University of Konstanz
Department of Mathematics and Statistics
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Fourier Analysis of Boolean Functions – Exercise Sheet 4

Exercise 1. For the whole exercise fix $n \in \mathbb{N}_0$. A *non-monomial* Boolean function on n bits is a function $f\colon \{-1,1\}^n \to \{-1,1\}$ such that $f \neq \chi_S$ for all $S \subseteq [n]$.

(a) Determine the (relative Hamming) distance between the set of monomial Boolean functions on n bits and the set of non-monomial Boolean functions on n bits.

(b) Parametrize the set of non-monomial Boolean functions that have minimal distance to the set of monomial Boolean functions. Compute the exact Fourier expansion of each element of this set depending on the two parameters.

(c) Compute

$$\mathop{\mathbf{Pr}}_{\substack{\mathbf{x},\mathbf{y}\sim\{-1,1\}^n\\ \text{independently}}}[f(\mathbf{xy}) = f(\mathbf{x})f(\mathbf{y})].$$

Is this probability independent from the parameters from (b)? Comment the result in view of §1.4 of the lecture notes.

Exercise 2. Show that there is no $c > 3$ such that for every $n \in \mathbb{N}_0$ in the situation of the robust version of 1.4.1 the implication $(\mathrm{a}_{c\varepsilon}) \implies (\mathrm{b}_\varepsilon)$ holds true.

Exercise 3. *A variant of §1.4*
Let $n \in \mathbb{N}_0$ and $f\colon \{-1,1\}^n \to \{-1,1\}$.

(a) Show that $f(x)f(y)f(z) = f(xyz)$ holds for all $x, y, z \in \{-1,1\}^n$ if and only if there exist $S \subseteq [n]$ and $\sigma \in \{-1,1\}$ such that $f = \sigma\chi_S$.

(b) Show

$$\mathop{\mathbf{E}}_{\substack{\mathbf{x},\mathbf{y},\mathbf{z}\sim\{-1,1\}^n\\ \text{independently}}}[f(\mathbf{x})f(\mathbf{y})f(\mathbf{z})f(\mathbf{xyz})] = \sum_{S\subseteq[n]} \widehat{f}(S)^4.$$

(c) Show $\delta := \min\{\operatorname{dist}(f, \sigma\chi_S) \mid S \subseteq [n], \sigma \in \{-1,1\}\} < \frac{1}{2}$.

Similar to Theorem 1.4.4 consider the following conditions for $\varepsilon \in \mathbb{R}_{\geq 0}$:

$$(\mathrm{a}_\varepsilon)\quad \mathop{\mathbf{Pr}}_{\substack{\mathbf{x},\mathbf{y},\mathbf{z}\sim\{-1,1\}^n\\ \text{independently}}}[f(\mathbf{x})f(\mathbf{y})f(\mathbf{z}) = f(\mathbf{xyz})] \geq 1-\varepsilon$$

$$(\mathrm{b}_\varepsilon)\ \exists S \subseteq [n] : \exists \sigma \in \{-1,1\} : \operatorname{dist}(f, \sigma\chi_S) \leq \varepsilon$$

(d) Show that (b_ε) implies $(\mathrm{a}_{4\varepsilon})$ for all $\varepsilon \in \mathbb{R}_{\geq 0}$.

(e) Show that $(\mathrm{a}_{4\varepsilon-16\varepsilon^2+28\varepsilon^3-16\varepsilon^4})$ implies (b_ε) for all $\varepsilon \in \mathbb{R}_{\geq 0}$.

Due Wednesday, Dezember 14, 2016, 11:44 Uhr. Post it in box 18 near room F411.