# Positive semidefinite quadratic forms on varieties defined by quadratic forms

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**Abstract:** For a fixed number of $n+1$ ($n \geq 1$) variables and even degree $2d$ ($d \geq 1$), the SOS cone $\Sigma_{n+1,2d}$ of all real forms representable as finite sums of squares (SOS) of half degree $d$ real forms is included in the PSD cone of all positive semidefinite (PSD) real forms $\mathcal{P}_{n+1,2d}$. Hilbert (1888) states that both cones coincide if and only if $n+1=2$, $d=1$ or $(n+1,2d)=(3,4)$. In this talk, we discuss necessary or sufficient conditions to extend local positive semidefiniteness of real quadratic forms along projective varieties generated by $s$ ($s \geq 0$) real quadratic forms. Those conditions allow us to construct an explicit filtration of intermediate cones $\Sigma_{n+1,2d} = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{s-1} \subseteq C_s = \mathcal{P}_{n+1,2d}$ (between the SOS and PSD cone) along the Veronese variety. Indeed, the latter is known to be a projective variety finitely induced by real quadratic forms. We analyze this filtration for proper inclusions. In fact, after applying an inductive argument, it suffices to investigate the situation for a truncated subfiltration of the former. A result of Blekherman et al. (2016) on projective varieties of minimal degree permits us to handle the inclusion $C_0 \subseteq C_1$. Generalizing this observation, we are able to show $\Sigma_{n+1,2d} = C_0 = \ldots = C_n$. Finally, we lay out the situation in the basic non Hilbert case of quaternary quartics by identify exactly two strictly separating intermediate cones in the particular filtration of $\Sigma_{4,4}$ and $\mathcal{P}_{4,4}$ via considerations of real forms based on techniques due to Robinson (1969) and Choi and Lam (1977a,b). This is a work in progress with Salma Kuhlmann and Charu Goel.

*Keywords:* Real quadratic forms, projective varieties generated by real quadratic forms, positive semidefinite forms, sums of squares, intermediate cones, varieties of minimal degree

## 1. INTRODUCTION

For $n \geq 0$, let $\mathbb{R}[X]$ be the polynomial ring in $n+1$ variables with coefficients in $\mathbb{R}$. If all monomials appearing in $f \in \mathbb{R}[X]$ are of the same total degree $d$ ($d \geq 1$), then $f$ is a *(real) form (of total degree d)*. The set of all real forms of total degree $d$ in $\mathbb{R}[X]$ is $\mathcal{F}_{n+1,d}$. In particular, $f \in \mathcal{F}_{n+1,2}$ is a *(real) quadratic form*. Moreover, if for $f \in \mathcal{F}_{n+1,2d}$ there exist some $t \geq 1$ and $g_1, \ldots, g_t \in \mathcal{F}_{n+1,d}$ such that $f = \sum_{i=1}^{t} g_i^2$, then $f$ is a *sum of squares (SOS)*. The cone of all SOS forms in $\mathcal{F}_{n+1,2d}$ is $\Sigma_{n+1,2d}$. Moreover, $f \in \mathcal{F}_{n+1,2d}$ is *locally positive semidefinite on* $W \subseteq \mathbb{R}^{n+1}$ if $f(x) \geq 0$ holds for all $x \in W$. In this case we write $f|_W \geq 0$, respectively, $f \geq 0$ for $W = \mathbb{R}^{n+1}$. In the latter case, $f$ is *(globally) positive semidefinite (PSD)*. The cone of all PSD forms in $\mathcal{F}_{n+1,2d}$ is $\mathcal{P}_{n+1,2d}$. It is clear that $\Sigma_{n+1,2d} \subseteq \mathcal{P}_{n+1,2d}$ always holds true and, especially, $\Sigma_{1,2d} = \mathcal{P}_{1,2d}$ in the univariate case (see Marshall (2008)). However, the situation is more evolved in the multivariate cases. Hence, from now on we assume $n \geq 1$.

*Theorem 1.* (Hilbert (1888)) Let $n$ and $d$ be positive integers. Then $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$ if and only if $n+1=2$ or $d=1$ or $(n+1,2d)=(3,4)$.

All cases in which the SOS and PSD cone coincide are called *Hilbert cases*, whereas all others are refered to as *non Hilbert cases*. The two simplest non Hilbert cases $(3,6)$ and $(4,4)$ are the *basic* non Hilbert cases.

Let $(n+1,2d)$ from now on denote a non Hilbert case and $\{m_0(X), \ldots, m_k(X)\}$ be an ordered monomial basis of $\mathcal{F}_{n+1,d}$ with $k := \dim(\mathcal{F}_{n+1,d}) - 1$. For $l \in \{n,k\}$, let $\mathbb{P}^l$ be the $l$-dimensional projective space of the complex numbers and the set of all real points of $W$ is denoted by $W(\mathbb{R})$ for any $W \subseteq \mathbb{P}^l$. Implicitly $x \in \mathbb{R}^{l+1}$ is assumed for any $[x] \in W(\mathbb{R})$. A form $f \in \mathcal{F}_{l+1,2d}$ is *locally positive semidefinite on* $W(\mathbb{R}) \subseteq \mathbb{P}^n(\mathbb{R})$ if $f(x_0, \ldots, x_l) \geq 0$ holds for any $[x_0 : \ldots : x_l] \in W(\mathbb{R})$ and we write $f|_{W(\mathbb{R}))} \geq 0$. This is a well defined expression due to the homogeneity of $f$ in even degree. In particular, the cone of all forms in $\mathcal{F}_{l+1,2d}$ which are locally positive semidefinite on $\mathbb{P}^n(\mathbb{R})$ is the former PSD cone $\mathcal{P}_{l+1,2d}$.

In a Gram matrix approach (see Choi et al. (1995), Powers and Wörmann (1998)), we consider the isomorphism

$$Q\colon \mathrm{Sym}_{k+1}(\mathbb{R}) \to \mathcal{F}_{k+1,2}$$
$$A \mapsto q_A,$$

where $q_A(Z_0, \ldots, Z_k) := (Z_0 \ldots Z_k) A (Z_0 \ldots Z_k)^t$, and the surjective linear *Gram map*

$$\mathcal{G}\colon \mathrm{Sym}_{k+1}(\mathbb{R}) \to \mathcal{F}_{n+1,2d}$$
$$A \mapsto f_A,$$

where $f_A(X) := (m_0(X) \dots m_k(X))A(m_0(X) \dots m_k(X))^t$ for the indeterminantes $X = (X_0, \dots, X_n)$, over the $\mathbb{R}$-vector space $\mathrm{Sym}_{k+1}(\mathbb{R})$ of real symmetric $(k+1) \times (k+1)$ matrices. Then a generic $A_f \in \mathcal{G}^{-1}(f)$ for any $f \in \mathcal{F}_{n+1,2d}$ can be fixed. In fact, any $A \in \mathcal{G}^{-1}(f)$ is a *Gram matrix associated to* $f$ and for any such, $q_A := Q(A) \in \mathcal{F}_{k+1,2}$ is a *(real) quadratic form associated to* $f$.

*Proposition 2.* A form $f \in \mathcal{F}_{n+1,2d}$ is SOS if and only if there exists a real quadratic form associated to $f$ which is locally positive semidefinite on $\mathbb{P}^k(\mathbb{R})$.

Under the consideration of the *(projective) Veronese embedding*

$$V\colon \mathbb{P}^n \to \mathbb{P}^k$$
$$[x] \mapsto [m_0(x) : \dots : m_k(x)]$$

and its image the *(projective) Veronese variety* $V(\mathbb{P}^n)$, the PSD forms in $\mathcal{F}_{n+1,2d}$ can be characterized.

*Proposition 3.* A form $f \in \mathcal{F}_{n+1,2d}$ is PSD if and only if there exists a real quadratic form associated to $f$ which is locally positive semidefinite on $V(\mathbb{P}^n)(\mathbb{R})$.

## 2. THE MAIN QUESTIONS

The previous two propositions reveal that the question of whether or not a given PSD form is SOS is equivalent to asking whether or not a given locally on $V(\mathbb{P}^n)(\mathbb{R})$ positive semidefinite real quadratic form can be extended to a real quadratic form locally positive semidefinite on $\mathbb{P}^k(\mathbb{R})$ over the set of real points of the Veronese variety. Indeed, the latter is a projective variety finitely generated by real quadratic forms of a specific structure imposed by the Gram map (see Plaumann (2020)). More precisely, the projective variety $V(\mathbb{P}^n)$ is induced by

$$\mathcal{S} := \{q(Z_0, \dots, Z_k) := Z_iZ_j - Z_sZ_t \mid \mathrm{LE}(m_i) + \mathrm{LE}(m_j) = \mathrm{LE}(m_s) + \mathrm{LE}(m_t)\} \subseteq \mathbb{R}[Z_0, \dots, Z_k],$$

where LE denotes the (leading) exponent of the indicated monomial. In general, the following question has to be answered.

*Question 1.* Let $W_0 \subseteq W_1$ be projective varieties finitely induced by real quadratic forms with non-empty sets of real points. Assume that a real quadratic form $q$ is locally positive semidefinite on $W_0(\mathbb{R})$. When exactly does there exist a real quadratic form $q_0$ vanishing on $W_0(\mathbb{R})$ such that $q + q_0$ is locally positive semidefinite on $W_1(\mathbb{R})$?

Under the assumption of $W_0$ being an irreducible projective variety with Zariski dense set of real points $W_0(\mathbb{R})$ and $W_1$ being the projective space $\mathbb{P}^k$, Blekherman et al. (2016) give an answer to the above question. They establish that any real quadratic form $q$ which is locally positive semidefinite on $W_0(\mathbb{R})$ is already SOS in the respective real homogeneous coordinate ring if and only if $W_0$ is a projective variety of minimal degree, i.e. a nondegenerate (not contained in any hyperplane of $\mathbb{P}^k$) irreducible projective variety with $\deg(W_0) = 1 + \mathrm{codim}(W_0)$. This result provides an alternative proof of Hilbert's 1888 Theorem by setting $W_0$ to be the Veronese variety and observing it being a projective variety of minimal degree exactly in the Hilbert cases.

Any subset $\mathcal{S}'$ of $\mathcal{S}$ naturally induces a supvariety $W$ of the Veronese variety, which is consequently again finitely induced by real quadratic forms of the specific structure imposed by the Gram map. The kernel of the Gram map can be described via the set of real points of the Veronese variety. Indeed, the set $Q(\mathcal{G}^{-1}(f))$ of all quadratic forms associated to $f \in \mathcal{F}_{n+1,2d}$ is completely determined by

$$\mathcal{G}^{-1}(f) = \{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A = q_{A_f} \text{ on } V(\mathbb{P}^n)(\mathbb{R})\}.$$

Hence, given a quadratic form locally positive semidefinite on $W(\mathbb{R})$, we can ask under exactly what conditions this form extends to a quadratic form locally positive semidefinite on $\mathbb{P}^k(\mathbb{R})$ over the set of real points of the Veronese variety. Set

$$\begin{aligned} C_W &:= \{f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f) \colon q_A|_{W(\mathbb{R})} \geq 0\} \\ &= \{f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f) \colon q_A|_{W(\mathbb{R})} \geq 0 \\ &\qquad \wedge q_A = q_{A_f} \text{ on } V(\mathbb{P}^n)(\mathbb{R})\}. \end{aligned}$$

Then by Proposition 2 and Proposition 3, it is clear that $C_W$ is an intermediate cone of the SOS and PSD cone. We especially investigate the inclusions in

$$\Sigma_{n+1,2d} \subseteq C_W \subseteq \mathcal{P}_{n+1,2d}$$

for strictness. Indeed, at least one of these inclusions has to be strict because $(n+1, 2d)$ is assumed to be a non Hilbert case. The following question has to be answered.

*Question 2.* Let $W_0 \subseteq W_1 \subseteq W_2$ be projective varieties finitely induced by real quadratic forms with non-empty sets of real points. Assume that a real quadratic form $q$ is locally positive semidefinite on $W_1(\mathbb{R})$. When exactly does there exist a real quadratic form $q_0$ vanishing on $W_0(\mathbb{R})$ such that $q + q_0$ is locally positive semidefinite on $W_2(\mathbb{R})$?

## 3. A FILTRATION OF INTERMEDIATE CONES

We algorithmically construct a particular $\mathcal{S}' \subseteq \mathcal{S}$ with a fixed numeration $\mathcal{S}' = \{p_1, \dots, p_s\}$ ($s := \#\mathcal{S}'$) such that the zero set of $\mathcal{S}'$ is the Veronese Variety and

$$V(\mathbb{P}^n) = W_s \subsetneq W_{s-1} \subsetneq \dots \subsetneq W_1 \subsetneq W_0 = \mathbb{P}^k$$

for $W_i := \mathcal{V}(p_1, \dots, p_i)$ ($i \in \{1, \dots, s\}$) and $W_0 := \mathbb{P}^k$. This leads to a corresponding strict filtration of sets of real points

$$V(\mathbb{P}^n)(\mathbb{R}) = W_s(\mathbb{R}) \subsetneq \dots \subsetneq W_0(\mathbb{R}) = \mathbb{P}^k(\mathbb{R}).$$

Setting $C_i := C_{W_i}$ we thus obtain a filtration of intermediate cones of the SOS and PSD cone, namely

$$\Sigma_{n+1,2d} = C_0 \subseteq C_1 \subseteq \dots \subseteq C_{s-1} \subseteq C_s = \mathcal{P}_{n+1,2d} \quad (1)$$

(see Goel (2020)). Since $(n+1, 2d)$ is a non Hilbert case by choice, at least one inclusion in (1) has to be strict. An answer to Question 2 in particularly provides a tool for identifying all strict inclusions in (1). Yet, it is not compulsory to investigate each inclusion in (1).

In the explicit construction of $\mathcal{S}'$, we determine

$$s = \#\mathcal{S}' = \sum_{m=1}^{n} 2(k(m) - m) - (k(m-1) + 1)$$

with

$$k\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$
$$m \mapsto \binom{m+d}{m} - 1.$$

Setting $\tau := 2(k(n) - n) - (k(n-1) + 1)$, we are able to identify the filtration of the last $s - \tau + 1$ cones

$$C_\tau \subseteq \ldots \subseteq C_s = \mathcal{P}_{n+1,2d} \quad (2)$$

with the $(n, 2d)$ case. This ensures the elimination of one variable in an inductive argument. Repeating that consideration, we at last arrive in the base case $(2, 2d)$. This is a Hilbert case and therefore fully understood. It thus remains to investigate the situation of the first $\tau + 1$ cones

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_\tau. \quad (3)$$

Indeed, putting (2) and (3) together recovers the initial filtration (1).

In (3), an immediate application of the main result from Blekherman et al. (2016) allows us to conclude that the SOS cone always coincides with $C_1$. Furthermore, a slight variation of this result ensures $\Sigma_{n+1,2d} = C_i$ for any $i \in \{1, \ldots, n\}$. Thus,

$$\Sigma_{n+1,2d} = C_0 = \ldots = C_n. \quad (4)$$

After that, for inclusions of the type $C_i \subseteq C_{i+1}$ with $i \in \{n, \ldots, \tau - 1\}$, the situation is more evolved and other methods have to be applied.

For example, in the basic non Hilbert case $(4, 4)$, exactly two strictly separating intermediate cone in (3) are identifiable. Indeed, the famous *Robinson form*

$$\begin{aligned} R(X_0, X_1, X_2, X_3) &:= X_0^2(X_0 - X_3)^2 + X_1^2(X_1 - X_3)^2 \\ &+ X_2^2(X_2 - X_3)^2 + 2X_0X_1X_2(X_0 + X_1 + X_2 - 2X_3) \end{aligned}$$

and the *Choi-Lam form*

$$\begin{aligned} W(X_0, X_1, X_2, X_3) &:= X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2 + X_3^4 \\ &- 4X_0X_1X_2X_3 \end{aligned}$$

both certify the proper containment $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ (see Robinson (1969) and Choi and Lam (1977a,b)). In particular, the Robinson form was alongside the *Motzkin form* (see Motzkin (1967)) one of the first forms found separating the SOS and PSD cone in a basic non Hilbert case. Both were firstly mentioned roughly nine decades after Hilbert's original abstract proof from 1888 in the late 1960's. The Choi-Lam form followed in 1977.

Now, a deeper reaching investigation of the Robinson form, the Choi-Lam form and a variation of the Choi-Lam form under permuation of variables reveals

$$C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 \quad (5)$$

in the basic non Hilbert case of quaternary quartics. Furthermore,

$$\Sigma_{4,4} = C_0 = C_1 = C_2 = C_3$$

by (4) and $C_6 \subseteq \ldots \subseteq C_{10} = \mathcal{P}_{4,4}$ corresponds to (2) and with that to the $(3, 4)$ case by our inductive argument. The ternary quartics describe a Hilbert case and, consequently, the latter subfiltration collapses to

$$C_6 = \ldots = C_{10} = \mathcal{P}_{4,4}.$$

Putting it all together, we thus fully understand the situation in the basic non Hilbert case of quaternary quartics.

In particular, we strengthen Hilbert's original observation from 1888 in the quaternary quartics case by testifying the existence of two distinct strictly separating intermediate cones between the SOS and the PSD cone.

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