Proof of Hilbert's 1888 Theorem for ternary quartics

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Abstract

In 1888, Hilbert proved that $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, where $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ denote the (closed, convex) cones of the positive semidefinite (psd) and sums of squares (sos) n-ary $2d$-ic forms respectively. We will discuss the original proof due to Hilbert of the fact that every psd ternary quartic is a sum of not more than three squares of quadratic forms. The central idea of this proof is to associate to any ternary quartic a curve in the complex projective plane and then use the classical theory of algebraic curves. We will present a modern simplified version of Hilbert's proof due to Cassels, which was given by Rajwade in 1993. Moreover, we will point out modern expositions of Hilbert's proof by Rudin and Swan.