sober for a Prüfer domain R, we will first show that the Ziegler spectrum of a valuation domain is sober. In order to do this we use a reformulation of Zg_V, where V is a valuation domain, in terms of equivalence classes of pairs of ideals (see [Pun99]).

We would like to draw attention to the simple but sometimes useful observation that the Ziegler spectrum of a valuation domain can be completely reformulated in terms of its value group and thus two valuation domains with isomorphic value groups have homeomorphic Ziegler spectra. This perspective can be very useful when working with examples.

2. Background

For general background on Model theory of Modules see [Pre88].

Let R be a ring. Let $\mathcal{L}_R := (0, +, (r)_{r\in R})$ be the language of (right) R-modules. A (right) **pp-formula** (in one variable) is a formula of the form $\exists \overline{y}\ \overline{y}A = x\overline{b}$ where n, m are natural numbers, A is an $n \times m$ matrix and $\overline{b}$ is a row vector of length m, both with entries from R, and $\overline{y}$ is an n-tuple of variables.

The solution set $\varphi(M)$ of a pp-formula φ in an R-module M is a subgroup of M. If we cosmetically weaken our definition of a pp-formula to include all formulae (in one variable) in the language of (right) R-modules, $\mathcal{L}_R$, which are equivalent over the theory of R-modules, T_R, to a pp-formula then the T_R-equivalence classes of pp-formulae become a lattice under implication with the join of two formulae φ, ψ given by

$$(\varphi + \psi)(x) := \exists y, z(x = y + z \wedge \varphi(y) \wedge \psi(z))$$

and the meet given by $\varphi \wedge \psi$.

A **pp-type** is a set of pp-formulae. If M is an R-module and $a \in M$ then the set of pp-formulae satisfied by a in M is called the pp-type of a. We say a pp-type is **complete** if it is the pp-type of an element of a module or equivalently if it is closed under implications (with respect to the theory of all R-modules) and conjunctions.

A **pure-embedding** between two modules is an embedding which preserves the solution sets of pp-formulae. We say a module N is pure-injective if for every pure-embedding $g : N \to M$, the image of N in M is a direct summand of M.

The **Ziegler spectrum** of a ring R, denoted Zg_R, is a topological space whose points are isomorphism classes of indecomposable pure-injective modules and which has a basis of open sets given by:

$$(\varphi/\psi) = \{M \mid \varphi(M) \supsetneq \psi(M) \wedge \varphi(M)\}$$

where φ, ψ range over pp-formulae.

A commutative integral domain V is called a **valuation domain** if the lattice of ideals of V is a chain. Note that this implies that a subset I of V is an ideal of V if and only if for all $r \in V$ and $a \in I$, $ar \in I$.

Lemma 2.1. [EH95][Pun92] *Every pp-formula over a valuation domain V is equivalent to a pp-formula of the form*

$$\bigwedge_{i=1}^{n} (a_i|x) + (xb_i = 0)$$

where $n \in \mathbb{N}$ and $a_i, b_i \in V$ and a pp-formula of the form

$$\sum_{j=1}^{m} (c_j|x) \wedge (xd_j = 0)$$

where $m \in \mathbb{N}$ and $c_j, d_j \in V$.

Lemma 2.2. [Pun99] *The collection of open sets*

$$\mathcal{W}_{a,b,g,h} := \left(\frac{(xag = 0) \wedge (b|x)}{(xa = 0) + (bh|x)} \right)$$

for non-zero $a, b \in V$ *and* $g, h \in \mathfrak{m}$ *form a basis for* Zg_V*.*

A **pair** over a valuation domain is a pair of proper ideals $\langle I, J\rangle$. To each pair over V, we can associate a pp-type

$$p\langle I, J\rangle = \{xb = 0 \mid b \in I\} \cup \{a|x \mid a \notin J\}.$$

Recall that every complete pp-type is realised in a (unique up to isomorphism) minimal pure-injective module, denoted $N(p)$. We say a complete pp-type is **indecomposable** if $N(p)$ is indecomposable. We say that $\langle I, J\rangle \sim \langle K, L\rangle$ if there exists non-zero $a, b \in R$ such that at least one of the following holds:

(1) $Ia = K$ and $J = La$ or
(2) $I = Ka$ and $Ja = L$.

Lemma 2.3. [Pun99] *Every pp-type* $p\langle I, J\rangle$ *has a unique extension to a complete indecomposable pp-type and every indecomposable pp-type arises in this way. We write* $N\langle I, J\rangle$ *for the unique (up to isomorphism) indecomposable pure-injective realising* $\langle I, J\rangle$*. Moreover, for two pairs* $\langle I, J\rangle$ *and* $\langle K, L\rangle$ *over* V*,* $N\langle I, J\rangle$ *is isomorphic to* $N\langle K, L\rangle$ *if and only if* $\langle I, J\rangle \sim \langle K, L\rangle$*.*

From now on we will write (I, J) for both the equivalence class of $\langle I, J\rangle$ and the corresponding isomorphism class of indecomposable pure-injective modules. We will refer to (I, J) as a point or a point in Zg_V. So, $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if there exists a pair $\langle K, L\rangle$ such that $\langle K, L\rangle \sim \langle I, J\rangle$ and $a \notin K, b \notin L, ag \in K$ and $bh \in L$.

3. Ideal quotients, attached primes, value groups and examples

Throughout this section R will be a commutative ring. For the rest of this paper V will be a valuation domain with maximal ideal $\mathfrak{m}$.

In this section we collect together results on the ideal theory of valuation domains which we will later need. We start by recalling facts about the ideal quotient.

Recall that given ideals $I, J \lhd R$, the **ideal quotient** is defined to be the ideal

$$(I : J) := \{a \in R \mid Ja \subseteq I\}.$$

For $I \lhd R$ and $a \in R$, we will write $(I : a)$ for $(I : Ra)$.

Directly from the definition of the ideal quotient we get that: for all $I, J, K \lhd R$,

$$IJ \subseteq K \text{ if and only if } I \subseteq (K : J). \tag{1}$$

For a valuation domain V we trivially also get that: for all $I, J, K \lhd V$,

(2) $$IJ \supsetneq K \text{ if and only if } I \supsetneq (K : J).$$

It is easy to see that for $I, J \lhd V$ proper ideals of a valuation domain and $a \notin J$, we have that:

(3) $$Ia = J \text{ if and only if } I = (J : a).$$

We can now reformulate $\sim$ in terms of ideal quotients: Let $\langle I, J\rangle$ and $\langle K, L\rangle$ be pairs over V. We have that $\langle I, J\rangle \sim \langle K, L\rangle$ if and only if at least one of the following holds:

(i) there exists $a \notin K$ such that $I = (K : a)$ and $J = La$
(ii) there exists $a \notin J$ such that $I = Ka$ and $J = (L : a)$.

Using the above observation we can now reformulate what it means for a point in Zg_V to be contained in a basic open set:

Lemma 3.1. *Let $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. A point $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if one of the following holds:*

(i) there exists $r \notin I$ such that $a \notin (I : r), b \notin Jr, ag \in (I : r)$ and $bh \in Jr$;
(ii) there exists $s \notin J$ such that $a \notin Is, b \notin (J : s), ag \in Is$ and $bh \in (J : s)$.

Let Q be the field of fractions of V, Γ the value group of V and $v : Q \to \Gamma \cup \{\infty\}$ the valuation map. Recall, [FS01, §II Prop3.4], the bijective correspondence between the set of the proper ideals of V and the strictly positive upsets of $\Gamma \cup \{\infty\}$ given by

$$I \lhd V \mapsto v(I).$$

The inverse map maps a strictly positive upset F to $v^{-1}(F)$.

Note that, if $I \lhd V$ is a proper ideal, $r \notin I$ and $s \in V\backslash\{0\}$ then

$$v((I : r)) = v(I) - v(r)$$

and

$$v(Is) = v(I) + v(s).$$

Thus, our equivalence relation $\sim$ on pairs over V corresponds exactly to an equivalence relation $\approx$ on pairs of strictly positive upsets of $\Gamma \cup \{\infty\}$ given by:

$\langle E, F\rangle \approx \langle G, H\rangle$ if at least one of the following holds:

(i) There exists positive $\gamma \notin G$ such that $E = G - \gamma$ and $F = H + \gamma$
(ii) There exists positive $\gamma \notin H$ such that $E = G + \gamma$ and $F = H - \gamma$

Moreover, given a pair $\langle I, J\rangle$ over V, $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$ we have that $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if at least one of the following holds:

(i) there exists $r \in V$ with $v(r) \notin v(I)$ such that
$v(a) \notin v(I) - v(r)$, $v(b) \notin v(J) + v(r)$, $v(ag) \in v(I) - v(r)$ and $v(bh) \in v(J) + v(r)$;

(ii) there exists $s \in V$ with $v(s) \notin v(J)$ such that
$v(a) \notin v(I) + v(s)$, $v(b) \notin v(J) - v(s), v(ag) \in v(I) + v(s)$ and $v(bh) \in v(J) - v(s)$.

Thus we have the following theorem.

Theorem 3.2. *Let Γ be the value group of V. For a pair $\langle E, F\rangle$ of strictly positive upsets of $\Gamma \cup \{\infty\}$ we denote by (E, F) the $\approx$-equivalence class of $\langle E, F\rangle$.*

For $\alpha, \beta \in \Gamma$ with $\alpha, \beta \geq 0$ and $\gamma, \delta \in \Gamma \cup \{\infty\}$ with $\gamma, \delta > 0$, let $\mathcal{U}_{\alpha,\beta,\gamma,\delta}$ be the set of $\approx$-equivalence classes of pairs of strictly positive upsets (E, F) such that there exists a pair $\langle G, H\rangle$ in the same $\approx$-equivalence class as $\langle E, F\rangle$ with $\alpha \notin G, \beta \notin H, \alpha + \gamma \in G$ and $\beta + \delta \in H$.

Let Zg_Γ be the topological space with points $\approx$-equivalence classes of pairs of strictly positive upsets of $\Gamma \cup \{\infty\}$ and basic open sets $\mathcal{U}_{\alpha,\beta,\gamma,\delta}$.

The map $T : \mathrm{Zg}_V \to \mathrm{Zg}_\Gamma$ given by:

$$(I, J) \mapsto (v(I), v(J))$$

is a homeomorphism.

Corollary 3.3. *If two valuation domains have isomorphic value groups then they have homeomorphic Ziegler spectra.*

We will now use this perspective to give two examples of Ziegler spectra of valuation domains, describing the specialisation between points. We say that a point a in a topological space T specialises to a point $b \in T$ if b is in the closure of a.

Example 3.4. *Let V be a valuation domain with value group $\mathbb{R}$ under addition. The strictly positive upsets of $\mathbb{R} \cup \{\infty\}$ are all of one of the following forms:*

(1) $F_{>\epsilon} := \{x \in \Gamma \mid x > \epsilon\} \cup \{\infty\}$ *for some* $\epsilon \geq 0$
(2) $F_{\geq\epsilon} := \{x \in \Gamma \mid x \geq \epsilon\} \cup \{\infty\}$ *for some* $\epsilon > 0$
(3) $\{\infty\}$

Thus pairs of upsets are all of one of the following forms:

(i) $\langle F_{>\epsilon_1}, F_{>\epsilon_2}\rangle$ *for* $\epsilon_1, \epsilon_2 \in \mathbb{R}_{\geq 0}$
(ii) $\langle F_{>\epsilon_1}, F_{\geq\epsilon_2}\rangle$ *for* $\epsilon_1 \in \mathbb{R}_{\geq 0}$ *and* $\epsilon_2 \in \mathbb{R}_{>0}$
(iii) $\langle F_{>\epsilon_1}, \{\infty\}\rangle$ *for* $\epsilon_1 \in \mathbb{R}_{\geq 0}$
(iv) $\langle F_{\geq\epsilon_1}, F_{\geq\epsilon_2}\rangle$ *for* $\epsilon_1 \in \mathbb{R}_{>0}$ *and* $\epsilon_2 \in \mathbb{R}_{>0}$
(v) $\langle F_{\geq\epsilon_1}, F_{>\epsilon_2}\rangle$ *for* $\epsilon_1 \in \mathbb{R}_{>0}$ *and* $\epsilon_2 \in \mathbb{R}_{\geq 0}$

(vi) $\langle F_{\geq\epsilon_1}, \{\infty\}\rangle$ *for* $\epsilon_1 \in \mathbb{R}_{>0}$
(vii) $\langle \{\infty\}, F_{>\epsilon_2}\rangle$ *for* $\epsilon_2 \in \mathbb{R}_{\geq 0}$
(viii) $\langle \{\infty\}, F_{\geq\epsilon_2}\rangle$ *for* $\epsilon_2 \in \mathbb{R}_{>0}$
(ix) $\langle \{\infty\}, \{\infty\}\rangle$

Each $\approx$*-equivalence class only contains pairs of at most one of the above forms.*

We start by discussing points of the form (iii), (vi), (vii), (viii) and (ix).

We have that: $\langle F_{>\epsilon_1}, \{\infty\}\rangle \approx \langle F_{>\epsilon_1'}, \{\infty\}\rangle$ *for all* $\epsilon_1, \epsilon_1' \in \mathbb{R}_{\geq 0}$ *and thus, for* $\alpha, \beta \in \mathbb{R}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$,

$$(F_{>\epsilon_1}, \{\infty\}) \in \mathcal{U}_{\alpha,\beta,\delta,\gamma}$$

if and only if

$$\gamma = \infty.$$

Symmetrically, we have that: $\langle \{\infty\}, F_{>\epsilon_1}\rangle \approx \langle \{\infty\}, F_{>\epsilon_1'}\rangle$ *for all* $\epsilon_1, \epsilon_1' \in \mathbb{R}_{\geq 0}$ *and thus, for* $\alpha, \beta \in \mathbb{R}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$,

$$(\{\infty\}, F_{>\epsilon_1}) \in \mathcal{U}_{\alpha,\beta,\delta,\gamma}$$

if and only if

$$\delta = \infty.$$

We have that: $\langle F_{\geq\epsilon_1}, \{\infty\}\rangle \approx \langle F_{\geq\epsilon_1'}, \{\infty\}\rangle$ *for all* $\epsilon_1, \epsilon_1' \in \mathbb{R}_{>0}$ *and thus, for* $\alpha, \beta \in \mathbb{R}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$,

$$(F_{\geq\epsilon_1}, \{\infty\}) \in \mathcal{U}_{\alpha,\beta,\delta,\gamma}$$

if and only if

$$\gamma = \infty.$$

Symmetrically, we have that: $\langle \{\infty\}, F_{\geq\epsilon_1}\rangle \approx \langle \{\infty\}, F_{\geq\epsilon_1'}\rangle$ *for all* $\epsilon_1, \epsilon_1' \in \mathbb{R}_{>0}$ *and thus, for* $\alpha, \beta \in \mathbb{R}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$,

$$(\{\infty\}, F_{\geq\epsilon_1}) \in \mathcal{U}_{\alpha,\beta,\delta,\gamma}$$

if and only if

$$\delta = \infty.$$

The point $(\{\infty\}, \{\infty\})$ *is a singleton* $\approx$*-equivalence class and thus, for* $\alpha, \beta \in \mathbb{R}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$,

$$(\{\infty\}, \{\infty\}) \in \mathcal{U}_{\alpha,\beta,\gamma,\delta}$$

if and only if

$$\gamma = \infty \text{ and } \delta = \infty.$$

Thus up to topological indistinguishability we have 3 *points of the forms (iii), (vi), (vii), (viii) and (ix). The point* $(\{\infty\}, \{\infty\})$ *is specialised to by the other two points. There is no other specialisation involving these points.*

We now consider the remaining points. We first state some facts which can be easily proved using the density of $\mathbb{R}$.

Let $\alpha, \beta \in \mathbb{R}$, $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$.

(1) There exists $\mu \in \mathbb{R}$ such that

$$\alpha \leq \epsilon_1 + \mu < \alpha + \gamma$$

and

$$\beta \leq \epsilon_2 - \mu < \beta + \delta$$

if and only if

$$\alpha + \beta \leq \epsilon_1 + \epsilon_2 < (\alpha + \gamma) + (\beta + \delta).$$

(2) There exists $\mu \in \mathbb{R}$ such that

$$\alpha < \epsilon_1 + \mu \leq \alpha + \gamma$$

and

$$\beta < \epsilon_2 - \mu \leq \beta + \delta$$

if and only if

$$\alpha + \beta < \epsilon_1 + \epsilon_2 \leq (\alpha + \gamma) + (\beta + \delta).$$

(3) There exists $\mu \in \mathbb{R}$ such that

$$\alpha \leq \epsilon_1 + \mu < \alpha + \gamma$$

and

$$\beta < \epsilon_2 - \mu \leq \beta + \delta$$

if and only if

$$\alpha + \beta < \epsilon_1 + \epsilon_2 < (\alpha + \gamma) + (\beta + \delta).$$

Using (1), a point of the form $(F_{>\epsilon_1}, F_{>\epsilon_2}) \in \mathcal{U}_{\alpha,\beta,\gamma,\delta}$, for $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$, if and only if

$$\epsilon_1 + \epsilon_2 \in [\alpha + \beta, (\alpha + \gamma) + (\beta + \delta)).$$

We label the point $(F_{>\epsilon_1}, F_{>\epsilon_2})$ as $(\epsilon_1 + \epsilon_2)^+$.

Using (2), a point of the form $(F_{\geq\epsilon_1}, F_{\geq\epsilon_2}) \in \mathcal{U}_{\alpha,\beta,\gamma,\delta}$, for $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$,

$$\epsilon_1 + \epsilon_2 \in (\alpha + \beta, (\alpha + \gamma) + (\beta + \delta)].$$

We label the point $(F_{\geq\epsilon_1}, F_{\geq\epsilon_2})$ as $(\epsilon_1 + \epsilon_2)^-$.

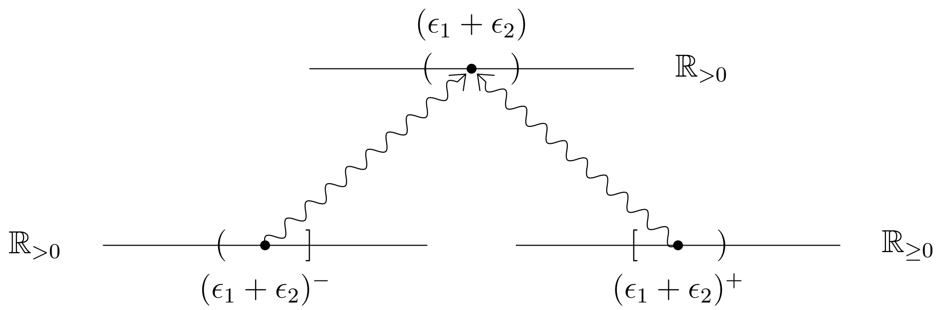
Using (3), a point of the form $\langle F_{\geq\epsilon_1}, F_{>\epsilon_2}\rangle$ or a point of the form $\langle F_{>\epsilon_1}, F_{\geq\epsilon_2}\rangle$, for $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\gamma, \delta \in \mathbb{R}_{>0} \cup \{\infty\}$, if and only if

$$\epsilon_1 + \epsilon_2 \in (\alpha + \beta, (\alpha + \gamma) + (\beta + \delta)).$$

We label the point $(F_{\geq\epsilon_1}, F_{>\epsilon_2})$ as $(\epsilon_1 + \epsilon_2)$.

Thus $(\epsilon_1+\epsilon_2)^+$ and $(\epsilon_1+\epsilon_2)^-$ specialise to $(\epsilon_1+\epsilon_2)$. The point 0^+ is a closed point. This is the only specialisation amongst points of this form.

The following diagram shows the Ziegler spectrum of a valuation domain with value group $\mathbb{R}$ with the points $(\{\infty\},\{\infty\})$, $(\{\infty\},F_{>\epsilon})$ and $(F_{>\epsilon},\{\infty\})$ removed. The oscillating lines show the specialisation between lines. The brackets show a typical open set.



Example 3.5. *Let V be a valuation domain with value group $\mathbb{Z}$ under addition. The strictly positive upsets of $\mathbb{Z}\cup\{\infty\}$ are all of one of the following forms:*

(1) $F_{\geq n} := \{x\in\mathbb{Z} \mid x\geq n\}\cup\{\infty\}$ *for* $n\in\mathbb{Z}_{>0}$
(2) $\{\infty\}$

Thus pairs of upsets are all of one of the following forms:

(i) $\langle F_{\geq n_1}, F_{\geq n_2}\rangle$ *for* $n_1,n_2\in\mathbb{Z}_{>0}$
(ii) $\langle F_{\geq n_1}, \{\infty\}\rangle$ *for* $n_1\in\mathbb{Z}_{>0}$
(iii) $\langle \{\infty\}, F_{\geq n_2}\rangle$ *for* $n_2\in\mathbb{Z}_{>0}$
(iv) $\langle \{\infty\}, \{\infty\}\rangle$

Each $\approx$-equivalence class only contains pairs of at most one of the above forms.

It is easy to see that, for $n_1,n_2,n_1',n_2'\in\mathbb{Z}_{>0}$,

$$\langle F_{\geq n_1}, F_{\geq n_2}\rangle \approx \langle F_{\geq n_1'}, F_{\geq n_2'}\rangle$$

if and only if

$$n_1+n_2=n_1'+n_2'.$$

Thus, for $\alpha,\beta\in\mathbb{Z}_{\geq 0}$ and $\gamma,\delta\in\mathbb{Z}_{>0}\cup\{\infty\}$,

$$(F_{\geq n_1}, F_{\geq n_2})\in\mathcal{U}_{\alpha,\beta,\gamma,\delta}$$

if and only if

$$\alpha+\beta+1<n_1+n_2 \text{ and } \alpha+\gamma+\beta+\delta\geq n_1+n_2.$$

For all $n_1,n_1'\in\mathbb{Z}_{>0}$,

$$\langle F_{\geq n_1}, \{\infty\}\rangle \approx \langle F_{\geq n_2}, \{\infty\}\rangle$$

and thus for $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{Z}_{>0} \cup \{\infty\}$,

$$(F_{\geq n_1}, \{\infty\}) \in \mathcal{U}_{\alpha,\beta,\delta,\gamma}$$

if and only if

$$\gamma = \infty.$$

Symmetrically, for all $n_1, n_1' \in \mathbb{Z}_{>0}$,

$$\langle \{\infty\}, F_{\geq n_1} \rangle \approx \langle \{\infty\}, F_{\geq n_2} \rangle$$

and thus for $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{Z}_{>0} \cup \{\infty\}$,

$$(\{\infty\}, F_{\geq n_1}) \in \mathcal{U}_{\alpha,\beta,\delta,\gamma}$$

if and only if

$$\delta = \infty.$$

The pair $\langle \{\infty\}, \{\infty\} \rangle$ *is a singleton* $\approx$*-equivalence class and for all* $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ *and* $\gamma, \delta \in \mathbb{Z}_{>0} \cup \{\infty\}$,

$$(\{\infty\}, \{\infty\}) \in \mathcal{U}_{\alpha,\beta,\gamma,\delta}$$

if and only if

$$\gamma = \infty \text{ and } \delta = \infty.$$

Thus we see that for a valuation domain with value group $\mathbb{Z}$*, the points of the form* $(F_{\geq n_1}, F_{\geq n_2})$ *(with* $n_1, n_2 \in \mathbb{Z}_{>0}$*) are closed points and the points* $((F_{\geq n_1}), \{\infty\})$ *and* $(\{\infty\}, F_{\geq n_2})$ *(with* $n_1, n_2 \in \mathbb{Z}_{>0}$*) specialise to the point* $(\{\infty\}, \{\infty\})$.

We now turn to the attached prime of an irreducible ideal. A proper ideal $I \lhd R$ is **irreducible** if it is not the intersection of any two ideals properly containing it.

Definition 3.6. *Let* R *be a commutative ring and* $I \lhd R$ *a proper irreducible ideal. We call the prime ideal*

$$I^{\#} = \bigcup_{r \notin I} (I : r)$$

the ***attached prime*** *of* I.

It is easy to see that for any irreducible ideal $I \lhd R$, $I^{\#}$ really is a prime ideal and that the attached prime of a prime ideal is itself [Fuc50]. Note that for valuation domains all proper ideals are irreducible.

We take our notation for the attached prime from [FS01]. Our definition of the attached prime of an ideal differs from the definition given in [FS01] (where it is only defined for valuation domains) but lemma 3.7(i) states that, for valuation domains, it is equivalent. The following lemma collects together properties of the attached prime of a proper ideal of a valuation domain.

Lemma 3.7. *Let V be a valuation domain and let I, J be proper ideals of V.*

(i) If $I \neq 0$ then $r \in I^{\#}$ if and only if $Ir \subsetneq I$;
(ii) If $I \neq 0$ then $r \in I^{\#}$ if and only if $I \subsetneq (I : r)$;
(iii) if $I^{\#} \supsetneq J^{\#}$ then there exists $r \notin I$ such that $(I : r) \supsetneq J$;
(iv) if $I \supsetneq J^{\#}$ then $IJ = J$;
(v) $I^{\#} \cap J^{\#} = (IJ)^{\#}$;
(vi) for $a \notin I$, $(I : a)^{\#} = I^{\#}$
(vii) for non-zero $a \in V$, $(aI)^{\#} = I^{\#}$

Proof. (i) This follows directly from the definition of attached prime and that the ideals of a valuation domain are totally ordered.

(ii) follows from (i) and the fact that, for all $r \notin I$, $(I : r)r = I$.

(iii) Take $s \in I^{\#} \backslash J^{\#}$. Since the ideals of V are totally ordered, there exists $r \notin I$ such that $s \in (I : r)$. So $(I : r) \supseteq sV \supsetneq J^{\#}$.

(iv) Suppose $j \in J$. Take $r \in I \backslash J^{\#}$. Since $r \notin J$, there exists $\mu \in V$ such that $r\mu = j$. Since $r \notin J^{\#}$, $\mu \in J$, so that $j = r\mu \in IJ$.

(v) is [FS01] Lemma 4.6.

(vi) and (vii) are encapsulated by [FS01] lemma 4.3 (ii). □

The following simple lemma will be used repeatedly without note.

Lemma 3.8. *Let $I \lhd V$ a proper ideal of V. The following are equivalent:*

(i) $r \notin I$
(ii) $r\mathfrak{m} \supseteq I$
(iii) $rI^{\#} \supseteq I$

Lemma 3.9. *Suppose that $\langle I, J\rangle, \langle K, L\rangle$ are pairs over V and $\langle I, J\rangle \sim \langle K, L\rangle$. Then $I^{\#} = K^{\#}$, $J^{\#} = L^{\#}$ and $IJ = KL$.*

Proof. Since $\langle I, J\rangle \sim \langle K, L\rangle$, one of the following holds:

(i) there exists $a \notin K$ such that $I = (K : a)$ and $J = La$
(ii) there exists $a \notin J$ such that $I = Ka$ and $J = (L : a)$.

Suppose (i) holds. Then $IJ = (K : a)La = KL$ because, by (3), page 5, $(K : a)a = K$ and $I^{\#} = (K : a)^{\#} = K^{\#}$ and $J^{\#} = (La)^{\#} = L^{\#}$ by lemma 3.7 (vi) and (vii). □

Lemma 3.10. *Let V be a valuation domain, $\mathfrak{p}$ a non-zero prime ideal and $a \in \mathfrak{p}$. Then*

$$I_a^{\mathfrak{p}} = \bigcup_{x \notin \mathfrak{p}} (aV : x)$$

is the pre-image of the ideal generated by a in $V_{\mathfrak{p}}$. Moreover, $I_a^{\mathfrak{p}}$ is an ideal in V and $I_a^{\mathfrak{p}} = I_b^{\mathfrak{p}}$ if and only if $a = bc$ or $b = ac$ for some $c \notin \mathfrak{p}$.

Proof. Let $\pi : V \to V_{\mathfrak{p}}$ be the localisation map. For all $x \in V$, $\pi(x) \in \pi(a)V_{\mathfrak{p}}$ if and only if $x = ar/s$ for some $r \in V$ and $s \notin \mathfrak{p}$, that is exactly if there exists $s \notin \mathfrak{p}$ such that $x \in (aV : s)$. So we have shown the first statement. That $I_a^{\mathfrak{p}}$ is an ideal of V follows from this.

Let $a, b \in \mathfrak{p}$. Then $I_a^{\mathfrak{p}} = I_b^{\mathfrak{p}}$ if and only if $\pi(a)V_{\mathfrak{p}} = \pi(b)V_{\mathfrak{p}}$, that is if and only if $a = bc$ or $b = ac$ for some $c \notin \mathfrak{p}$. □

Lemma 3.11. *Let V be a valuation domain, $\mathfrak{p}$ a non-zero prime ideal $b \in V\backslash\{0\}$ and $a \in \mathfrak{p}\backslash\{0\}$. Then $I_{ab}^{\mathfrak{p}} = bI_a^{\mathfrak{p}}$, $(I_a^{\mathfrak{p}})^{\#} = \mathfrak{p}$ and $I_a^{\mathfrak{p}}\mathfrak{p} = a\mathfrak{p}$.*

Proof. First we show that $I_{ab}^{\mathfrak{p}} = bI_a^{\mathfrak{p}}$. Note that, by (3), page 5, if $a, b \in V\backslash\{0\}$ and $x \notin aV$ then $(abV : x) = b(aV : x)$. Using the original definition of $I_a^{\mathfrak{p}}$, we have that

$$I_a^{\mathfrak{p}} \cdot b = \left(\bigcup_{x \notin \mathfrak{p}} (aV : x)\right) \cdot b = \bigcup_{x \notin \mathfrak{p}} ((aV : x) \cdot b) = \bigcup_{x \notin \mathfrak{p}} (abV : x).$$

So $I_{ab}^{\mathfrak{p}} = I_a^{\mathfrak{p}} b$.

Now $c \cdot I_a^{\mathfrak{p}} = I_{ca}^{\mathfrak{p}} = I_a^{\mathfrak{p}}$ if and only if $\pi(ca)$ and $\pi(a)$ generate the same ideal in $V_{\mathfrak{p}}$ where π is the localisation map. So $c \notin (I_a^{\mathfrak{p}})^{\#}$ if and only if $c \notin \mathfrak{p}$. Therefore $\mathfrak{p} = (I_a^{\mathfrak{p}})^{\#}$.

Since $a \in I_a^{\mathfrak{p}}$, we have that $a\mathfrak{p} \subseteq I_a^{\mathfrak{p}}\mathfrak{p}$. Suppose $c \in I_a^{\mathfrak{p}}$. There exists $x \notin \mathfrak{p}$ such that $cx \in aV$. Therefore $cxp \in a\mathfrak{p}$ for all $p \in \mathfrak{p}$. But since $x \notin (a\mathfrak{p})^{\#} = \mathfrak{p}$, $cp \in a\mathfrak{p}$. □

4. Open sets

In this section we will simplify the conditions for a point to be contained in a basic open set. We will first show that every basic open set is equal to a basic open set of the form $\mathcal{W}_{1,\lambda,g,h}$. We will then reformulate the condition $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ in terms of conditions on IJ, $I^{\#}$, $J^{\#}$ and whether $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Finally, we will then consider the condition $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

Lemma 4.1. *Let $a, b \in V\backslash\{0\}$, $g, h \in \mathfrak{m}$ and (I, J) a point in Zg_V. The following statements are equivalent:*

(i) $(I, J) \in \mathcal{W}_{a,b,g,h}$.
(ii) $(I, J) \in \mathcal{W}_{1,ab,g,h}$.
(iii) $(I, J) \in \mathcal{W}_{ab,1,g,h}$.

Proof. We will show that (i) and (ii) are equivalent. That (iii) is equivalent to (i) follows from the left-right symmetry of $\mathcal{W}_{a,b,g,h}$, that is $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if $(J, I) \in \mathcal{W}_{b,a,h,g}$.

First suppose that $(I, J) \in \mathcal{W}_{a,b,g,h}$. Therefore, there exists a pair $\langle K, L\rangle$ equivalent to $\langle I, J\rangle$ such that $a \notin K$, $ag \in K$, $b \notin L$ and $bh \in L$.

Since $a \notin K$, $\langle (K : a), aL\rangle \sim \langle K, L\rangle \sim \langle I, J\rangle$. Now $1 \notin (K : a)$, $g \in (K : a)$, $ab \notin aL$ and $abh \in aL$. Therefore $(I, J) \in \mathcal{W}_{1,ab,g,h}$.

Now suppose that $(I, J) \in \mathcal{W}_{1,ab,g,h}$. As above this means that there exists $\langle K, L\rangle$ equivalent to $\langle I, J\rangle$ such that $1 \notin K$, $g \in K$, $ab \notin L$ and $abh \in L$. Clearly $a \notin L$. Therefore $\langle aK, (L : a)\rangle$ is equivalent to $\langle I, J\rangle$. We have that $a \notin aK$ because V is a domain and it is clear that $ag \in aK$, $b \notin (L : a)$ and $bh \in (L : a)$. Therefore $(aK, (L : a)) \in \mathcal{W}_{a,b,g,h}$. So $(I, J) \in \mathcal{W}_{a,b,g,h}$. □

The lemma above means that from now on we need only consider basic open sets of the form $\mathcal{W}_{1,\lambda,g,h}$.

Lemma 4.2. *Let (I, J) be a point in Zg_V. If $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then $\lambda \notin IJ$, $\lambda gh \in IJ$, $g \in I^\#$ and $h \in J^\#$.*

Proof. Lemma 3.9 states that, if $\langle I, J\rangle \sim \langle K, L\rangle$ then $IJ = KL$. If $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then there exists $\langle K, L\rangle$ equivalent to $\langle I, J\rangle$ such that $g \in K$, $\lambda \notin L$ and $\lambda h \in L$. Therefore $\lambda gh \in KL = IJ$ and $\lambda \notin KL = IJ$. Lemma 3.9 states that, if $\langle I, J\rangle \sim \langle K, L\rangle$ then $I^\# = K^\#$ and $J^\# = L^\#$. If $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then there exists $\langle K, L\rangle$ equivalent to $\langle I, J\rangle$ such that $g \in K$, $\lambda \notin L$ and $\lambda h \in L$. Therefore $g \in K^\# = I^\#$ and $h \in L^\# = J^\#$. □

Theorem 4.3. *Let $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Let (I, J) be a point in Zg_V. The following are equivalent:*

(i) $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$

(ii) $\lambda gh \in IJ$, $g \in I^\#$, $h \in J^\#$ *and* $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

Proof. (i)⇒ (ii)

Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. By lemma 4.2, $\lambda gh \in IJ$, $g \in I^\#$ and $h \in J^\#$. So it remains to show that $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Since $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ there exists a pair $\langle K, L\rangle$ over V such that $\langle I, J\rangle \sim \langle K, L\rangle$ and $\lambda \notin L$. Hence $(K, L) \in \mathcal{W}_{1,\lambda,0,0}$. So $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

(ii)⇒ (i)

Replace $\langle I, J\rangle$ by $\langle K, L\rangle$ such that $\lambda \notin L$. We can do this since $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. By lemma 3.9, $KL = IJ$, $I^\# = K^\#$ and $J^\# = L^\#$.

Case 1: $\lambda h \in L$

If $g \in K$ then we are done. So, suppose $g \notin K$. First note that since $g \in K^\#$, $(K : g) \supsetneq K$ by 3.7(ii).

Suppose for a contradiction that $(K : g) \subseteq (\lambda h\mathfrak{m} : L)$. By (1), page 4, $(K : g)L \subseteq \lambda h\mathfrak{m}$. Since $g \notin K$, we get that $KL \subseteq \lambda gh\mathfrak{m}$. This implies that $\lambda gh \notin KL$ and thus contradicts our assumptions. So $(K : g) \supsetneq (\lambda h\mathfrak{m} : L)$.

Now take $x \in V$ such that $x \in (K : g)$, $x \notin (\lambda h\mathfrak{m} : L)$ and $x \notin K$. Then $g \in (K : x)$ and $xL \supsetneq \lambda h\mathfrak{m}$, so $\lambda h \in xL$. Since $\lambda \notin L$, $\lambda \notin Lx$. Therefore $((K : x), Lx) \in \mathcal{W}_{1,\lambda,g,h}$ and since $x \notin K$, $\langle (K : x), Lx\rangle \sim \langle K, L\rangle$.

Case 2: $\lambda h \notin L$

First note that since $\lambda \notin L$ and $h \in L^{\#} = (L : \lambda)^{\#}$,

$$(L : \lambda) \subsetneq ((L : \lambda) : h) = (L : \lambda h).$$

Suppose for a contradiction that $(g\mathfrak{m} : K) \supseteq (L : \lambda h)$. Then by (1), page 4, $g\mathfrak{m} \supseteq K(L : \lambda h)$. Since $\lambda h \notin L$, we get that $\lambda gh\mathfrak{m} \supseteq KL$, which contradicts our assumption that $\lambda gh \in KL$. Therefore

$$(L : \lambda h) \supsetneq (g\mathfrak{m} : K).$$

Now take $x \in V$ such that $x \in (L : \lambda h)$, $x \notin (L : \lambda)$ and $x \notin (g\mathfrak{m} : K)$. Then $x \notin L$, $\lambda \notin (L : x)$ and $\lambda h \in (L : x)$. Since $x \notin (g\mathfrak{m} : K)$, $xK \supsetneq g\mathfrak{m}$. So $g \in xK$. Therefore $(Kx, (L : x)) \in \mathcal{W}_{1,\lambda,g,h}$ and since $x \notin L$ $\langle K, L\rangle \sim \langle Kx, (L : x)\rangle$. □

Corollary 4.4. *Let $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Let $\mathfrak{p} \lhd V$ be a prime ideal and $T \lhd V$ be such that $T^{\#} \subseteq \mathfrak{p}$. Then $(\mathfrak{p}, T) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda \notin T$, $\lambda gh \in \mathfrak{p}T$, $g \in \mathfrak{p}$ and $h \in T^{\#}$.*

Proof. Suppose $(\mathfrak{p}, T) \in \mathcal{W}_{1,\lambda,g,h}$. By lemma 4.3, $(\mathfrak{p}, T) \in \mathcal{W}_{1,\lambda,0,0}$, $\lambda gh \in \mathfrak{p}T$, $g \in \mathfrak{p}$ and $h \in T^{\#}$. Since $(\mathfrak{p}, T) \in \mathcal{W}_{1,\lambda,0,0}$, there exists $a \notin \mathfrak{p}$ such that $\lambda \notin aT$. However, since $a \notin \mathfrak{p} \supseteq T^{\#}$, we have that $T = aT$. So $\lambda \notin T$.

Conversely, suppose $\lambda \notin T$, $\lambda gh \in \mathfrak{p}T$, $g \in \mathfrak{p}$ and $h \in T^{\#}$. Since $\lambda \notin T$, $(\mathfrak{p}, T) \in \mathcal{W}_{1,\lambda,0,0}$. So, by proposition 4.3, $(\mathfrak{p}, T) \in \mathcal{W}_{1,\lambda,g,h}$. □

We will now investigate when a point (I, J) in Zg_V is in a basic open set $\mathcal{W}_{1,\lambda,0,0}$. In terms of modules, the condition $(I, J) \in \mathcal{W}_{1,\lambda,0,0} = \mathcal{W}_{\lambda,1,0,0}$ is equivalent to $\lambda \notin \mathrm{ann}_V N\langle I, J\rangle$, since $N\langle I, J\rangle \in \mathcal{W}_{\lambda,1,0,0}$ means exactly that there exists an element in $N\langle I, J\rangle$ not annihilated by λ.

Lemma 4.5. *Let (I, J) be a point in Zg_V such that $I^{\#} \neq J^{\#}$. Then for all $\lambda \in V\backslash\{0\}$, $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin IJ$.*

Proof. The forward implication is always true, see lemma 4.2.

Without loss of generality, we may assume $I^{\#} \supsetneq J^{\#}$. Since $I^{\#} \supsetneq J^{\#}$, lemma 3.7(iii), there exists $a \notin I$ such that $(I : a) \supsetneq J^{\#}$. Therefore, $IJ = (I : a)Ja = Ja$. So, $\langle (I : a), Ja\rangle \sim \langle I, J\rangle$ and $\lambda \notin Ja = IJ$. Therefore $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

□

Corollary 4.6. *Let (I, J) be a point in Zg_V such that $I^\# \neq J^\#$. If $I^\# \supsetneq J^\#$ then $(I^\#, IJ)$ is topologically indistinguishable from (I, J) and if $J^\# \supsetneq I^\#$ then $(IJ, J^\#)$ is topologically indistinguishable from (I, J).*

Proof. We prove the first conjunct, the second follows analogously. Suppose $I^\# \supsetneq J^\#$. Then, by lemma 3.7(v), $(IJ)^\# = I^\# \cap J^\# = J^\#$. Suppose $k \in IJ$. Take $r \in (I)^\# \backslash (IJ)^\#$. Since $r \notin (IJ)^\#$, we have that $k = \mu r$ for some $\mu \in IJ$. Therefore $k \in (IJ)I^\#$. Thus $IJ \subseteq (IJ)I^\#$. Therefore $IJ = (IJ)I^\#$. In view of lemma 4.5, $(I^\#, IJ)$ is topologically indistinguishable from (I, J). □

Lemma 4.7. *Let (I, J) be a point in Zg_V such that $I^\# = J^\#$. Then for all $\lambda \in V\backslash\{0\}$, $\lambda I^\# \supsetneq IJ$ implies $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.*

Proof. We prove the contrapositive. Suppose that $(I, J) \notin \mathcal{W}_{1,\lambda,0,0}$. Then for all $x \notin I$, $\lambda \in Jx$. Take $a \in I^\#$. There exists $t \notin I$ such that $at \in I$. Since $t \notin I$, $\lambda \in tJ$. Therefore $a\lambda \in atJ \subseteq IJ$. Therefore $\lambda I^\# \subseteq IJ$.

□

Definition 4.8. *We say a point (I, J) in Zg_V is **normal** if for all $\lambda \notin IJ$, $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Otherwise we say (I, J) is **abnormal**.*

In example 3.4, the points of form (ii) and (v) are abnormal, all other points are normal. In example 3.5, the points of form (i) are abnormal, all other points are normal.

In terms of modules, bearing in mind the comment after corollary 4.4, $N\langle I, J\rangle$ is abnormal if and only if $\mathrm{ann}_V N\langle I, J\rangle \supsetneq IJ$.

Note that by lemma 4.5, if (I, J) is abnormal then $I^\# = J^\#$. The following lemma gives a necessary and sufficient condition for an abnormal point to be contained in a basic open set $\mathcal{W}_{1,\lambda,0,0}$.

Lemma 4.9. *Let (I, J) be an abnormal point with $I^\# = J^\# = \mathfrak{p}$. Then $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda\mathfrak{p} \supsetneq IJ$, $\lambda gh \in IJ$ and $g, h \in \mathfrak{p}$.*

Proof. Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. By proposition 4.3, $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$, $\lambda gh \in IJ$ and $g, h \in \mathfrak{p}$. There exists $t \notin I$ such that $\lambda \notin Jt$ because $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. On the other hand, since (I, J) is abnormal, there exists $a \in V\backslash\{0\}$ such that $a \notin IJ$ and $(I, J) \notin \mathcal{W}_{1,a,0,0}$. Thus $a\mathfrak{p} \supseteq IJ$ and $a \in Jt$. Therefore

$$\lambda\mathfrak{p} \supseteq Jt \supsetneq a\mathfrak{p} \supseteq IJ.$$

The reverse direction follows directly from lemma 4.7 and proposition 4.3. □

Lemma 4.9 means that, up to topological indistinguishability, a point (I,J) is completely determined by $I^\#$, $J^\#$, IJ and whether or not the point is abnormal. We already know, lemma 4.5, that if $I^\# \neq J^\#$ then (I,J) is normal. Note that this means points of the form $(I,0)$ and $(0,I)$ are normal whenever $I \neq 0$. The equivalence class of the pair $(0,0)$ only contains $(0,0)$. The point $(0,0)$ is in $\mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \neq 0$. Thus $(0,0)$ is a normal point. We now show that every abnormal point (I,J) with $I^\# = J^\# = \mathfrak{p}$ has $IJ = a\mathfrak{p}$ for some $a \in \mathfrak{p}\backslash\{0\}$ and conversely for each prime ideal $\mathfrak{p}$ and each non-zero $a \in \mathfrak{p}$ there is an abnormal point (I,J) with $I^\# = J^\# = \mathfrak{p}$ and $IJ = a\mathfrak{p}$.

Proposition 4.10. *Let $\mathfrak{p} \lhd V$ be a prime ideal.*

(i) If $\mathfrak{p}^2 = \mathfrak{p}$ and $a \in \mathfrak{p}\backslash\{0\}$ then the point $(\mathfrak{p}, I_a^{\mathfrak{p}})$ is abnormal.
(ii) If $\mathfrak{p}^2 \neq \mathfrak{p}$ and $a \in V\backslash\{0\}$ then the point $(\mathfrak{p}, a\mathfrak{p})$ is an abnormal.
(iii) For all non-zero $a \in \mathfrak{p}$ there is a point (I,J) such that $IJ = a\mathfrak{p}$ and $I^\# = J^\# = \mathfrak{p}$.
(iv) Let (I,J) be an abnormal point with $I^\# = J^\# = \mathfrak{p}$. There exists non-zero $a \in \mathfrak{p}$ such that $IJ = a\mathfrak{p}$.

Proof. (i) For all $b \notin \mathfrak{p}$, $b \cdot I_a^{\mathfrak{p}} = I_a^{\mathfrak{p}}$ since $I_a^{\mathfrak{p}}$ has attached prime $\mathfrak{p}$. Therefore $(\mathfrak{p}, I_a^{\mathfrak{p}}) \notin \mathcal{W}_{1,a,0,0}$. As $a \notin I_a^{\mathfrak{p}} \cdot \mathfrak{p} = a\mathfrak{p}$, $(\mathfrak{p}, I_a^{\mathfrak{p}})$ is an abnormal point.

(ii) For all $b \notin \mathfrak{p}$, $a\mathfrak{p} \cdot b = a\mathfrak{p}$ since the attached prime of $a\mathfrak{p}$ is $\mathfrak{p}$. Take $k \in \mathfrak{p}\backslash\mathfrak{p}^2$. Then $(\mathfrak{p}, a\mathfrak{p}) \notin \mathcal{W}_{1,ak,0,0}$ by corollary 4.4 but $ak \notin a\mathfrak{p}^2$. So $(\mathfrak{p}, a\mathfrak{p})$ is abnormal.

(iii) Suppose $\mathfrak{p}^2 = \mathfrak{p}$. Part (i) states that for all $a \in \mathfrak{p}\backslash\{0\}$, $(\mathfrak{p}, I_a^{\mathfrak{p}})$ is abnormal and by lemma 3.11, $\mathfrak{p} \cdot I_a^{\mathfrak{p}} = a\mathfrak{p}$.

Suppose $\mathfrak{p}^2 \neq \mathfrak{p}$ and $a \in \mathfrak{p}\backslash\{0\}$. Take $k \in \mathfrak{p}\backslash\mathfrak{p}^2$. If $a \in \mathfrak{p}^2$ then $a = k\mu$ for some $\mu \in \mathfrak{p}$. Part (ii) states that the point $(\mathfrak{p}, \mu\mathfrak{p})$ is abnormal and $\mu\mathfrak{p}^2 = a\mathfrak{p}$. If $a \in \mathfrak{p}\backslash\mathfrak{p}^2$ then $a\mathfrak{p} = \mathfrak{p}^2$ and part (ii) states that $(\mathfrak{p}, \mathfrak{p})$ is abnormal.

(iv) Suppose $a \notin IJ$ and $(I,J) \notin \mathcal{W}_{1,a,0,0}$. Then $a\mathfrak{p} \supseteq IJ$ and by lemma 4.7, $a\mathfrak{p} \subseteq IJ$. $\square$

In the above lemma we have exhibited all abnormal points up to topological indistinguishability. When $\mathfrak{p}^2 \neq \mathfrak{p}$ all abnormal points (I,J) with $I^\# = J^\# = \mathfrak{p}$ are of the form $(\mathfrak{p}, a\mathfrak{p})$ for some $a \in V\backslash\{0\}$. However, when $\mathfrak{p}^2 = \mathfrak{p}$, there may be more abnormal points than those of the form $(\mathfrak{p}, I_a^{\mathfrak{p}})$ and $(I_a^{\mathfrak{p}}, \mathfrak{p})$ where $a \in \mathfrak{p}\backslash\{0\}$. For a prime ideal $\mathfrak{p} \lhd V$, we call an ideal $I \lhd V$ with $I^\# = \mathfrak{p}$ a proper $\mathfrak{p}$-cut if it is not of the form $a\mathfrak{p}$ for any $a \in V\backslash\{0\}$ and not of the form $I_b^{\mathfrak{p}}$ for any $b \in \mathfrak{p}\backslash\{0\}$. A point (I,J) with $I^\# = J^\# = \mathfrak{p}$ is an abnormal point if and only if $I = I_a^{\mathfrak{p}}$ and $J = b\mathfrak{p}$ for some $a \in \mathfrak{p}\backslash\{0\}$ and $b \in V\backslash\{0\}$; $I = a\mathfrak{p}$ and $J = I_b^{\mathfrak{p}}$

for some $a \in V\backslash\{0\}$ and $b \in \mathfrak{p}\backslash\{0\}$ or I and J are proper $\mathfrak{p}$-cuts with $IJ = a\mathfrak{p}$ for some $a \in \mathfrak{p}\backslash\{0\}$.

5. The Ziegler spectrum of a valuation domain is sober

Recall that a closed subset C of a topological space is called **irreducible** if whenever C is contained in the union of two closed sets A and B then C is contained in A or in B. Equivalently, a closed subset C is irreducible if and only if for all open sets $\mathcal{U}_1, \mathcal{U}_2$, if $\mathcal{U}_1 \cap C \neq \emptyset$ and $\mathcal{U}_2 \cap C \neq \emptyset$ then $\mathcal{U}_1 \cap \mathcal{U}_2 \cap C \neq \emptyset$.

By lemma 4.2, for all $s, t \in \mathfrak{m}$, we have that $\mathcal{W}_{1,ts,0,0} \cap \mathcal{W}_{1,1,t,s} = \emptyset$. Thus, for the Ziegler spectrum of a valuation domain we have the following:

Lemma 5.1. *Let $s, t \in \mathfrak{m}$ and C be an irreducible closed subset of Zg_V. Either $\mathcal{W}_{1,ts,0,0} \cap C = \emptyset$ or $\mathcal{W}_{1,1,t,s} \cap C = \emptyset$.*

Definition 5.2. *Let C be an irreducible closed subset of Zg_V. We define T_C to be the set containing 0 and all non-zero $\lambda \in V$ such that $\mathcal{W}_{1,\lambda,0,0} \cap C = \emptyset$.*

As noted above Lemma 4.5, we have that $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin \mathrm{ann}_R N\langle I, J\rangle$. Thus, for an irreducible closed set C, T_C is the intersection of the annihilators of the indecomposable pure-injective modules contained in C.

Lemma 5.3. *Let C be an irreducible closed set in Zg_V. Then T_C is an ideal with the following properties: For all $(I, J) \in C$,*

(i) $T_C(IJ)^{\#} \subseteq IJ \subseteq T_C$

(ii) $I^{\#} \cap J^{\#} = (IJ)^{\#} \subseteq T_C^{\#}$

(iii) if (I, J) is normal then $T_C = IJ$.

Proof. Since T_C is the intersection of the annihilator ideals of the modules contained in C, T_C is an ideal.

(i) Suppose that $(I, J) \in C$. Let $\lambda \in IJ$ be non-zero and let $i \in I$ and $j \in J$ be such that $ij = \lambda$. Then $(I, J) \in \mathcal{W}_{1,1,i,j}$. By lemma 5.1 $\mathcal{W}_{1,\lambda,0,0} \cap C = \emptyset$. Therefore $\lambda \in T_C$. So $IJ \subseteq T_C$. Suppose $t \in T_C$. Then $(I, J) \notin \mathcal{W}_{1,\lambda,0,0}$ since $\mathcal{W}_{1,t,0,0} \cap C = \emptyset$. If (I, J) is normal, this means $t \in IJ$. So $t(IJ)^{\#} \subseteq IJ$. If (I, J) is abnormal, $t(IJ)^{\#} \subseteq IJ$ by lemma 4.8 and that $(IJ)^{\#} = I^{\#} = J^{\#}$.

(ii) From part (i), either $T_C(IJ)^{\#} \subsetneq T_C$ or $T_C(IJ)^{\#} = IJ = T_C$. Either case implies $(IJ)^{\#} \subseteq T_C^{\#}$.

(iii) Suppose that $(I, J) \in C$ is normal. Then $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin IJ$. So $\lambda \notin IJ$ implies $\lambda \notin T_C$. Therefore $T_C \subseteq IJ$. So, by (i), $T_C = IJ$.

$\square$

Corollary 5.4. *Suppose that C is an irreducible closed subset of Zg_V containing at least one normal point. Then each abnormal point in C is in the closure of every normal point in C.*

Proof. Suppose that $(K, L) \in C$ is normal and that $(I, J) \in C$ is abnormal with $I^{\#} = J^{\#} = \mathfrak{p}$. In order to show that (I, J) is in the closure of (K, L), it is enough to show that, for all basic open sets $\mathcal{W}_{1,\lambda,g,h}$, if $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$.

Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. Then $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$, $\lambda gh \in IJ$ and $g, h \in \mathfrak{p}$. By definition of T_C, $\lambda \notin T_C$ and by lemma 5.3 (i), $\lambda gh \in T_C$. Since (K, L) is normal, by lemma 5.3(iii), $KL = T_C$. So $\lambda \notin KL$ and $\lambda gh \in KL$.

Lemma 5.3(ii) states that $\mathfrak{p} = I^{\#} \cap J^{\#} \subseteq T_C^{\#}$. So $g, h \in \mathfrak{p}$ implies $g, h \in T_C^{\#}$. Therefore $g, h \in K^{\#} \cap L^{\#} = (KL)^{\#} = T_C^{\#}$. So $g \in K^{\#}$ and $h \in L^{\#}$.

Therefore $\lambda \notin KL$, $\lambda gh \in KL$, $g \in K^{\#}$ and $h \in L^{\#}$. Since (K, L) is normal, $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$.

$\square$

We are now left with two possible situations. An irreducible closed set C either contains a normal point, in which case we must show that C contains a normal point such that all other normal points in C are in its closure, or, C only contains abnormal points. We will deal with these two possibilities separately.

The following lemma gives restrictions on the image of an irreducible closed set under the map

$$(I, J) \mapsto (I^{\#}, J^{\#})$$

in $\mathrm{Spec}^* V \times \mathrm{Spec}^* V$.

Lemma 5.5. *Let C be an irreducible closed subset of Zg_V containing at least one normal point. Then one of the following is true:*

1. *For all normal points $(I, J) \in C$, either $I^{\#} = T_C^{\#}$ and $J^{\#} = T_C^{\#}$, or (I, J) is topologically indistinguishable from $(T_C, \mathfrak{p})$ for some prime ideal $\mathfrak{p} \supsetneq T_C^{\#}$.*
2. *For all normal points $(I, J) \in C$, either $I^{\#} = T_C^{\#}$ and $J^{\#} = T_C^{\#}$, or (I, J) is topologically indistinguishable from $(\mathfrak{p}, T_C)$ for some prime ideal $\mathfrak{p} \supsetneq T_C^{\#}$.*

Proof. Let C be an irreducible closed set containing at least one normal point.

By lemma 5.3(iii), for all normal points $(I, J) \in C$, we have that $I^{\#} \cap J^{\#} = T_C^{\#}$ since $IJ = T_C$. Therefore, either $I^{\#} = T_C^{\#}$ or $J^{\#} = T_C^{\#}$. So, suppose for a contradiction that there exists $(I, J) \in C$ and $(K, L) \in C$ such that $I^{\#} \supsetneq T_C^{\#}$ and $L^{\#} \supsetneq T_C^{\#}$. Then $I^{\#} \cap L^{\#} \supsetneq T_C^{\#}$. Take $t \in (I^{\#} \cap L^{\#}) \backslash T_C^{\#}$ and $\mu \in T_C$. Then $\mu = tr$ for some $r \in T_C^{\#}$. Then $(I, J) \in \mathcal{W}_{1,1,t,r}$ and $(K, L) \in \mathcal{W}_{1,1,r,t}$. Since C is irreducible, $C \cap \mathcal{W}_{1,1,t,r} \cap \mathcal{W}_{1,1,r,t} \neq \emptyset$. Therefore, there exists $(M, N) \in C$ such that $t \in M^{\#}$ and $t \in N^{\#}$. So $N^{\#} \cap M^{\#} \supsetneq T_C^{\#}$. But this contradicts 5.3(ii).

By corollary 4.5, for all normal points (I, J) with $I^{\#} \supsetneq J^{\#}$ (respectively with $J^{\#} \supsetneq I^{\#}$) we have that (I, J) is topologically indistinguishable from $(I^{\#}, IJ)$ (respectively $(IJ, J^{\#})$). Thus we have proved the lemma. □

Lemma 5.6. *Let $\{\mathfrak{p}_i \mid i \in \mathcal{I}\}$ be a set of prime ideals of V and let T be an ideal of V with $T^{\#} \subsetneq \mathfrak{p}_i$ for all $i \in \mathcal{I}$.*

If C is a closed set in Zg_V such that $(T, \mathfrak{p}_i) \in C$ for all $i \in \mathcal{I}$ then $(T, \cup_{i \in \mathcal{I}} \mathfrak{p}_i) \in C$.

If C is a closed set in Zg_V such that $(\mathfrak{p}_i, T) \in C$ for all $i \in \mathcal{I}$ then $(\cup_{i \in \mathcal{I}} \mathfrak{p}_i, T) \in C$.

Proof. Suppose $\lambda \in V \backslash \{0\}$ and $g, h \in \mathfrak{m}$ are such that $(T, \cup_{i \in \mathcal{I}} \mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$. By lemma 4.5, $(T, \cup_{i \in \mathcal{I}} \mathfrak{p}_i)$ is a normal point and since $\cup_{i \in \mathcal{I}} \mathfrak{p}_i \supsetneq T^{\#}$, we have that $T \cup_{i \in \mathcal{I}} \mathfrak{p}_i = T$. Thus $\lambda \notin T$, $\lambda gh \in T$, $g \in T^{\#}$ and $h \in \cup_{i \in \mathcal{I}} \mathfrak{p}_i$. Therefore $h \in \mathfrak{p}_i$ for some $i \in \mathcal{I}$. Since $\mathfrak{p}_i \supsetneq T^{\#}$, $T\mathfrak{p}_i = T$. Therefore $(T, \mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$.

The second statement follows symmetrically since $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $(J, I) \in \mathcal{W}_{1,\lambda,h,g}$. □

When an irreducible closed set C contains a normal point (I, J) with $I^{\#} \subsetneq J^{\#}$, the point $(T_C, \cup_{i \in \mathfrak{I}} \mathfrak{p}_i)$ in the above lemma will be a generic point of C and symmetrically, when C contains a normal point (I, J) with $I^{\#} \supsetneq J^{\#}$, the point $(\cup_{i \in \mathfrak{I}} \mathfrak{p}_i, T_C)$ will be the generic point of C (this will be fully justified in proposition 5.12). In the case that the only normal points (I, J) contained in an irreducible closed set C are such that $I^{\#} = J^{\#}$, lemma 5.8 will show that C only contains one normal point up to topological indistinguishability; in view of lemma 5.4, this point will be generic in C.

Definition 5.7. *Let $\mathfrak{p}, \mathfrak{q} \lhd V$ be prime ideals. Then*

$$X_{\mathfrak{p},\mathfrak{q}} = \{(I, J) \in \mathrm{Zg}_V \mid I^{\#} = \mathfrak{p} \text{ and } J^{\#} = \mathfrak{q}\}.$$

Lemma 5.8. *Let $\mathfrak{p} \lhd V$ be a prime ideal. Suppose C is an irreducible closed subset in Zg_V. Then all normal points in $X_{\mathfrak{p},\mathfrak{p}} \cap C$ are topologically indistinguishable.*

Proof. Suppose $(I, J) \in C \cap X_{\mathfrak{p},\mathfrak{p}}$ is a normal point in Zg_V. For all $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda \notin IJ = T_C$, $\lambda gh \in IJ = T_C$ and $g, h \in I^{\#} = J^{\#} = \mathfrak{p}$. Thus, whether $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ is dependent only on T_C and $\mathfrak{p}$. Therefore, all normal points in $C \cap X_{\mathfrak{p},\mathfrak{p}}$ are topologically indistinguishable. □

Proposition 5.9. *Let C be an irreducible closed subset containing at least one normal point. Then C has a generic point.*

Proof. First suppose that for all normal points $(I, J) \in C$, $I^{\#} = J^{\#}$. Then, by lemma 5.8, C contains only one normal point up to topological indistinguishability. Thus, by lemma 5.4, this point is generic.

Now suppose that C contains a normal point (I, J) such that $J^{\#} \supsetneq I^{\#}$. By lemma 5.5, all normal $(K, L) \in C$ either have $T_C^{\#} = K^{\#} = L^{\#}$ or are topologically indistinguishable from a point of the form $(T_C, \mathfrak{p})$ for some prime ideal $\mathfrak{p} \lhd V$. Let $\mathfrak{I}$ index the prime ideals $\mathfrak{p}_i$ such that $(T, \mathfrak{p}_i) \in C$ with $\mathfrak{p}_i \supsetneq T^{\#}$. By lemma 5.6, $(T_C, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i) \in C$. So it remains to show that $(T_C, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$ is a generic point of C. This follows for abnormal points by lemma 5.4. Suppose $(I, J) \in C$ is a normal point. Then $IJ = T_C$ and $I^{\#} = T_C^{\#}$ and $J^{\#} \subseteq \cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. So, if $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then $\lambda \notin T_C$, $\lambda gh \notin T_C$, $g \in T_C^{\#}$ and $h \in J^{\#} \subseteq \cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. Thus $(T_C, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$. Therefore (I, J) is in the closure of $(T_C, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$.

When C contains a normal point (I, J) such that $I^{\#} \supsetneq J^{\#}$, the generic point of C is $(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i, T_C)$ where the prime ideals $\mathfrak{p}_i$ are exactly those prime ideals such that $(\mathfrak{p}_i, T_C) \in C$. This follows symmetrically to the argument above. □

It remains to consider irreducible closed sets of Zg_V containing only abnormal points.

Lemma 5.10. *Let C be an irreducible closed subset of Zg_V containing only abnormal points. For each prime ideal $\mathfrak{p} \lhd V$, all points in $C \cap X_{\mathfrak{p},\mathfrak{p}}$ are topologically indistinguishable from each other.*

Proof. Suppose $\mathfrak{p} \lhd V$ is a prime ideal such that $C \cap X_{\mathfrak{p},\mathfrak{p}} \neq \emptyset$. Let $(I, J), (K, L) \in C \cap X_{\mathfrak{p},\mathfrak{p}}$. We will show that $IJ = KL$ and thus, by lemma 4.9, (I, J) and (K, L) are topologically indistinguishable.

Seeking a contradiction, suppose that $IJ \subsetneq KL$. Since (I, J) and (K, L) are abnormal, there exists $a, b \in \mathfrak{p}$ such that $a\mathfrak{p} = IJ \subsetneq KL = b\mathfrak{p}$.

We consider the cases $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{p} \neq \mathfrak{p}^2$ separately.

If $\mathfrak{p}^2 = \mathfrak{p}$ then there exists $c, \gamma_1, \gamma_2 \in \mathfrak{p}$ such that $a\mathfrak{p} \subsetneq c\mathfrak{p} \subsetneq b\mathfrak{p}$ and $\gamma_1\gamma_2 = c$. By lemma 4.9, $(I, J) \in \mathcal{W}_{1,c,0,0}$ and $(K, L) \in \mathcal{W}_{1,1,\gamma_1,\gamma_2}$. But $\mathcal{W}_{1,c,0,0} \cap \mathcal{W}_{1,1,\gamma_1,\gamma_2} = \emptyset$. This contradicts the irreducibility of C. Thus $IJ = KL$.

Now suppose $\mathfrak{p}^2 \neq \mathfrak{p}$. Take $k \in \mathfrak{p}\backslash\mathfrak{p}^2$. By lemma 4.9, $(I, J) \in \mathcal{W}_{1,b,0,0}$ and $(K, L) \in \mathcal{W}_{1,1,b,k}$. Therefore, since C is irreducible, there exists $(T, S) \in C \cap \mathcal{W}_{1,b,0,0} \cap \mathcal{W}_{1,1,b,k}$. Since C contains only abnormal points, $TS = \gamma\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ and some $\gamma \in \mathfrak{q}$. Moreover,

$$b\mathfrak{q} \supsetneq \gamma\mathfrak{q} \supseteq bkV.$$

Hence $\mathfrak{q} \supsetneq \mathfrak{p}$.

Claim: Either $\gamma\mathfrak{q} \supsetneq b\mathfrak{p}$ or there exists $\mu \in V$ such that $\mu\mathfrak{q} \supsetneq \gamma\mathfrak{q}$ and $\mu \in b\mathfrak{p}$.

Suppose $\gamma\mathfrak{q} \subseteq b\mathfrak{p}$. Then $\gamma\mathfrak{q} \subsetneq b\mathfrak{p}$ as $\mathfrak{p} \neq \mathfrak{q}$. Take $s \in b\mathfrak{p}\backslash\gamma\mathfrak{q}$ and $t \in \mathfrak{q}\backslash\mathfrak{p}$. Since $s \in \mathfrak{p}$ and $t \notin \mathfrak{p}$, there exists $\mu \in \mathfrak{p}$ such that $s = t\mu$. Now $\mu \in b\mathfrak{p}$ since $s \in b\mathfrak{p}$ and $t \notin \mathfrak{p}$ and $\mu\mathfrak{q} \supsetneq s\mathfrak{q} \supseteq \gamma\mathfrak{q}$.

Having proved the claim, we can now consider the two possible cases in turn and show that either case contradicts the irreducibility of C.

Case 1: $\gamma\mathfrak{q} \supsetneq b\mathfrak{p}$

Take $q \in \mathfrak{q}$ such that $\gamma q \notin b\mathfrak{p}$. So we have that

$$\gamma q\mathfrak{p} \supseteq b\mathfrak{p} \supsetneq a\mathfrak{p}.$$

Thus $(I, J) \in \mathcal{W}_{1,\gamma q,0,0}$ and $(T, S) \in \mathcal{W}_{1,1,\gamma,q}$. But $\mathcal{W}_{1,\gamma q,0,0} \cap \mathcal{W}_{1,1,\gamma,q} = \emptyset$. This contradicts the irreducibility of C.

Case 2: There exists $\mu \in V$ such that $\mu\mathfrak{q} \supsetneq \gamma\mathfrak{q}$ and $\mu \in b\mathfrak{p}$.

Take $p \in \mathfrak{p}$ such that $\mu = bp$. We have that $(T, S) \in \mathcal{W}_{1,\mu,0,0}$ and $(K, L) \in \mathcal{W}_{1,1,b,p}$. But $\mathcal{W}_{1,\mu,0,0} \cap \mathcal{W}_{1,1,b,p} = \emptyset$. This contradicts the irreducibility of C. □

It is worth noting that without the assumption that C only contains abnormal points the above lemma is not true i.e. the specialisation order is not a root system.

Lemma 5.11. *Let $\mathfrak{p}, \mathfrak{q} \lhd V$ be prime ideals such that $\mathfrak{p} \supsetneq \mathfrak{q}$. Suppose $(I, J) \in X_{\mathfrak{p},\mathfrak{p}}$ is abnormal and $IJ \subseteq \mathfrak{q}$. Then there exists an abnormal point $(K, L) \in X_{\mathfrak{q},\mathfrak{q}}$ such that (K, L) is in the closure of (I, J).*

Proof. Since (I, J) is abnormal, there exists $a \in \mathfrak{p}$ such that $a\mathfrak{p} = IJ$. Since $a\mathfrak{p} \subseteq \mathfrak{q}$, $a \in \mathfrak{q}$. By lemma 4.10(iii), there exists an abnormal point $(K, L) \in X_{\mathfrak{q},\mathfrak{q}}$ with $KL = a\mathfrak{q}$.

We will now show that (K, L) is in the closure of (I, J). Suppose $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$. Then $\lambda\mathfrak{q} \supsetneq KL = a\mathfrak{q}$, $\lambda gh \in a\mathfrak{q}$ and $g, h \in \mathfrak{q}$. Since $\mathfrak{p} \supseteq \mathfrak{q}$, $g, h \in \mathfrak{p}$. Since $a\mathfrak{q} \subseteq a\mathfrak{p}$, $\lambda gh \in a\mathfrak{p}$.

It remains to show that $\lambda\mathfrak{p} \supsetneq a\mathfrak{p}$. By lemma 3.8, $\lambda\mathfrak{q} \supsetneq a\mathfrak{q}$ implies $a \in \lambda\mathfrak{q}$. Therefore $a \in \lambda\mathfrak{p}$. So, again by lemma 3.8, $\lambda\mathfrak{p} \supsetneq a\mathfrak{p}$.

Putting all this together, we get that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. Therefore (K, L) is in the closure of (I, J). $\square$

Proposition 5.12. *Let C be an irreducible closed subset of Zg_V containing only abnormal points and let $\mathcal{I}$ index the prime ideals $\mathfrak{p}_i$ such that $X_{\mathfrak{p}_i,\mathfrak{p}_i} \cap C \neq \emptyset$. Then:*

(i) $T_C^{\#} = \cup\mathfrak{p}_i$

(ii) $(T_C, \cup\mathfrak{p}_i) \in C$

(iii) Every point in C is in the closure of $(T_C, \cup\mathfrak{p}_i)$, that is $(T_C, \cup\mathfrak{p}_i)$ is a generic point for C.

Proof. First order $\mathcal{I}$ by $i \geq j$ if and only if $\mathfrak{p}_i \supseteq \mathfrak{p}_j$. By lemma 5.10, $X_{\mathfrak{p}_i,\mathfrak{p}_i} \cap C$ contains one point up to topological indistinguishability. For each $i \in \mathcal{I}$, let (I_i, J_i) be such a point and let $a_i \in \mathfrak{p}_i$ be such that $a_i\mathfrak{p}_i = I_iJ_i$. To make reading easier, let $\mathfrak{P} := \cup_{i\in\mathcal{I}}\mathfrak{p}_i$.

(i) By lemma 5.3 (ii), $\mathfrak{p}_i \subseteq T_C^{\#}$ for all $i \in \mathcal{I}$. Suppose $t \in T_C^{\#}$. There exists $s \notin T_C$ such that $ts \in T_C$. Thus $\mathcal{W}_{1,ts,0,0} \cap C = \emptyset$ and $\mathcal{W}_{1,s,0,0} \cap C \neq \emptyset$. Therefore, there exists $i \in \mathcal{I}$ such that $(I_i, J_i) \in \mathcal{W}_{1,s,0,0}$ and $(I_i, J_i) \notin \mathcal{W}_{1,ts,0,0}$. Thus $s\mathfrak{p}_i \supsetneq I_iJ_i = a_i\mathfrak{p}_i$ and $I_iJ_i = a_i\mathfrak{p}_i \supseteq st\mathfrak{p}_i$. Hence $t \in \mathfrak{p}_i$. Therefore $T_C^{\#} = \mathfrak{P}$.

(ii) First we show that for $i \geq j$, $a_i\mathfrak{p}_i \supseteq a_j\mathfrak{p}_j$. In order to do this we first show that $a_i \in \mathfrak{p}_j$. Suppose to the contrary that $a_i \notin \mathfrak{p}_j$. Take $t \in \mathfrak{p}_i\backslash\mathfrak{p}_j$. Since $a_it \notin \mathfrak{p}_j$, $a_it\mathfrak{p}_j = \mathfrak{p}_j \supsetneq a_j\mathfrak{p}_j$. Therefore $(I_i, J_i) \in \mathcal{W}_{1,1,a_i,t}$ and $(I_j, J_j) \in \mathcal{W}_{1,a_it,0,0}$. But this contradicts the irreducibility of C. Thus $a_i \in \mathfrak{p}_j$.

Let (K, L) be an abnormal point in $X_{\mathfrak{p}_j,\mathfrak{p}_j}$ such that $KL = a_i\mathfrak{p}_j$. Lemma 5.11 tells us that (K, L) is in the closure of (I_i, J_i) and by lemma 5.10 (K, L) and (I_j, J_j) are topologically indistinguishable. Thus $a_i\mathfrak{p}_i \supseteq a_i\mathfrak{p}_j = KL = I_jJ_j = a_j\mathfrak{p}_j$.

We now show that if $\lambda \in V\backslash\{0\}$ and $i, j \in \mathcal{I}$ with $i \geq j$, then

$$\lambda\mathfrak{p}_j \supsetneq a_j\mathfrak{p}_j \text{ implies } \lambda\mathfrak{p}_i \supsetneq a_i\mathfrak{p}_i.$$

Suppose that $\lambda\mathfrak{p}_j \supsetneq a_j\mathfrak{p}_j$ and $\lambda\mathfrak{p}_i \subseteq a_i\mathfrak{p}_i$. Take $t \in \mathfrak{p}_i\backslash\mathfrak{p}_j$. So $t^2 \in \mathfrak{p}_i\backslash\mathfrak{p}_j$. Since $t^2 \notin \mathfrak{p}_j$, $\lambda t^2\mathfrak{p}_j = \lambda\mathfrak{p}_j \supsetneq a_j\mathfrak{p}_j$. Therefore $(I_j, J_j) \in \mathcal{W}_{1,\lambda t^2,0,0}$. Since $\lambda t \in \mathfrak{p}_i$, $t \in \mathfrak{p}_i$ and $\lambda t^2 \in a_i\mathfrak{p}_i$, $(I_i, J_i) \in \mathcal{W}_{1,1,\lambda t,t}$. But $\mathcal{W}_{1,\lambda t^2,0,0} \cap \mathcal{W}_{1,1,\lambda t,t} = \emptyset$. This contradicts the irreducibility of C. Therefore $\lambda\mathfrak{p}_i \supsetneq a_i\mathfrak{p}_i$.

Putting this all together, we show that $(T_C, \mathfrak{P}) \in C$. Suppose $(T_C, \mathfrak{P}) \in \mathcal{W}_{1,\lambda,g,h}$. As stated in lemma 4.4, $\lambda \notin T_C$, $\lambda gh \in T_C\mathfrak{P}$ and $g, h \in \mathfrak{P} = T_C^{\#}$. So, since $\lambda \notin T_C$, there exists $i \in \mathcal{I}$ such that

$(I_i, J_i) \in \mathcal{W}_{1,\lambda,0,0}$. So $\lambda\mathfrak{p}_i \supsetneq a_i\mathfrak{p}_i$. So for all $l \geq i$, $(I_l, J_l) \in \mathcal{W}_{1,\lambda,0,0}$. Since $\lambda gh \in T_C\mathfrak{P}$, there exists some $j \in \mathcal{I}$, $p \in \mathfrak{p}_j$ and $t \in T_C$ such that $\lambda gh = tp$. Since $t \in T_C$, $t\mathfrak{p}_j \subseteq a_j\mathfrak{p}_j$. Therefore $\lambda gh = tp \in a_j\mathfrak{p}_j$. So, for all $l \geq j$, $\lambda gh \in a_l\mathfrak{p}_l$. Since $g, h \in \mathfrak{P}$, there exists $k \in \mathcal{I}$ such that $g, h \in \mathfrak{p}_k$ and thus for all $l \geq k$, $g, h \in \mathfrak{p}_l$. Now let m be greater than or equal to i, j and k. Then $\lambda\mathfrak{p}_m \supsetneq a_m\mathfrak{p}_m$, $\lambda gh \in a_m\mathfrak{p}_m$ and $g, h \in \mathfrak{p}_m$. So $(I_m, J_m) \in \mathcal{W}_{1,\lambda,g,h}$. Thus $(T_C, \mathfrak{P}) \in C$.

(iii) By lemmas 5.11 and 5.10, it is enough to show that $T_C\mathfrak{P} \subseteq \mathfrak{p}_i$ for all $i \in \mathcal{I}$. Suppose $x \in T_C$. Then $\mathcal{W}_{1,x,0,0} \cap C = \emptyset$. Therefore $x\mathfrak{p}_i \subseteq a_i\mathfrak{p}_i \subsetneq \mathfrak{p}_i$. Thus $x \in \mathfrak{p}_i$. Therefore $T_C \subseteq \mathfrak{p}_i$. So $T_C\mathfrak{P} \subseteq \mathfrak{p}_i$. □

Proposition 5.9 and proposition 5.12 prove our theorem.

Theorem 5.13. *Every irreducible closed subset of* Zg_V *has a generic point.*

6. Localisation and generalising to Prüfer domains

In this section we will show that the Ziegler spectrum of a Prüfer domain is sober.

An integral domain R is called a Prüfer domain if for all $\mathfrak{p} \lhd R$ prime, $R_\mathfrak{p}$ is a valuation domain.

Definition 6.1. *Let R be a commutative ring and N an indecomposable pure-injective module. The set of $r \in R$ whose action on N by multiplication is not bijective, denoted* $\mathrm{Att}N$*, is called the attached prime of* N*.*

Recall that, for a commutative ring R, Spec^*R is the Hochster dual of the prime ideal spectrum, $\mathrm{Spec}R$, of R, that is, it is the space we get by declaring the complements of compact open sets in $\mathrm{Spec}R$ to be open.

Proposition 6.2. *Let R be a commutative domain. The map taking an indecomposable pure-injective to its attached prime induces a continuous map from* Zg_R *to* Spec^*R*.*

Proof. In order to check that

$$f : \mathrm{Zg}_R \to \mathrm{Spec}^*R, \; f : N \to \mathrm{Att}N$$

is continuous, it is enough to check the preimage of subbasic open sets are open. First note that the collection of open sets $V(aR) = \{\mathfrak{p} \in \mathrm{Spec}^* \mid a \in \mathfrak{p}\}$ with $a \in R$ are a sub-basis for Spec^*R, so it is enough to check that the pre-image under f of each $V(aR)$ is open. Suppose N is an indecomposable pure-injective module and $a \in R$. Observe that the following 3 statements are equivalent:

(i) $f(N) \in V(aR)$.

(ii) Either there exists $n \in N\backslash\{0\}$ such that $na = 0$ or there exists $n \in N$ such that a does not divide n.

(iii) $N \in \left(\frac{xa=0}{x=0}\right) \cup \left(\frac{x=x}{a|x}\right)$.

Hence for any $a \in R$ the pre-image of $V(aR)$ under f is $\left(\frac{xa=0}{x=0}\right) \cup \left(\frac{x=x}{a|x}\right)$. Thus f is continuous. $\square$

Theorem 6.3. [Pre09, pg67] *Suppose that $f : R \to S$ is an epimorphism of rings. If N is an indecomposable pure injective S-module then as an R-module, N is indecomposable pure-injective. The induced map from Zg_S to Zg_R continuously embeds Zg_S into Zg_R as closed set.*

Lemma 6.4. *Let R be a commutative ring, $\mathfrak{p} \lhd R$ be a prime ideal and $\tau : R \to R_\mathfrak{p}$ be the localisation map. The image of the map induced by τ from $\mathrm{Zg}_{R_\mathfrak{p}}$ to Zg_R is the set of indecomposable pure-injectives with attached prime contained in $\mathfrak{p}$.*

Proof. Suppose N has attached prime $\mathfrak{q} \subseteq \mathfrak{p}$. Then for all $r \notin \mathfrak{p}$, multiplication by r is a bijective map. Hence we may define multiplication by $1/r$ to be the inverse of this map. So N can be endowed with the structure of an $R_\mathfrak{p}$-module. It is clear that N remains indecomposable as an $R_\mathfrak{p}$-module. In order to see that N remains pure-injective, note that a pure-embedding between $R_\mathfrak{p}$-modules remains a pure-embedding when viewed as a map between R-modules and that if A, B are $R_\mathfrak{p}$-modules and a map $A \to B$ is an R-module map then it is also an $R_\mathfrak{p}$-module map.

Suppose N is an $R_\mathfrak{p}$-module. Then N may be viewed via τ as an R module. For any $t \notin \mathfrak{p}$, since N is an $R_\mathfrak{p}$-module, the action of t is invertible. Hence $t \notin \mathrm{Att}N_R$. Therefore $\mathfrak{p} \supseteq \mathrm{Att}N_R$. $\square$

Proposition 6.5. *Let R be a commutative ring. The following are equivalent:*

(i) Zg_R *is sober.*

(ii) For all $\mathfrak{p} \lhd R$ prime, $\mathrm{Zg}_{R_\mathfrak{p}}$ is sober.

(iii) For all $\mathfrak{m} \lhd R$ maximal, $\mathrm{Zg}_{R_\mathfrak{m}}$ is sober.

Proof. (i) $\Rightarrow$ (ii) Suppose Zg_R is sober. For any prime ideal $\mathfrak{p} \lhd R$, $\mathrm{Zg}_{R_\mathfrak{p}}$ is homeomorphic to a closed subset of Zg_R and hence is sober.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Suppose $C \subseteq \mathrm{Zg}_R$ is an irreducible closed set. Its image $f(C)$ in Spec^*R is irreducible. So, the closure of $f(C)$ has a generic point $\mathfrak{p}$ in Spec^*R. Hence, $N \in C$ implies $f(N) \subseteq \mathfrak{p}$. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then, $N \in f(C)$ implies $f(N) \subseteq \mathfrak{m}$. Therefore, by lemma 6.4, C is contained in a closed set homeomorphic to $\mathrm{Zg}_{R_\mathfrak{m}}$. Hence, if $\mathrm{Zg}_{R_\mathfrak{m}}$ is sober then C has a generic point. $\square$

Corollary 6.6. *Let R be a Prüfer domain. The Ziegler spectrum of R is sober.*

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