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Fachbereich Mathematik und Statistik
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FACHSEMINAR MA ON REAL ALGEBRAIC GEOMETRY–SS2019

Description

The aim of this cycle of seminars is to develop in more details some topics that were introduced in the course "Real Algebraic Geometry I" from the WS 2018-19 and that could be a starting point for a master thesis in the research stream investigated in the Schwerpunkt Reelle Geometrie und Algebra. The talks could focus for instance on the topics listed below, but we are also open to related thematic proposals from the participants. Each talk will provide a review of the results already known in literature about one of these topics and point out some related open questions.

List of possible topics

1. **Cylindrical decomposition of semialgebraic sets**
This seminar will discuss how a semialgebraic subset of R^n (where R is a real closed field) can be decomposed as a disjoint union of finitely many semialgebraic sets such that each of them is semialgebraically homeomorphic to an open hypercube of R^d. (See [9, Lecture 16, 17], [6, Sec 2.3])

2. **Topological definition of dimension of semialgebraic sets**
Using the cylindrical decomposition it is possible to give a topological definition of dimension of a semialgebriac set K. In this seminar, this notion will be introduced, discussed and compared with the dimension of K as an algebraic set. (See [9, Lecture 22], [6, Section 2.8])

3. **The necessity part of Hilbert's 1888 Theorem**:
$\Sigma_{n,m} \subsetneq \mathcal{P}_{n,m}$ for all $n \geq 3$, $m \geq 4$ and $(n, m) \neq (3, 4)$ with m even.
Using algebraic geometry, Hilbert showed that there exist psd quaternary quartics and ternary sextic which are not sos forms and also that this is sufficient to get psd not sos forms in all the other cases mentioned above. This seminar will present an explicit example of psd ternary sextic which is not a sos provided by Schmüdgen in 1979 without applying the theory of algebraic curves. The connections between the existence of such examples and the multivariate moment problem might also be highlighted. (See [15] and [8], [14, Chapter 7])

4. **Theorems of Polya and Reznick**
This seminar will discuss two interesting results which can be both derived from the Representation Theorem for Archimedean T-modules. Namely, Polya's theorem establishes that if f is a homogenous polynomial which is positive on $\mathbb{R}^n \setminus \{0\}$, then there exists $k \in \mathbb{N}_0$ s.t. the polynomial $(\sum_{i=1}^n X_i)^k f$ has non-negative coefficients. With similar methods, it is possible to show a similar result due of Reznick which states that if f is a positive homogenous polynomial, then there exists $k \in \mathbb{N}_0$ s.t. the polynomial $(\sum_{i=1}^n X_i^2)^k f$ is a sum of squares. (See [10, Section 5.5], [12], [13])

5. **The core variety in the truncated moment problem**
In [7] Fialkow introduced an alternate approach to the truncated moment problem based on a geometric invariant called the core variety. This seminar will give an introduction to this approach and show how the core variety can be used to establish a general solvability criterion for the multivariate truncated moment problem, [3] (see also [16, Section 18.3]).

6. **Preservation of moments**
   This seminar will discuss the problem of classifying all functions which, when applied term by term, leave invariant the sequences of moments of positive measures on the real line. Equivalently, the classification of all entrywise endomorphisms of the cone of positive Hankel matrices with real entries. (See [1], [2, Section 3])

7. **Continuous logic**
   This seminar will provide a general presentation of continuous first order logic as a formalism for the model theory of complete metric structures, such as may arise in probability theory, functional analysis, topological dynamics, and other areas of mathematics. The main purpose is to discuss separably categorical theories, analogous to countably categorical theories in classical first order logic. (see [4], [5]).

8. **Counting rational points using o-minimality.**
   This seminar will be concerned with the distribution of rational points on certain nonalgebraic subsets of $\mathbb{R}^n$. In particular, it will focus on a recent result in [11] showing that there are very few rational points of a definable set X in an o-minimal structure over $\mathbb{R}$ which do not lie on some connected semialgebraic subset of X of positive dimension.

References

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