The necessity part of Hilbert's 1888 Theorem: an explicit example of psd not sos ternary sextic

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Abstract

Using results from algebraic geometry, Hilbert showed that $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$, where $\mathcal{P}_{n,m}$ and $\Sigma_{n,m}$ denote the cones of positive semidefinite (psd) and sums of squares (sos) n-ary m-ic forms respectively. He also showed that this is sufficient to construct counter examples for all the other cases i.e. $\Sigma_{n,m} \subsetneq \mathcal{P}_{n,m}$ for all $n \geq 3$, $m \geq 4$ and $(n, m) \neq (3, 4)$ with m even. This is exactly the necessity part of the celebrated Hilbert's 1888 Theorem. In this seminar we will present an explicit example of a ternary sextic form that is psd but not sos. This example was given by Schmüdgen in 1979, who used in his proof only elementary computations without applying the theory of algebraic curves.