Algebra Beritseminar

A proof that every convex quarternary quartic is a sum of squares

Notations: Let $n, k \in \mathbb{N}$.

- The polynomial ring over $\mathbb{R}$ in n variables:
$$\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \dots, x_n].$$
- The vector space of homogenous polynomials or forms in n variables of degree k
$$H_{n,k} = \{ p = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = k} p_\alpha \underline{x}^\alpha \in \mathbb{R}[\underline{x}] \}$$
- The set of positive semidefinite (PSD) forms
$$P_{n,k} = \{ p \in H_{n,k} : p \geq 0 \}$$

If $k = 2d$ is even

- The set of all forms, which are a (finite) sum of squares (SOS)
$$\Sigma_{n,2d} = \{ p \in H_{n,2d} : p = \sum_{i=1}^{m} \sigma_i^2, \ \sigma_i \in H_{n,d} \}$$

Since squares are nonnegative, it is clear that every SOS form is already PSD, i.e. for $n, d \in \mathbb{N}$ it holds

$$\Sigma_{n,2d} \subseteq P_{n,2d}$$

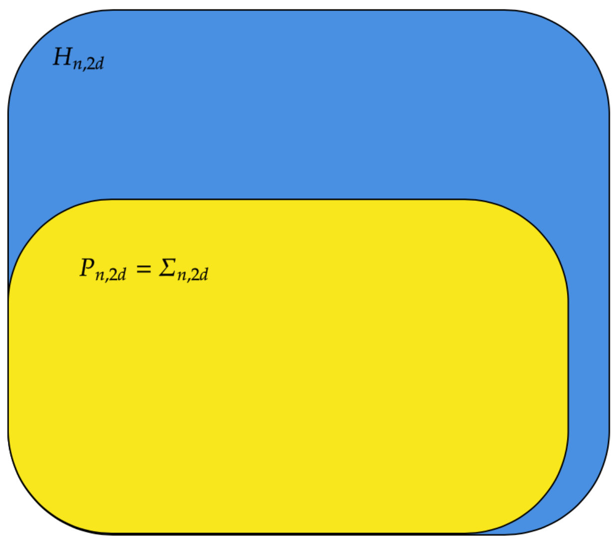
The obvious question that arises whether or not the converse is true as well. This was answered by Hilbert in the following Theorem.

Thm. (1888 Hilbert): Let $n, d \in \mathbb{N}$ be arbitrary. Then it holds

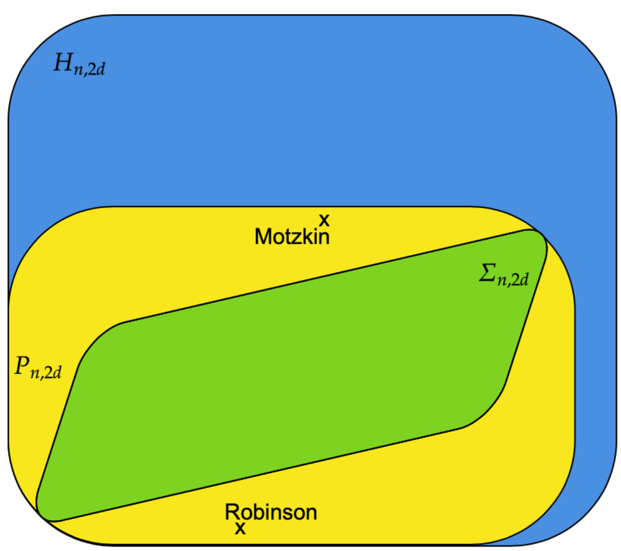
$$\Sigma_{n,2d} = P_{n,2d} \iff n=1 \text{ or } d=1 \text{ or } (n,2d)=(3,4).$$

Hence, we have the following picture:

If $n = 1$ or $d = 1$ or $(n, 2d) = (3, 4)$:



Else:



Now that the relation between PSD and SOS forms is fully characterized, one could ask what happens, if we consider additional properties. One could for example ask what happens in the picture if we consider in addition convex forms, since convexity plays an important role e.g. in optimization.

Def.: Let $n \in \mathbb{N}$. A multivariate function $f: \mathbb{R}^n \to \mathbb{R}$ is **convex**

$$:\Longleftrightarrow \quad \forall x,y \in \mathbb{R}^n \ \forall \lambda \in [0,1]: \quad f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

There is also an equivalent characterization when the function is twice differentiable (see [7, Satz 6.4])

Prop. Let $f: \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Then

$$f \text{ is convex} \iff \forall x \in \mathbb{R}^n: \ \nabla^2 f(x) \succeq 0.$$

This motivates the following definition

Def.: The set of **convex forms** in n variables of degree k is

$$C_{n,k} := \{ p \in H_{n,k} : \ \forall x \in \mathbb{R}^n: \ \nabla^2 p(x) \succeq 0 \}.$$

First Results

It turns out that it can be seen rather easily that for a degree $k>1$, indeed every convex form is PSD.

Prop.: Let $n\in\mathbb{N}$ and $k\in\mathbb{N}_{>1}$. Then it holds $C_{n,k}\subseteq P_{n,k}$.

proof Let $p\in C_{n,k}$ be arbitrary. Clearly, it holds $p(0)=0$. Hence, it suffices to show that p has a global minimum at 0.

Assume not. Then there is some $x\in\mathbb{R}^n$ s.t. $p(x)\lneq 0$.

Take $\lambda\in(0,1)$ arbitrary. It holds

$$\lambda^k p(x) \overset{p\text{ hom.}}{=} p(\lambda x) = p(\lambda x+(1-\lambda)0) \overset{p\text{ convex}}{\leq} \lambda p(x) + \underbrace{(1-\lambda)p(0)}_{=0} = \lambda p(x).$$

Hence, dividing by $p(x)\lneq 0$ shows

$$\lambda^k \geq \lambda,$$

which is a contradiction since $\lambda\in(0,1)$. This shows the claim.

$\square$

On the other hand, not every PSD form is convex, as one can see in the following.

Example: (a) Consider $p(x,y)=x^2y^2\in\Sigma_{2,4}\subseteq P_{2,4}$. Then it holds

$$\nabla p(x,y)=\begin{pmatrix}2xy^2\\2yx^2\end{pmatrix},\quad \nabla^2 p(x,y)=\begin{pmatrix}2y^2 & 4xy\\4xy & 2x^2\end{pmatrix}.$$

Hence we have $\nabla^2 p(1,1) = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \not\succeq 0$, since $\det(\nabla^2 p(1,1)) = 4 - 16 = -12 < 0$. This shows that $p \in \Sigma_{2,4} \setminus C_{2,4}$. In particular, not even every SOS form is convex. More general, it can be seen that $x^{2d-2} y^2 \in \Sigma_{2,2d}$ for arbitrary $d \in \mathbb{N}$.

(b) Neither the Motzkin nor the Robinson form is convex. The Robinson form for example is given by

$$p_R = w^4 + x^2y^2 + y^2z^2 + x^2z^2 - 4xyzw \in P_{4,4} \setminus \Sigma_{4,4}$$

The following values are computed using some MATLAB code which can be found in my thesis. Take

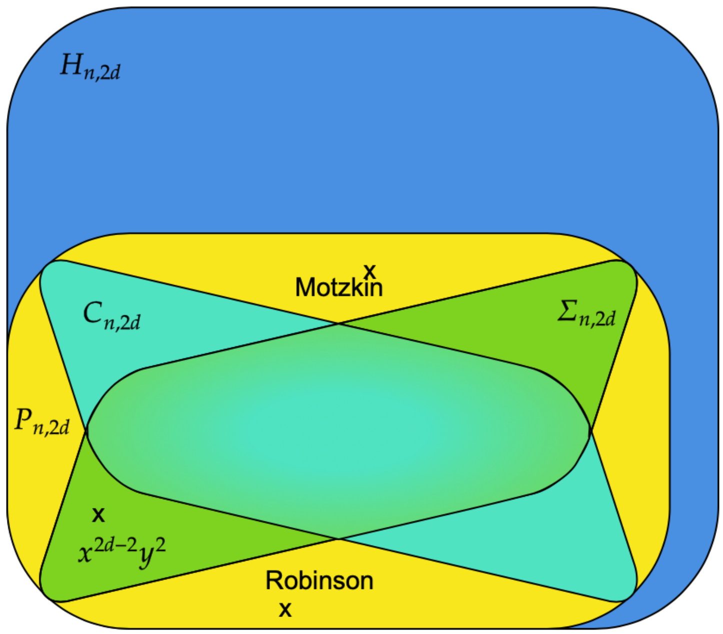
$$x = (0.2785, 0.5468, 0.9575, 0.9649) \in \mathbb{R}^4$$

Then $\nabla^2 p_M(x)$ has a strictly negative eigenvalue, more precisely its eigenvalues are

$$-2.0291, \quad -0.6403, \; 3.2654, \; 8.9224.$$

This shows that the Robinson form p_R is not convex

So far, we have the following picture for $n,d>1$ and $(n,2d)\neq(3,4)$:



The question which arises is whether or not every convex form is SOS. This was asked by Parillo in 2007. Two years later, Blekherman answered in [2] the question to the negative by using some volume arguments. However, no explicit example of a convex but not SOS form is known until today. The easiest cases where such an example could exist are for $(n,2d)\in\{(4,4),(3,6)\}$. In 2019,

El Khadir proved in [4] that for $(n,2d)=(4,4)$, indeed every convex quaternary quartic is a SOS.

For understanding the proof, we need some more definitions (for more details consider e.g. [5, Section 6.4] and [4, Section 2])

Tensors

In the following, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be arbitrary.

Definition: Let $n, k \in \mathbb{N}$. The set of all n-dimensional (real or complex) tensors of order k is defined as

$$T^k = T^k(\mathbb{K}^n) = \{ (x^{j_1 \cdots j_k})_{j_1, \dots, j_k = 1, \dots, n} \mid x^{j_1 \cdots j_k} \in \mathbb{K} \}.$$

Example: (a) For $n \in \mathbb{N}, k = 1$: $T^1(\mathbb{K}^n)$ is the set of all vectors in $\mathbb{K}^n$.

(b) For $n \in \mathbb{N}, k = 2$: $T^2(\mathbb{K}^n)$ is the set of all (real or complex) $n \times n$ matrices.

Rmk.: Let $T = (T^{j_1 \cdots j_k})_{j_1, \dots, j_k = 1, \dots, n} \in T^k(\mathbb{K}^n)$ be arbitrary. Then

$$f_T : \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{k \times} \to \mathbb{K}, \quad f_T(x^1, \dots, x^k) = \sum_{j_1, \dots, j_k = 1, \dots, n} T^{j_1 \cdots j_k} x^1_{j_1} \cdots x^k_{j_k}$$

is multilinear. Further, $T \mapsto f_T$ is a bijection between the set of tensors and the set of multilinear mappings $f : \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{k \times} \to \mathbb{K}$.

Def.: A tensor $T \in T^k(\mathbb{R}^n)$ is symmetric

$$:\Leftrightarrow \quad \forall \pi \in S_k \ \forall j_1, \dots, j_k = 1, \dots, n : \quad T^{j_1 \cdots j_k} = T^{j_{\pi(1)} \cdots j_{\pi(k)}}.$$

Equivalently, $f_T : \mathbb{K}^n \times \dots \times \mathbb{K}^n \to \mathbb{K}$ is invariant under permutation of its arguments. Further, the set of symmetric tensors of order k is denoted by $S^k(\mathbb{K}^n) = S^k$.

Def. For $n, k \in \mathbb{N}$, $x^1, \dots, x^k \in \mathbb{K}^n$, we define the outer product

$$x^1 \otimes \dots \otimes x^k = \left(x^1_{j_1} \cdots x^k_{j_k}\right)_{j_1, \dots, j_k = 1, \dots, n} \in T^k(\mathbb{K}^n).$$

Further, the symmetric outer product is defined as

$$x^1 \odot \dots \odot x^k = \frac{1}{k!} \sum_{\pi \in S_k} x^{\pi(1)} \otimes \dots \otimes x^{\pi(k)} \in S^k(\mathbb{K}^n).$$

Remark: (a) For $x \in \mathbb{K}^n$, it holds $x \otimes x = x \odot x$. We will further write $x^k = \underbrace{x \odot \dots \odot x}_{k \times} = \underbrace{x \otimes \dots \otimes x}_{k \times}$ $(k \in \mathbb{N})$.

(b) It holds $x \odot y = x \cdot y^T$ for all $x, y \in \mathbb{K}^n$.

We will also write $xy \equiv x \odot y$.

Def.: For $n \in \mathbb{N}$, $j_1, \dots, j_\ell \in \{1, \dots, n\}$ s.t. $\sum_{i=1}^{\ell} j_i = n$, we define the generalized binomial coefficient $\binom{n}{j_1, \dots, j_\ell} = \frac{n!}{j_1! \cdots j_\ell!}$

Now we can prove the following Theorem.

Thm.: There is a bijection $\varphi : H_{n,k} \to S^k(\mathbb{R}^n)$ between the set of homogeneous polynomials in n variables of degree k and the set $S^k(\mathbb{R}^n)$ of real, symmetric n-dim.l tensors of order k.

Proof. Let $p = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = k} p_\alpha x^\alpha \in H_{n,k}$ be arbitrary. Writing each monomial of degree k as the product of k single variables leads to

$$p = \sum_{j_1, \dots, j_k} \tilde{p}^{j_1 \dots j_k} x_{j_1} \cdots x_{j_k}.$$

Goal: Choose the coefficients $\tilde{p}^{j_1 \dots j_k} \in \mathbb{R}$ in such a way that they are invariant under permutation of indices.

Therefore, let $j_1, \dots, j_k \in \{1, \dots, n\}$ be arbitrary. For $i \in \{1, \dots, n\}$, take

$$\begin{aligned} \alpha_i &:= \#\{\ell \in \{1, \dots, k\} : j_\ell = i\} \\ &= \#\{\ell \in \{1, \dots, k\} : x_{j_\ell} = x_i\} \end{aligned}$$

and define

$$p^{j_1 \dots j_k} = \binom{n}{\alpha_1, \dots, \alpha_n}^{-1} p_\alpha = \binom{n}{\alpha}^{-1} p_\alpha.$$

Then, it clearly holds $p^{j_1 \dots j_k} = p^{j_{\pi(1)} \dots j_{\pi(k)}}$ for all $\pi \in S_k$ and the coefficients $p^{j_1 \dots j_k}$ are uniquely determined by this property, since it must hold

$$p_\alpha = \sum_{j_1, \dots, j_k \in \{1, \dots, n\},\ x^\alpha = x_{j_1} \cdots x_{j_k}} \tilde{p}^{j_1 \dots j_k}$$

for all α. Therefore, the mapping

$$\varphi : H_{n,k} \to S^k(\mathbb{R}^n),\ p \mapsto \left(\tilde{p}^{j_1 \dots j_k}\right)_{j_1, \dots, j_k = 1, \dots, n}$$

with $\tilde{p}^{j_1\cdots j_k} \in \mathbb{R}$ as above is welldefined and injective. Clearly, it is also surjective, since for $(T^{j_1\cdots j_k})_{j_1,\dots,j_k=1,\dots,n} \in S^k$, it holds $\sum_{j_1,\dots,j_k=1,\dots,n} T^{j_1\cdots j_k} x_{j_1}\cdots x_{j_k} \in H_{n,k}$. This shows the claim. $\square$

Def.: For $p \in H_{n,k}$, we say that $T_p := \varphi(p)$ is the tensor associated to p. Further, if $k = 2d \in \mathbb{N}$ is even, we define

$$Q_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\quad Q_p(x,y) = T_p(\underbrace{x,\dots,x}_{d\times},\underbrace{y,\dots,y}_{d\times})$$

as the biform associated to p.

These notions are crucial in the next section.

The Generalized Cauchy Schwarz Inequalities

Let us start by giving the regular Cauchy Schwarz Inequalities.

Thm (CSI): Let $Q \in \mathbb{R}^{n\times n}$ be psd. Then, it holds:

$$\forall x,y \in \mathbb{R}^n:\quad x^T Q y \le \sqrt{x^T Q x \cdot y^T Q y}$$

$$\forall z \in \mathbb{C}^n:\quad z^T Q \bar{z} \ge \sqrt{z^T Q z \cdot \bar{z}^T Q \bar{z}}$$

El Khadir managed to prove the following generalization for convex forms.

Thm (Generalized CSI): Let $p \in C_{n,2d}$ be an arbitrary convex form in n variables of degree $2d$, $Q_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the associated biform. Then there are $A_d, B_d > 0$ that depend only on the degree $2d$ s.t.

$$\forall x,y \in \mathbb{R}^n: \quad Q_p(x,y) \le A_d \sqrt{p(x) \cdot p(y)}$$

$$\forall z \in \mathbb{C}^n: \quad |p(z)| \le B_d \cdot Q_p(z, \bar{z})$$

Prop: Let $Q \in \mathbb{R}^{n \times n}$ be symmetric and define the quadratic form $p_Q(x) = x^T Q x$, where $x = (x_1, \dots, x_n)^T$ is the vector of variables. Then:

$$p_Q \text{ is convex} \iff Q \succeq 0.$$

proof. This is clear, since $\nabla p_Q(x) = 2Qx$ and hence $\nabla^2 p(x) = 2Q$. Hence, p_Q being convex coincides with Q being psd. $\square$

Remark: Further, the biform associated to p_Q is given by

$$Q_{p_Q} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad Q_{p_Q}(x,y) = x^T Q y.$$

This shows that the Generalized CSI's are indeed a generalization of the regular ones, where $2d = 2$, $p = p_Q \in C_{n,2}$ (Q psd) and constants $A_d = B_d = 1$.

Optimal Constants

Def: For $d\in\mathbb{N}$, the optimal constants of the GCSI's are determined by

$$A_d^* = \inf_{A>0} A \text{ s.t. } \forall n\in\mathbb{N}\ \forall p\in C_{n,2d}\ \forall x,y\in\mathbb{R}^n:\ Q_p(x,y)\le A\sqrt{p(x)\,p(y)}$$

$$B_d^* = \inf_{B>0} B \text{ s.t. } \forall n\in\mathbb{N}\ \forall p\in C_{n,2d}\ \forall z\in\mathbb{C}^n:\ |p(z)|\le B\, Q_p(z,\bar z)$$

Remark: Using the primal and dual structure of the optimization problems in the definition of A_d^* and B_d^*, one can prove the following results:

- $\forall d\in\mathbb{N}$: $A_d^*, B_d^* \ge 1$.
- $A_1^* = A_2^* = A_3^* = 1$.
- For all even $d\in\mathbb{N}_{>2}$: $A_d^* \ne 1$.
- $\forall d\in\mathbb{N}$: $B_d^* = \dfrac{\binom{2(d-1)}{d-1}}{d}$ (Catalan numbers (see [6])).

Further, there is the following conjecture formulated by El Khadir.

Conjecture: For all odd $d\in\mathbb{N}$, it holds $A_d^*=1$.

Remark: In particular, we know that for $d=2$, it holds

$$A_2^*=1,\quad B_2^* = \frac{\binom{2\cdot(2-1)}{2-1}}{2} = 1.$$

This will be used in the proof of the inclusion $C_{4,4}\subseteq \Sigma_{4,4}$.

What separates the SOS forms $\Sigma_{4,4}$ from the non SOS forms inside $P_{4,4}$

The following result is based on the work of Blekherman, which uses the socalled Cayley Bacharach relations.
El Khadir managed to further simplify those results under the usage of classical tools from optimization such as the Karush Kuhn Tucker (KKT) conditions. This way, he obtained the following conditions on a PSD form $p \in P_{4,4}$ to be SOS.

Thm.: A nonnegative quaternary quartic $p \in P_{4,4}$ is SOS, i.e. $p \in \Sigma_{4,4}$ if and only if the following two conditions hold:

(1.) $\forall r^1, \dots, r^8 \in \mathbb{R}^4 \; \forall a_2, \dots, a_8 \in \{-1,1\}$ s.t. $ev_{r^1} + \sum_{i=2}^{8} a_i \, ev_{r^i} = 0$:

$$p(r^1) \leq \left(\sum_{i=2}^{8} \sqrt{p(r^i)} \right)^2 .$$

(2.) $\forall r^3, \dots, r^8 \in \mathbb{R}^4 \; \forall z \in \mathbb{C}^4$ s.t. $\bar{z} \neq z \; \forall a_3, \dots, a_8 \in \{-1,1\}$

s.t. $z z^T + \bar{z} \bar{z}^T = \sum_{i=3}^{8} a_i \, r^i (r^i)^T$:

$$2(|p(z)| + \mathrm{Re}(p(z))) \leq \left(\sum_{i=3}^{8} \sqrt{p(r^i)} \right)^2 .$$

Main Theorem

In this Section, we use the GCSI's to prove that every convex quarternary quartic fulfills the two conditions from the previous Theorem, which yields that $C_{4,4} \subseteq \Sigma_{4,4}$.

Theorem: It holds $C_{4,4} \subseteq \Sigma_{4,4}$.

proof. Let $p \in C_{4,4}$ be arbitrary. We want to show that p satisfies the conditions (1.) & (2.) of the previous Theorem.

First of all, we know that p satisfies the GCSI's with optimal constants equal to one, i.e.

$$\forall\, x,y \in \mathbb{R}^4: \quad Q_p(x,y) \leq \sqrt{p(x)\cdot p(y)} \qquad (1)$$

$$\forall\, z \in \mathbb{C}^4: \quad |p(z)| \leq Q_p(z,\bar{z}). \qquad (2)$$

Now, take a look at the conditions (1.) & (2.):

(1.) Let $r^1,\dots,r^8 \in \mathbb{R}^4$ and $a_2,\dots,a_8 \in \{-1,1\}$ be s.t. $r^1(r^1)^T = \sum_{i=2}^{8} a_i r^i (r^i)^T$.

T.S.: $p(r^1) \leq \left(\sum_{i=2}^{8} \sqrt{p(r^i)}\right)^2$

Remember that the matrix product of a vector x with its transpose is the same as the (symmetric) outer product of x with itself, i.e. $xx^T = x \odot x = x^2$. Hence, we have

$$(r^1)^2 = \sum_{i=2}^{8} a_i (r_i)^2$$

Squaring both sides, i.e. taking again the symmetric outer product of each side with itself shows

$$(r^1)^4 = \sum_{i,j=2,\dots,8} a_i a_j (r^i)^2 (r^j)^2. \qquad (3)$$

Now, write $p = \sum_{j_1,\dots,j_4=1,\dots,4} \tilde{p}^{j_1\cdots j_4} x_{j_1}\cdots x_{j_4}$ s.t. the coefficients

$\tilde{p}^{j_1\cdots j_4} \in \mathbb{R}$ are invariant under permutation of indices.

This shows for $x,y\in\mathbb{R}^4$:

$$\begin{aligned} Q_p(x,y) &= \sum_{j_1,\dots,j_4=1,\dots,4} \tilde{p}^{j_1\cdots j_4} \, x_{j_1} x_{j_1} y_{j_2} y_{j_2} \\ &= \sum_{j_1,\dots,j_4=1,\dots,4} \tilde{p}^{j_1\cdots j_4} (x^2 y^2)_{j_1\cdots j_4} \end{aligned} \qquad (4)$$

Hence, we have

$$\begin{aligned} p(r^1) &= \sum_{j_1,\dots,j_4=1,\dots,4} \tilde{p}^{j_1\cdots j_4} ((r^1)^4)_{j_1\cdots j_4} \\ &\overset{(3)}{=} \sum_{i,j=2}^{8} a_i a_j \sum_{j_1,\dots,j_4=1,\dots,4} \tilde{p}^{j_1\cdots j_4} ((r^i)^2 (r^j)^2)^{j_1\cdots j_4} \\ &\overset{(4)}{=} \sum_{i,j=2}^{8} a_i a_j \; Q_p(r^i, r^j) \\ &\leq \sum_{i,j=2}^{8} \underbrace{|a_i a_j|}_{=1} \, |Q_p(r^i, r^j)| \\ &\overset{GCSI}{\leq} \sum_{i,j=2}^{8} \sqrt{p(r^i)\, p(r^j)} = \left(\sum_{i=1}^{8} \sqrt{p(r^i)} \right)^2 . \end{aligned}$$

This shows that (1) holds.

(2.) Let $r^3, \dots, r^8 \in \mathbb{R}^4$, $z \in \mathbb{C}^4$ s.t. $z \neq \bar{z}$ and $a_3, \dots, a_8 \in \{-1, 1\}$ be arbitrary such that $z z^T + \bar{z} \bar{z}^T = \sum_{i=3}^{8} a_i r_i r_i^T$. As in (1), one can see that

$$z^4 + 2 z^2 \bar{z}^2 + \bar{z}^4 = \sum_{i,j=3}^{8} a_i a_j (r^i)^2 (r^j)^2 \qquad (5)$$

Further we have

$$\begin{aligned}
& p(z) + 2 Q_p(z, \bar{z}) + p(\bar{z}) \\
\overset{\text{def. } p \,\&\, (4)}{=} & \sum_{j_1, \dots, j_4 = 1, \dots, 4} \tilde{p}^{j_1 \dots j_4} \left(z^4 + 2 z^2 \bar{z}^2 + \bar{z}^4 \right)_{j_1 \dots j_4} \\
\overset{(5)}{=} & \sum_{i,j=3}^{8} a_i a_j \sum_{j_1, \dots, j_4 = 1, \dots, 4} \tilde{p}^{j_1 \dots j_4} \left((r^i)^2 (r^j)^2 \right)_{j_1 \dots j_4} \qquad (6) \\
\overset{(4)}{=} & \sum_{i,j=3}^{8} a_i a_j \, Q_p(r^i, r^j).
\end{aligned}$$

This yields

$$\begin{aligned}
2\left(|p(z)| + \mathrm{Re}(p(z)) \right) & \overset{(2)}{\leq} 2 \left(Q_p(z, \bar{z}) + \mathrm{Re}(p(z)) \right) \\
& = p(z) + 2 Q_p(z, \bar{z}) + p(\bar{z}) \\
& \overset{(6)}{=} \sum_{i,j=3}^{8} a_i a_j \, Q_p(r^i, r^j) \\
& \leq \sum_{i,j=3}^{8} \underbrace{|a_i a_j|}_{=1} \, |Q_p(r^i, r^j)| \\
& \overset{(1)}{\leq} \sum_{i,j=3}^{8} \left(\sqrt{p(r^i)\, p(r^j)} \right)^2 = \left(\sum_{i=3}^{8} \sqrt{p(r^i)} \right)^2 .
\end{aligned}$$

This shows the claim.

$\square$

References

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