University of Konstanz
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Fourier Analysis of Boolean Functions – Exercise Sheet 3

Exercise 1. *Fast Walsh-Hadamard Transform*
Define matrices $H_n \in \{-1,1\}^{2^n \times 2^n}$ for $n \in \mathbb{N}_0$ inductively by $H_0 := (1)$ and

$$H_n := \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix}$$

for $n \in \mathbb{N}$.

(a) Let us index the rows and columns of H_n by the integers $\{0,1,\ldots,2^n-1\}$ rather than $[2^n]$. Further, let us identify such an integer $i \in \{0,1,\ldots,2^n-1\}$ with its binary expansion $(i_0,i_1,\ldots,i_{n-1}) \in \{0,1\}^n$, i.e., $i = \sum_{k=0}^{n-1} i_k 2^k$. Show that for $\alpha, \beta \in \{0,1,\ldots,2^n-1\}$ the (α,β) entry of H_n is $(-1)^{\sum_{k=0}^{n-1}\alpha_k\beta_k}$.

(b) Show that if $f\colon \mathbb{F}_2^n \to \mathbb{R}$ is represented as a column vector in $\mathbb{R}^{2^n}$ (according to the indexing scheme from part (a)), then $H_n f = 2^n \widehat{f}$. Here we think of $\widehat{f}$ as also being a function $\mathbb{F}_2^n \to \mathbb{R}$, identifying subsets $S \subseteq [n]$ with their indicator vectors.

(c) Show how to compute $H_n v$ for a vector $v \in \mathbb{R}^{2^n}$ using just $n2^n$ additions and subtractions (rather than 2^{2n} additions and subtractions as the usual matrix-vector multiplication algorithm would require).

Exercise 2. Let $A \subseteq \{-1,1\}^n$ have "volume" δ, meaning $\mathbf{E}[1_A] = \delta$, where

$$1_A\colon \{-1,1\}^n \to \{0,1\},\ x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Suppose φ is a probability density function on $\{-1,1\}^n$ which is *supported* on A, i.e., $\varphi(x) = 0$ when $x \in \{-1,1\}^n \setminus A$. Show that $\|\varphi\|_2^2 \geq \frac{1}{\delta}$ with equality if and only $\varphi = \varphi_A$.

Exercise 3. Let φ and ψ be probability density functions on $\{-1,1\}^n$.

(a) Show that the *total variation distance* between φ and ψ defined by

$$d_{\mathrm{TV}}(\varphi,\psi) := \max_{A \subseteq \{-1,1\}^n} \left\{ \left| \Pr_{\mathbf{x}\sim\varphi}[\mathbf{x} \in A] - \Pr_{\mathbf{x}\sim\psi}[\mathbf{x} \in A] \right| \right\},$$

is equal to $\frac{1}{2}\|\varphi - \psi\|_1$ where $\|f\|_1 := \mathbf{E}_{\mathbf{x}}[|f(x)|]$ for all $f \in \mathbb{R}^{\{-1,1\}^n}$.

(b) Show that the total variation distance of φ from (the probability density function of) the uniform distribution is at most $\frac{1}{2}\sqrt{\mathbf{Var}[\varphi]}$.

(c) The χ^2-*distance* of φ from ψ is defined by

$$d_{\chi^2}(\varphi, \psi) := \mathop{\mathbf{E}}_{\mathbf{x}\sim\psi}\left[\left(\frac{\varphi(\mathbf{x})}{\psi(\mathbf{x})} - 1\right)^2\right].$$

Show that the χ^2-distance of φ from uniform is equal to $\mathbf{Var}[\varphi]$.

(d) Show that the *collision probability* of φ defined by

$$\mathop{\mathbf{Pr}}_{\substack{\mathbf{x},\mathbf{y}\sim\varphi \\ \text{independently}}}[\mathbf{x} = \mathbf{y}],$$

is equal to $\frac{\|\varphi\|_2^2}{2^n}$.

Due Wednesday, November 30, 2016, 11:44 Uhr. Post it in box 18 near room F411.