A constructive proof of the Helton-Vinnikov Theorem

Sarah-Tanja Hess

University of Konstanz

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Outline

Hyperbolicity

Definition

Let n be a positive integer. A homogenous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is called **hyperbolic with respect to** $e \in \mathbb{R}^n$, if $p(e) \neq 0$ and for any $v \in \mathbb{R}^n$ the univariate polynomial $g(S) := p(v - Se) \in \mathbb{R}[S]$ only has real roots.

Example

Consider $p(x, y, z) := x^2 - y^2 - z^2 \in \mathbb{R}[x, y, z]$. Then p is hyperbolic with respect to $e := (1, 0, 0) \in \mathbb{R}^3$.
Indeed, $p(e) := p(1, 0, 0) = 1 \neq 0$. For any $v := (v_1, v_2, v_3) \in \mathbb{R}^3$ we have

$$g(S) := p(v - Se) := (v_1 - S)^2 - v_2^2 - v_3^2 = v_1^2 - 2v_1 S + S^2 - v_2^2 - v_3^2$$

has a non negative discriminant

$$\Delta = (-2v_1)^2 - 4(v_1^2 - v_2^2 - v_3^2) = 4\underbrace{(v_2^2 + v_3^2)}_{\geq 0} \geq 0.$$

So all roots of g are real. ■

The Lax Conjecture

The Lax Conjecture was posed in 1958 by P. D. Lax in [7].

Lax Conjecture

Let d be a positive integer and $p \in \mathbb{R}[x, y, z]$, then p is hyperbolic of degree d with respect to $e := (1, 0, 0) \in \mathbb{R}^3$ such that $p(e) = 1$ if and only if there exist two Hermitian complex matrices $Y, Z \in \mathbb{C}^{d\times d}$ such that

$$p(x, y, z) = \det(xI_d + yY + zZ).$$

The trivial direction of the Lax Conjecture

Let $p \in \mathbb{R}[x, y, z]$ be of the form

$$p(x, y, z) = \det(xI_d + yY + zZ)$$

with two complex Hermitian matrices $Y, Z \in \mathbb{C}^{d\times d}$. Then p is hyperbolic of degree d with respect to $e := (1, 0, 0) \in \mathbb{R}^3$ and $p(e) = 1$.

The Lax Conjecture

Proof.

Indeed, $p(e) := p(1,0,0) = \det(1 * I_d + 0 * Y + 0 * Z) = \det(I_d) = 1$. Moreover, for $v := (v_1, v_2, v_3) \in \mathbb{R}^3$ consider the univariate polynomial $g(S) := p(v - Se) \in \mathbb{R}[S]$. Let $s \in \mathbb{C}$ be a root of g i.e.

$$\begin{aligned} 0 &= g(s) := p(v - se) = \det((v_1 - s)I_d + v_2 Y + v_3 Z) \\ &= \det((v_1 I_d + v_2 Y + v_3 Z) - sI_d) = \chi_{v_1 I_d + v_2 Y + v_3 Z}(s). \end{aligned}$$

Clearly, $v_1 I_d + v_2 Y + v_3 Z$ is a Hermitian matrix and so any Eigenvalue of $v_1 I_d + v_2 Y + v_3 Z$ is real. Thence, $s \in \mathbb{R}$. ∎

Real zero polynomials

Definition

A bivariate polynomial $q \in \mathbb{R}[y, z]$ is called a **real zero polynomial**, if for any $w := (w_2, w_3) \in \mathbb{R}^2$ the univariate polynomial

$$f(T) := q(Tw_2, Tw_3) \in \mathbb{R}[T]$$

only has real roots.

Example

Consider $q(y, z) := 1 - y^2 - z^2 \in \mathbb{R}[y, z]$. Let $w := (w_2, w_3) \in \mathbb{R}^2$ and observe that

$$f(T) := q(Tw_2 Tw_3) := 1 - T^2 w_2^2 - T^2 w_3^2 = T^2(-w_2^2 - w_3^2) + 1,$$

which has a non negative discriminant

$$\Delta = 4(w_2^2 + w_3^2) \geq 0.$$

So f only has real roots. ■

Motivation of real zero polynomials

1. If $p \in \mathbb{R}[x, y, z]$ is hyperbolic of degree $d \in \mathbb{Z}_{\geq 0}$ with respect to $e := (1, 0, 0) \in \mathbb{R}^3$, then we can reduce p to a bivariate polynomial $q \in \mathbb{R}[y, z]$ by setting
$$q(y, z) := p^D(y, z) := p(1, y, z).$$
2. Since p is hyperbolic of degree $d \in \mathbb{Z}_{\geq 0}$ with respect to $e := (1, 0, 0) \in \mathbb{R}^3$ we know for any $v \in \mathbb{R}^3$ the univariate polynomial $g(S) := p(v - Se) \in \mathbb{R}[S]$ only has real roots. Under dehomogenization this translates as follows:
For any $w := (w_2, w_3) \in \mathbb{R}^2$.
$$\begin{aligned} f(T) &:= q(Tw_2, Tw_3) := p(1, Tw_2, Tw_3) \\ &= p(T(0, w_2, w_3) + (1, 0, 0)) \\ &= p(T((0, w_2, w_3) + \frac{1}{T}(1, 0, 0)) \\ &= T^d p(\underbrace{(0, w_2, w_3)}_{"=v"} + \underbrace{\frac{1}{T}}_{"=-S"}(1, 0, 0)) \in \mathbb{R}[T] \end{aligned}$$

only has real roots.

The two-sided medal

Lemma

1. *Let* $p \in \mathbb{R}[x, y, z]$ *be a hyperbolic polynomial with respect to* $e := (1, 0, 0) \in \mathbb{R}^3$ *such that* $p(e) = 1$. *Then the bivariate polynomial*

$$q(y, z) := p^D(y, z) := p(1, y, z) \in \mathbb{R}[y, z]$$

is a real zero polynomial of degree no more than the degree of p *and* $q(0, 0) = 1$.

2. *Vice versa, let* $q \in \mathbb{R}[y, z]$ *be a real zero polynomial of degree* $d \in \mathbb{Z}_{\geq 0}$ *such that* $q(0, 0) = 1$. *Then the homogeneous polynomial*

$$p(x, y, z) := q^H(x, y, z) := x^d q\left(\frac{y}{x}, \frac{z}{x}\right) \in \mathbb{R}[x, y, z]$$

is hyperbolic with respect to $e := (1, 0, 0) \in \mathbb{R}^3$ *and* $p(e) = 1$. *Moreover,* $\deg(p) = \deg(q) =: d$.

The two-sided medal

Example

Recall

$$p(x, y, z) := x^2 - y^2 - z^2 \in \mathbb{R}[x, y, z]$$

is hyperbolic with respect to $e := (1, 0, 0) \in \mathbb{R}^3$ and

$$q(y, z) := 1 - y^2 - z^2 \in \mathbb{R}[y, z]$$

is a real zero polynomial. Obviously

$$p^D(y, z) := p(1, y, z) := 1 - y^2 - z^2 =: q(y, z)$$

and

$$q^H(x, y, z) := x^2 q\left(\frac{y}{x}, \frac{z}{x}\right) := x^2 - y^2 - z^2 =: p(x, y, z).$$

So they are the homogeneous respectively dehomogeneous version of one another.

The Helton-Vinnikov Theorem

Theorem (Helton-Vinnikov)

Let d be a positive integer, then a bivariate polynomial $q \in \mathbb{R}[y, z]$ of degree d is a real zero polynomial with $q(0,0) = 1$ if and only if there exist two Hermitian complex matrices $Y, Z \in \mathbb{C}^{d\times d}$ such that

$$q(y, z) = \det(I_d + yY + zZ).$$

This was first proven in [4, 2002, J. W. Helton and V. Vinnikov].

Theorem

Lax Conjecture $\Leftrightarrow$ Helton-Vinnikov Theorem

This was observed in [8, 2005, A. S. Lewis et al.].

The trivial direction of the Helton-Vinnikov

Let d be a positive integer, $q \in \mathbb{R}[y, z]$ of degree d such that there exist two Hermitian complex matrices $Y, Z \in \mathbb{C}^{d\times d}$ with $q(y, z) = \det(I_d + yY + zZ)$, then q is a real zero polynomial with $q(0, 0) = 1$.

The Helton-Vinnikov Theorem

Proof.

Clearly, $q(0,0) = \det(I_d + 0 * Y + 0 * Z) = \det(I_d) = 1$.
Fix $w := (w_2, w_3) \in \mathbb{R}^2$ and let $t \in \mathbb{C}\backslash\{0\}$ be a root of $f(T) := q(Tw_2, Tw_3) \in \mathbb{R}[T]$. Then

$$\begin{aligned} 0 &= f(t) := q(tw_2, tw_3) = \det(I_d + t(w_2 Y) + t(w_3 Z)) \\ &= \det(I_d + t(w_2 Y + w_3 Z)) \\ &= \det\left(t\left(\frac{1}{t} I_d + (w_2 Y + w_3 Z)\right)\right) \\ &= t^d \det\left(\frac{1}{t} I_d + (w_2 Y + w_3 Z)\right) = \underbrace{t^d}_{\neq 0} \chi_{w_2 Y + w_3 Z}\left(-\frac{1}{t}\right). \end{aligned}$$

Hence, we have $-\frac{1}{t} \in \mathbb{R}\backslash\{0\}$, since $w_2 Y + w_3 Z$ is Hermitian.
Therefore, $t \in \mathbb{R}$. ■

The Proof

We will now give a constructive proof of the Helton-Vinnikov Theorem following [3, 2016, Grinshpan et al.].

The non-trivial direction.

Clearly, q is of the form

$$q(y,z) = \sum_{i=0}^{d}\sum_{j=0}^{d-i} a_{ij} y^i z^j,$$

where $a_{ij} \in \mathbb{R}$. For any $s \in \mathbb{R}$ fix

$$\tilde{q}_s(T) := T^d q\left(\frac{1}{T}, \frac{s}{T}\right) \in \mathbb{R}[T].$$

Note

$$y^d \tilde{q}_{\frac{z}{y}}\left(\frac{1}{y}\right) = q(y,z).$$

Example

Let us consider

$$q(y,z) := 1 - y^2 - z^2 \in \mathbb{R}[y,z].$$

For any $s \in \mathbb{R}$ set

$$\begin{aligned}\tilde{q}_s(T) &:= T^2 q\left(\frac{1}{T}, \frac{s}{T}\right)\\ &= T^2\left(1 - \frac{1}{T^2} - \frac{s^2}{T^2}\right)\\ &= T^2 - 1 - s^2.\end{aligned}$$

The Proof

The non-trivial direction.

With the given polynomial representation of q observe

$$\tilde{q}_s(T) = \sum_{m=0}^{d} q_{d-m}(s) T^m$$

for some $q_{d-m} \in \mathbb{R}[S]_{\leq d-m}$ and

$$q_{d-d}(s) = q(0,0) = 1.$$

Therefore, $\tilde{q}_s$ is a monic univariate polynomial of degree d.

Example

As for any $s \in \mathbb{R}$ we have

$$\begin{aligned} \tilde{q}_s(T) &:= T^2 q\left(\frac{1}{T}, \frac{s}{T}\right) \\ &= T^2\left(1 - \frac{1}{T^2} - \frac{s^2}{T^2}\right) \\ &= T^2 - 1 - s^2, \end{aligned}$$

with

$$\begin{aligned} q_0 &:\equiv 1 \in \mathbb{R}[S]_{\leq 0} \\ q_1 &:\equiv 0 \in \mathbb{R}[S]_{\leq 1} \\ q_2(S) &:= -1 - S^2 \in \mathbb{R}[S]_{\leq 2}. \end{aligned}$$

The Proof

The non-trivial direction.

Let $a \in \mathbb{C} \backslash \{0\}$ be a root of $\tilde{q}_s$. Hence,

$$\begin{aligned} 0 &= \tilde{q}_s(a) := a^d * q\left(\frac{1}{a}, \frac{s}{a}\right) \\ &= \underbrace{a^d}_{\neq 0} * q\left(\frac{1}{a} * 1, \frac{1}{a} * s\right) \end{aligned}$$

gives $\frac{1}{a} \in \mathbb{R}$ respectively $a \in \mathbb{R}$, due to q being a real zero polynomial.

Let us for now assume that any root of $\tilde{q}_s$ is simple. Denote these distinct roots of $\tilde{q}_s$ by $\lambda_1(s), \ldots, \lambda_d(s) \in \mathbb{R}$.

Example

Furthermore, we have to determine the roots of $\tilde{q}_s$. Clearly, for any $a \in \mathbb{C}$ we have

$$0 = \tilde{q}_s(a) = a^2 - 1 - s^2$$

if and only if

$$a^2 = 1 + s^2 > 0.$$

So immediately $a \in \mathbb{R}$ and we set

$$\lambda_1(s) := \sqrt{1 + s^2} \in \mathbb{R}$$

and

$$\lambda_2(s) := -\sqrt{1 + s^2} \in \mathbb{R}.$$

The Proof

The non-trivial direction.

Let us now consider the companion matrix $C(s)$ of $\tilde{q}_s$ given by

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & -q_d(s) \\ 1 & 0 & \cdots & 0 & -q_{d-1}(s) \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -q_2(s) \\ 0 & \cdots & 0 & 1 & -q_1(s) \end{pmatrix}.$$

Claim 1 For any positive integer d and any $s \in \mathbb{R}$ we have

$$\tilde{q}_s(T) = \det(TI_d - C(s)).$$

Example

We can easily give the companion matrix of $\tilde{q}_s$ namely

$$C(s) := \begin{pmatrix} 0 & 1+s^2 \\ 1 & 0 \end{pmatrix}$$

and clearly

$$\begin{aligned} \tilde{q}_s(T) &= T^2 - 1 - s^2 \\ &= \det \begin{pmatrix} T & -1-s^2 \\ -1 & T \end{pmatrix} \\ &= \det(TI_2 - C(s)). \end{aligned}$$

The Proof

The non-trivial direction.

For $j \in \{0, \ldots, 2(d-1)\}$ set

$$s_j(S) := \sum_{k=1}^{d} \lambda_k(S)^j.$$

Clearly, for any $j \in \{0, \ldots, 2(d-1)\}$ we have $\deg(s_j) \leq j$.
For any $s \in \mathbb{R}$ the Hermite matrix of $\tilde{q}_s$ is given by

$$H(s) := (s_{i+j}(s))_{i,j=0,\ldots,d-1}.$$

Clearly, $H \in \mathbb{R}^{d \times d}[S]$ is a matrix of polynomials of degree $2(d-1)$

Example

For the newton sums we get

$$\begin{aligned} s_0 &:\equiv 2 \in \mathbb{R}[S]_{\leq 0} \\ s_1 &:\equiv 0 \in \mathbb{R}[S]_{\leq 1} \\ s_2(S) &:= 2(1+S^2) \in \mathbb{R}[S]_{\leq 2}. \end{aligned}$$

and so

$$\begin{aligned} H(S) &:= \begin{pmatrix} s_0(S) & s_1(S) \\ s_1(S) & s_2(S) \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2(1+S^2) \end{pmatrix}. \end{aligned}$$

The Proof

The non-trivial direction.

Let

$$V(S) := (\lambda_{i+1}(S)^j)_{i,j=0,\dots,d-1}.$$

Obviously

$$\begin{aligned} V(S)^T V(S) &= \left(\sum_{k=0}^{d-1} \lambda_{k+1}(S)^i \lambda_{k+1}(S)^j\right)_{i,j\in\{0,\dots,d-1\}} \\ &= \left(\sum_{k=1}^{d} \lambda_k(S)^{i+j}\right)_{i,j\in\{0,\dots,d-1\}} \\ &= (s_{i+j}(S))_{i,j\in\{0,\dots,d-1\}} = H(S). \end{aligned}$$

The Proof

The non-trivial direction.

For any $s \in \mathbb{R}$ the matrix $H(s)$ is Hermitian.
Indeed,

$$\begin{aligned} H(s)^H &= (V(s)^T V(s))^H \\ &= V(s)^H (V(s)^T)^H \\ &\underset{V(s)\in\mathbb{R}^{d\times d}}{=} V(s)^T (V(s)^T)^T \\ &= V(s)^T V(s) \\ &= H(s). \end{aligned}$$

Example

Recall for any $s \in \mathbb{R}$

$$H(s) := \begin{pmatrix} 2 & 0 \\ 0 & 2(1+s^2) \end{pmatrix}.$$

The Proof

The non-trivial direction.

Furthermore, for any $s \in \mathbb{R}$ the matrix $H(s)$ is positive definite. Indeed, for $v \in \mathbb{C}^d \setminus \{0\}$ observe

$$\begin{aligned} v^H H(s) v &= v^H (V(s)^T V(s)) v \\ &= v^H (V(s)^H V(s)) v \\ &= (V(s)v)^H (V(s)v) \\ &> 0, \end{aligned}$$

where we exploit all roots of $\tilde{q}_s$ being distinct.

Example

For $v := (v_1, v_2) \in \mathbb{C}^2 \setminus \{0\}$ and $s \in \mathbb{R}$ we have

$$\begin{aligned} v^H H(s) v &= 2|v_1|^2 + 2 \underbrace{(1 + s^2)}_{\neq 0} |v_2|^2 \\ &> 0. \end{aligned}$$

The Proof

The non-trivial direction.

Therefore, there exists some matrix of polynomials $Q \in \mathbb{C}^{d \times d}[S]$ such that for any $s \in \mathbb{R}$

$$H(s) = Q(s)^H Q(s),$$

where Q is a matrix of polynomial of degree $d-1$ and $Q(s)$ is invertible for any $s \in \mathbb{C}$ with $\mathrm{Im}(s) \geq 0$ (see [10] and [1]).

Example

An appropriate factorization of H is given by Q with

$$Q(S) := \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2}(1 - iS) \end{pmatrix} \in \mathbb{C}^{2 \times 2}[S].$$

The Proof

The non-trivial direction.

For any $s \in \mathbb{R}$ set

$$M(s) := Q(s)C(s)Q(s)^{-1}.$$

Observe

$$\begin{aligned}\det(TI_d - M(s)) &:= \det(TI_d - (Q(s)C(s)Q(s)^{-1})) \\ &= \det(TI_d - C(s)) \\ &= \tilde{q}_s(T).\end{aligned}$$

Claim 2 For any $s \in \mathbb{R}$ it holds

$$C(s)^H H(s) = H(s)C(s).$$

Example

Now for any $s \in \mathbb{R}$ we set

$$\begin{aligned}M(s) &:= Q(s)C(s)Q(s)^{-1} \\ &= \begin{pmatrix} 0 & 1+is \\ 1-is & 0 \end{pmatrix}.\end{aligned}$$

Moreover, for any $s \in \mathbb{R}$

$$\begin{aligned}C(s)^H H(s) &= \begin{pmatrix} 0 & 2(1+s^2) \\ 2(1+s^2) & 0 \end{pmatrix} \\ &= H(s)C(s).\end{aligned}$$

The Proof

The non-trivial direction.

Now for any $s \in \mathbb{R}$ observe

$$\begin{aligned} M(s)^H &= (Q(s)^{-1})^H C(s)^H Q(s)^H \\ &\quad * \underbrace{Q(s)Q(s)^{-1}}_{=I_d} \\ &= (Q(s)^{-1})^H (C(s)^H H(s)) \\ &\quad * Q(s)^{-1} \\ &= Q(s)C(s)Q(s)^{-1} \\ &= M(s) \end{aligned}$$

i.e. $M(s)$ is Hermitian.
Claim 3 M is a matrix of polynomials over $\mathbb{C}$ of degree at most 1.

Example

Recall, for any $s \in \mathbb{R}$

$$M(s) \quad := \quad \begin{pmatrix} 0 & 1 + is \\ 1 - is & 0 \end{pmatrix}.$$

and so $M(s)$ is clearly Hermitian. Moreover, M is clearly a matrix of polynomials over $\mathbb{C}$ of degree 1 when considered in the variable S.

The Proof

The non-trivial direction.

Therefore, there exist two Hermitian matrices $Y, Z \in \mathbb{C}^{d\times d}$ such that

$$M(S) = -Y - ZS.$$

Thus for any $s \in \mathbb{R}$

$$\begin{aligned} \tilde{q}_s(T) &= \det(TI_d - M(s)) \\ &= \det(TI_d + Y + Zs). \end{aligned}$$

Example

Moreover,

$$\begin{aligned} M(S) &:= \begin{pmatrix} 0 & 1+iS \\ 1-iS & 0 \end{pmatrix} \\ &= -\underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{=:Y} \\ &\quad -\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{=:Z} S \\ &= -Y - ZS. \end{aligned}$$

Obviously, $Y, Z \in \mathbb{C}^{2\times 2}$ are Hermitian matrices.

The Proof

The non-trivial direction.

Altogether

$$\begin{aligned} q(y,z) &= y^d \tilde{q}_{\frac{z}{y}}\left(\frac{1}{y}\right) \\ &= y^d \det\left(\frac{1}{y}I_d + Y + \frac{z}{y}Z\right) \\ &= \det\left(y\left(\frac{1}{y}I_d + Y + \frac{z}{y}Z\right)\right) \\ &= \det(I_d + yY + zZ). \end{aligned}$$

Example

Obviously,

$$\begin{aligned} & \det(I_2 + yY + zZ) \\ = & \det\begin{pmatrix} 1 & -y - iz \\ -y + iz & 1 \end{pmatrix} \\ = & 1 - (-y + iz)(-y - iz) \\ = & 1 - ((-y)^2 + z^2) \\ = & 1 - y^2 - z^2 =: q(y,z). \end{aligned}$$

Altogether, we found an appropriate representation of q via Hermitian matrices.

The Proof

The non-trivial direction.

Claim 4 For any $\varepsilon > 0$ there exists a real zero polynomial $q_\varepsilon \in \mathbb{R}[y,z]$ of degree d with $q_\varepsilon(0,0) = 1$ such that any coefficient of q_ε is within an Euclidean distance of ε from the corresponding coefficient of q and for any $s \in \mathbb{R}$

$$\tilde{q}_{\varepsilon,s}(T) := T^d q_\varepsilon\left(\frac{1}{T}, \frac{s}{T}\right)$$

only has simple real roots (see [9]).

Hence, in the general case of $q \in \mathbb{R}[y,z]$, for any $\varepsilon > 0$ we can fix such corresponding q_ε with two Hermitian matrices $Y_\varepsilon, Z_\varepsilon \in \mathbb{C}^{d\times d}$ such that $q_\varepsilon(y,z) = \det(I_d + yY_\varepsilon + zZ_\varepsilon)$.

Clearly, $(q_\varepsilon)_{\varepsilon>0}$ converges to q as ε tends to zero and the sequence $(Y_\varepsilon, Z_\varepsilon)_{\varepsilon>0} \subseteq (\mathbb{C}^{d\times d})^2$ converges to a tuple $(Y,Z) \in (\mathbb{C}^{d\times d})^2$ of Hermitian matrices.

The Proof

The non-trivial direction.

Altogether,

$$q(y,z) = \lim_{\varepsilon \searrow 0} q_\varepsilon(y,z) = \lim_{\varepsilon \searrow 0} \det(I_d + yY_\varepsilon + zZ_\varepsilon) = \det(I_d + yY + zZ)$$

and so q has an appropriate representation via Hermitian matrices as claimed. ■

Outlook

1. The Lax Conjecture fails in more than 3 variables (see [8]).
2. In more than 3 variables, there exists a generalized counterpart to the Lax Conjecture, called the **generalized Lax Conjecture**, and it is still open up to today.
3. In my master thesis I relate the Lax Conjecture to the (multiplicative) Horn's Problem.

End

- Questions? -

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The two-sided medal

$\Rightarrow$.

It is clear that q cannot be of a degree larger than the degree $d \in \mathbb{Z}_{\geq 0}$ of p. Moreover,

$$q(0,0) := p(1,0,0) = p(e) = 1.$$

Fix $w := (w_2, w_3) \in \mathbb{R}^2$ and consider the univariate polynomial $f(T) := q(Tw_2, Tw_3) \in \mathbb{R}[T]$. Let $t \in \mathbb{C}\backslash\{0\}$ be a root of f. Then

$$0 = f(t) := q(tw_2, tw_3) := p(1, tw_2, tw_3) = p\left(\frac{t}{t}, tw_2, tw_3\right) = t^d p\left(\frac{1}{t}, w_2, w_3\right).$$

Since p is hyperbolic with respect to $e := (1,0,0) \in \mathbb{R}^3$, we have $-\frac{1}{t} \in \mathbb{R}\backslash\{0\}$ respectively $t \in \mathbb{R}\backslash\{0\}$. ■

The two-sided medal

$\Leftarrow$.

Clearly p is of the same degree $d \in \mathbb{Z}_{\geq 0}$ as q and

$$p(e) = p(1,0,0) := 1^d q\left(\frac{0}{1}, \frac{0}{1}\right) = 1^d q(0,0) = 1^d * 1 = 1.$$

Fix $v := (v_1, v_2, v_3) \in \mathbb{R}^3$ and consider $g(S) := p(v - Se) \in \mathbb{R}[S]$. Let $s \in \mathbb{C} \backslash \{v_1\}$ be a root of g i.e.

$$0 \quad = \quad g(s) := p(v - se) := \underbrace{(v_1 - s)}_{\neq 0}{}^d q\left(\frac{v_2}{v_1 - s}, \frac{v_3}{v_1 - s}\right).$$

So we have $\frac{1}{w_1 - s} \in \mathbb{R}$, since q is a real zero polynomial. Therefore, $s \in \mathbb{R}$. ∎

◂ The two-sided medal

The equivalence

Lax Conjecture ⇒ Helton-Vinnikov theorem.

Set

$$p(x,y,z) := q^H(x,y,z) := x^d q\left(\frac{y}{x}, \frac{z}{x}\right) \in \mathbb{R}[x,y,z]$$

and so p is hyperbolic with respect to $e := (1,0,0)$ and $p(e) = 1$. Moreover, $\deg(p) = \deg(q)$. Thus by the Lax Conjecture there exist two Hermitian complex matrices $Y, Z \in \mathbb{C}^{d\times d}$ such that

$$p(x,y,z) = \det(xI_d + yY + zZ).$$

So

$$q(y,z) = (q^H)^D(y,z) =: p^D(y,z) := p(1,y,z) = \det(I_d + yY + zZ)$$

i.e. there exist two Hermitian complex matrices $Y, Z \in \mathbb{C}^{d\times d}$ such that

$$q(x,y) = \det(I_d + yY + zZ).$$

The other direction is the trivial direction of the Helton-Vinnikov Theorem. ■

The equivalence

Helton-Vinnikov Theorem $\Rightarrow$ Lax Conjecture.

Set $q(y,z) := p^D(y,z) := p(1,y,z) \in \mathbb{R}[y,z]$. So q is a real zero polynomial with $q(0,0) = 1$ and set $d := \deg(q) \in \mathbb{Z}_{\geq 0}$.
Hence, by the Helton-Vinnikov Theorem there exist two Hermitian complex matrices $Y, Z \in \mathbb{C}^{d \times d}$ such that

$$q(y,z) = \det(I_d + yY + zZ).$$

W.l.o.g we can assume $d = \deg(p)$ by completing Y, Z to matrices of dimension $\deg(p)$ via trivial block matrices i.e. we could consider $\mathrm{diag}(Y, 0_{\mathbb{C}^{d-\deg(p))\times(d-\deg(p))}})$ and $\mathrm{diag}(Z, 0_{\mathbb{C}^{d-\deg(p))\times(d-\deg(p))}})$ instead of Y and Z. Then

$$\begin{aligned} p(x,y,z) &= (p^D)^H(x,y,z) := q^H(x,y,z) := x^d q\left(\frac{y}{x}, \frac{z}{x}\right) \\ &= x^d \det\left(I_d + \frac{y}{x}Y + \frac{z}{x}Z\right) = \det\left(x\left(I_d + \frac{y}{x}Y + \frac{z}{x}Z\right)\right) \\ &= \det(xI_d + yY + zZ). \end{aligned}$$

The other direction is the trivial direction of the Lax Conjecture. ■

Claim 1

For any positive integer d and any $s \in \mathbb{R}$ we have $\tilde{q}_s(T) = \det(TI_d - C(s))$.

We will prove this by induction on d.
Fix $s \in \mathbb{R}$ and clearly the base case holds.
Now assume that for any $t < d$ we have $\det(TI_t - C(s)) = \tilde{q}_s(T)$. For d, using Laplace formula for determinants on the first row of $TI_d - C(s)$, we conclude

$$\det(TI_d - C(s)) = \det \begin{pmatrix} T & 0 & \cdots & \cdots & 0 & q_d(s) \\ -1 & T & 0 & \cdots & 0 & q_{d-1}(s) \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & T & q_2(s) \\ 0 & \cdots & \cdots & 0 & -1 & T + q_1(s) \end{pmatrix}$$

Claim 1

For any positive integer d and any $s \in \mathbb{R}$ we have $\tilde{q}_s(T) = \det(TI_d - C(s))$.

$$
\begin{aligned}
= \quad & T \det \begin{pmatrix} T & 0 & \cdots & 0 & q_{d-1}(s) \\ -1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & T & q_2(s) \\ 0 & \cdots & 0 & -1 & T + q_1(s) \end{pmatrix} \\
+ \quad & (-1)^{d+1} q_d(s) \det \begin{pmatrix} -1 & T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & T \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

For any positive integer d and any $s \in \mathbb{R}$ we have $\tilde{q}_s(T) = \det(TI_d - C(s))$.

According to the hypothesis of induction we have

$$\det \begin{pmatrix} T & 0 & \cdots & 0 & q_{d-1}(s) \\ -1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & T & q_2(s) \\ 0 & \cdots & 0 & -1 & T + q_1(s) \end{pmatrix} = \sum_{m=0}^{d-1} q_{d-1-m}(s) T^m$$

and

$$\det \begin{pmatrix} -1 & T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & T \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix} = (-1)^{d-1}.$$

For any positive integer d and any $s \in \mathbb{R}$ we have $\tilde{q}_s(T) = \det(TI_d - C(s))$.

Putting both together we have

$$\begin{aligned}
\det(TI_d - C(s)) &= \underbrace{T * \sum_{m=0}^{d-1} q_{d-1-m}(s) T^m}_{=\sum_{m=0}^{d-1} q_{d-1-m}(s) T^{m+1}} + \underbrace{(-1)^{d+1}(-1)^{d-1}}_{=(-1)^{d+1+d-1}=(-1)^{2d}=1} * q_d(s) \\
&= q_d(z) + \sum_{m=0}^{d-1} q_{d-1-m}(s) T^{m+1} \\
&= q_d(s) T^0 + \sum_{m=1}^{d} q_{d-m}(s) T^m \\
&= \sum_{m=0}^{d} q_{d-m}(s) T^m = \tilde{q}_s(T). \blacksquare
\end{aligned}$$

◂ The Proof

Claim 2

For any $s \in \mathbb{R}$ it holds $C(s)^H H(s) = H(s)C(s)$.

Recall, for any $s \in \mathbb{R}$, $H(s) = V(s)^T V(s) = V(s)^H V(s)$ is the product of two matrices of Vandermonde type with entries being the zeros of the univariate polynomial $\tilde{q}_s$, which were all proven to be real and assumed to be distinct. Since $C(s)$ is the companion matrix of $\tilde{q}_s$ we thus have

$$\begin{aligned} V(s)C(s)V(s)^{-1} &= \operatorname{diag}(\lambda_1(s), \dots, \lambda_d(s)) = \operatorname{diag}(\lambda_1(s), \dots, \lambda_d(s))^H \\ &= (V(s)C(s)V(s)^{-1})^H = (V^{-1}(s))^H C(s)^H V(s)^H \\ &= (V(s)^H)^{-1} C(s)^H V(s)^H. \end{aligned}$$

Now multiplying $V(s)^H$ from the left and $V(s)$ from the right yields

$$H(s)C(s) = V(s)^H V(s)C(s) = C(s)^H V(s)^H V(s) = C(s)^H H(s). \blacksquare$$

◂ The Proof

Claim 3

M is a matrix of polynomials over $\mathbb{C}$ of degree at most 1.

Clearly, poles can only arise in roots of $Q \in \mathbb{R}^{d\times d}[S]$.
Hence, let $a \in \mathbb{C}$ be a root of Q and with that necessarily $\mathrm{Im}(a) < 0$ (see [1]).
We thus have

$$\begin{aligned} C(\bar{a})^T(Q^H(\bar{a}))Q(a) &= C(\bar{a})^T H(a) = C(a)^H H(a) = H(a)C(a) \\ &= (Q^H(\bar{a}))Q(a)C(a) \end{aligned}$$

respectively

$$(Q^H(\bar{a}))^{-1}C(\bar{a})^T(Q^H(\bar{a})) = M(a),$$

as $Q(\bar{a})$ is invertible, due to $\mathrm{Im}(\bar{a}) > 0$ (see [1]). So $M(a)$ is the product of three invertible matrices i.e. $M(a)$ is regular. Hence, a was proven to not be a pole of M.
Altogether, $M \in \mathbb{C}^{d\times d}[S]$ is a matrix of polynomials.

$M(S)$ is a matrix of polynomials over $\mathbb{C}$ of degree at most 1.

Let k denote the degree of the matrix of polynomials M. Hence there exists some $B_1, \ldots, B_k \in \mathbb{C}^{d\times d}$ such that

$$-M(S) := \sum_{i=0}^{k} B_i S^i.$$

Observe for any $s \in \mathbb{R}$

$$\det(TI_d + \sum_{j=0}^{k} B_k s^k) = \det(TI_d - M(s)) = \tilde{q}_s(T) = \sum_{m=0}^{d} q_{d-m}(s) T^m.$$

Therefore, for any $j \in \{0, \ldots, d-1\}$ the sum of all principal $j \times j$ minors of $-M(s)$ gives exactly $p_j(s)$ and so the coefficient for s^{kj} in p_j is given by the sum of all $j \times j$ principal minors of B_k.

Furthermore, $B_k \neq 0_{\mathbb{C}^{d\times d}}$ and M is Hermitian. So B_k must be Hermitian as well. Hence, not for all $j \in \{1, \ldots, d-1\}$ the sum of all $j \times j$ principal minors of B_k can be zero, as else B_k would be nilpotent. ↯

$M(S)$ is a matrix of polynomials over $\mathbb{C}$ of degree at most 1.

So for some $j \in \{1, \ldots, d-1\}$ the coefficient of s^{kj} cannot vanish and thence $kj \leq \deg(p_j)$ on the one hand.
Recalling, $\deg(p_j) \leq j$ on the other hand, yields $k \leq 1$. ■

◂ The Proof

The density argument

The sequence $(Y_\varepsilon, Z_\varepsilon)_{\varepsilon>0} \subseteq (\mathbb{C}^{d\times d})^2$ converges to a tuple $(Y, Z) \in (\mathbb{C}^{d\times d})^2$.

Set

$$\mu := \min\{|t| \mid q(t,0)q(0,t) = 0\} \geq 0.$$

Since, $q(0,0) = 1$, we have $q(0,0)q(0,0) = 1$ and so $\mu > 0$ necessarily. Moreover,

$$\mu := \min\{|t| \mid q(t,0)q(0,t) = 0\} = \min(\{|t| \mid q(t,0) = 0\} \cup \{|t| \mid q(0,t) = 0\}).$$

So for $\varepsilon > 0$ sufficiently small we have $\lambda_Y, \lambda_Z \in]-\frac{2}{\mu}, \frac{2}{\mu}[$.

Since Y_ε and Z_ε are Hermitian by choice, their spectral radii coincide with their spectral norms i.e. the norms $||Y_\varepsilon||_2$ of Y_ε and $||Z_\varepsilon||_2$ of Z_ε are bounded by $\frac{2}{\mu}$ for sufficiently small ε.

Therefore, the sequence $(Y_\varepsilon, Z_\varepsilon)_{\varepsilon>0} \subseteq (\mathbb{C}^{d\times d})^2$ converges to a tuple $(Y, Z) \in (\mathbb{C}^{d\times d})^2$. ■

◂ The Proof

Homogeneity

Definition

Let n be a positive integer.

1. Let $q \in \mathbb{R}[x_1, \ldots, x_n]$ be of degree $d \in \mathbb{Z}_{\geq 0}$. Then

$$q^H(x_0, x_1, \ldots, x_n) := x_0^d p\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{R}[x_0, \ldots, x_n]$$

   is called the **homogenization of** q.
2. Let $p \in \mathbb{R}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d \in \mathbb{Z}_{\geq 0}$, then

$$p^D(x_1, \ldots, x_n) := p(1, x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$$

   is called the **dehomogenization of** p.

Homogeneity

Fact

1. *For any* $q \in \mathbb{R}[x_1, \ldots, x_n]$ *of degree* $d \in \mathbb{Z}_{\geq 0}$ *we have* $(q^H)^D = p$.
2. *The degree of the dehomogenization* p^D *of any homogeneous polynomial* p *of degree* $d \in \mathbb{Z}_{\geq 0}$ *can never outreach the degree of* p
i.e. $\deg(p^D) \leq d := \deg(p)$.
3. *Vice versa, for any polynomial* q *of degree* d*, the homogenization* q^H *of* q *is exactly of degree* d *i.e.* $\deg(q^H) = d := \deg(q)$.

Motivation of real zero polynomials

The Lax Conjecture fails in more than 3 variables

Proof.

Let $n \in \mathbb{Z}_{>3}$ be a positive integer greater than 3. Set

$$p(x_1, \ldots, x_n) := x_1^2 - \sum_{i=2}^{n} x_i^2 \in \mathbb{R}[x_1, \ldots, x_n].$$

Clearly, p is homogeneous of degree $d := 2$. Set $e := (1, \underbrace{0, \ldots, 0}_{(n-1)\text{many}}) \in \mathbb{R}^n$, then obviously $p(e) = 1$.

For $v := (v_1, \ldots, v_n) \in \mathbb{R}^n$ consider

$$\begin{aligned} m(S) &:= p(v - Se) = p((v_1 - S, v_2, \ldots, v_n)) := (v_1 - S)^2 - \sum_{i=2}^{n} v_i^2 \\ &= S^2 - 2v_1 S + v_1^2 - \sum_{i=2}^{n} v_i^2 \in \mathbb{R}[S]. \end{aligned}$$

The Lax Conjecture fails in more than 3 variables

Proof.

Clearly m has a non negative discriminant

$$\Delta := (-2v_1)^2 - 4 * (v_1^2 - \sum_{i=2}^{n} v_i^2) = 4 \sum_{i=2}^{n} v_i^2 \geq 0.$$

Let $X_2 := (X_{ij}^{(2)})_{i,j\in\{1,\dots,n\}}, \dots, X_n := (X_{ij}^{(n)})_{i,j\in\{1,\dots,n\}} \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Consider,

$$n(x_2, \dots, x_n) := \sum_{i=2}^{n} x_i (X_{1j}^{(i)})_{j=1,\dots,n}.$$

Clearly, we can find $v_2, \dots, v_n \in \mathbb{R}$ not all equal 0 such that $n(v_2, \dots, v_n) = 0_{\mathbb{R}^n}$. Fix such $(v_2, \dots, v_n) \in \mathbb{R}^{n-1}$ and consider $v := (0, v_2, \dots, v_n) \in \mathbb{R}^n$.

The Lax Conjecture fails in more than 3 variables

Proof.

Then on the one hand

$$p(v) := p(0, v_2, \ldots, v_n) = 0^2 - \sum_{i=2}^{n} v_i^2 = -\underbrace{\sum_{i=2}^{n} v_i^2}_{>0} < 0,$$

because at least for one $i \in \{2, \ldots, n\}$ we have $v_i \neq 0$. But on the other hand

$$\det(0 * I_n + \sum_{i=2}^{n} v_i X_i) = \det(\sum_{i=2}^{n} v_i X_i) = 0,$$

as the first row of $\sum_{i=2}^{n} v_i X_i$ equals $n(v_2, \ldots, v_n) = 0_{\mathbb{R}^n}$ by the choice of $v_2, \ldots, v_n$.

Therefore, clearly $p(x_1, \ldots, x_n) \neq \det(x_1 I_n + \sum_{i=2}^{n} x_i X_i)$. ■

◂ Outlook