# THE CORE VARIETY
## A NEW APPROACH TO THE TRUNCATED MOMENT PROBLEM

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Abstract. The classical truncated Moment Problem asks whether a real-valued linear functional defined on the space $\mathbb{R}[X_1, \ldots, X_n]_d$ of polynomials in $n$ variables with real coefficients and total degree at most $d$ can be represented as integration with respect to a non-negative Radon measure on $\mathbb{R}^n$. While huge progress has been made in finding solvability criteria for the full case, i.e. when the starting functional is defined on the whole polynomial algebra $\mathbb{R}[X_1, \ldots, X_n]$, new approaches are needed for getting solutions to the truncated case. In 2017 L. A. Fialkow introduced a new approach to the classical truncated Moment Problem using the core variety, which we are going to investigate in this report. We will derive the most important properties of the core variety along with an illustrative example and see how the core variety can be used to establish a necessary and sufficient condition for solving the truncated Moment Problem. Furthermore, we will pose some open questions and emphasize the great potential of this approach.

Contents

## 1. Introduction

Let $(S, \tau)$ be a non-empty topological T1 space and $V$ a finite dimensional $\mathbb{R}$-vector space consisting of real valued Borel measurable functions on $(S, \mathcal{B}(S))$, where $\mathcal{B}(S)$ is the Borel $\sigma$-algebra induced by $\tau$ on $S$. This report gives an existence criterion for a fixed $L\colon V \to \mathbb{R}$ linear to be representable as integration w.r.t. a non-negative Borel measure $\mu$ on $(S, \tau)$ i.e.

$$L(f) = \int_S f \,\mathrm{d}\mu, \;\; \forall f \in V.$$

For $S = \mathbb{R}^n$ endowed with the Euclidean topology and $V$ equal to the finite dimensional $\mathbb{R}$-vector space $\mathbb{R}[X_1, \ldots, X_n]_d$ of polynomials in $n$ variables with real coefficients and total degree at most $d$, this criterion provides a solution to the *truncated Moment Problem* $(n, d \in \mathbb{N})$. More precisely, the *truncated $K$-Moment Problem* asks whether a linear functional $L\colon \mathbb{R}[X_1, \ldots, X_n]_d \to \mathbb{R}$ can be represented as integration w.r.t. a non-negative Radon measure $\mu$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ supported on a fixed closed subset $K$ of $\mathbb{R}^n$, said $K$-representing measure for $L$, i.e.

$$L(f) = \int_K f \,\mathrm{d}\mu, \;\; \forall f \in \mathbb{R}[X_1, \ldots, X_n]_d.$$

If $K = \mathbb{R}^n$, then we refer to this problem just as the truncated Moment Problem.

The *full $K$-Moment Problem* consists of the same question but for $V := \mathbb{R}[X_1, \ldots, X_n]$, i.e. for the infinite dimensional $\mathbb{R}$-vector space consisting of polynomials in $n$ variables with coefficients in $\mathbb{R}$ of any degree. Several criteria for the existence of a solution to the full $K$-Moment Problem are known in the literature. The most famous of these criteria is the Riesz-Haviland theorem, which states that the non-negativity of $L$ on any non-negative polynomial on $K$ is both necessary and sufficient for the existence of a representing measure for $L$ supported on $K$. However, this result as well as many other criteria for the full case do not remain valid in the truncated case, since clearly we lose information by the truncation and so new approaches are needed for the truncated $K$-Moment Problem.

The approach presented in this report has been introduced in [2] and makes use of a geometric invariant called the *core variety*. In fact, in [2] the author proves that the core variety contains the support of any representing measure for $L$. Therefore, the core variety being non-empty is clearly necessary for the existence of a representing measure for $L$. It is then natural to ask, if this also suffices for the existence of a representing measure for $L$. This question is positively answered in the main theorem of this report based on [4]. Hence, a necessary and sufficient criterion for solving the truncated Moment Problem is given. This result can also be used to establish a truncated version of the Riesz-Haviland theorem in the general setting defined above (see [2, Theorem 3.1]). A truncated version of the Riesz-Haviland theorem in the classical setting over $\mathbb{R}[X_1, \ldots, X_n]_d$ was already established in [3, Theorem 2.2].

This new approach to the truncated $K$-Moment Problem also sheds some light on the full $K$-Moment Problem. Indeed, J. Stochel proves in [10] that a given full $K$-Moment Problem is solvable if and only if any corresponding truncated version is solvable.

Let us shortly describe the structure of this report. In Section 2 we recall some preliminary notions based on [5], [6], [7] and [8]. In Section 3 we introduce the core variety using a sequence of zero sets $\{S_i\}_{i \in \mathbb{N}_0}$ associated to a fixed linear functional and observe some of the most important properties of this sequence, such as the fact that it stabilizes. This is useful to give a more explicit description of the core variety, as in [4]. In Section 4 we prove the main theorem of [4], giving a criterion for the existence of a representing measure in the general setting introduced above. In Section 5 this result is applied to the truncated Moment Problem and yields a necessary and sufficient condition for a linear functional on $\mathbb{R}[X_1, \ldots, X_n]_d$ to admit a $K$-representing measure where $K \subseteq \mathbb{R}^n$. In Section 6 the potential of this new approach is pointed out, by posing some open questions related to the truncated Moment Problem.

## 2. Preliminaries

The aim of this section is to recall the most basic notions from (real) algebraic geometry and measure theory needed to understand the main result presented in this report.

For any $n \in \mathbb{N}$ and any field $k$ we set $k[\underline{X}] := k[X_1, \ldots, X_n]$ to be the ring of polynomials in $n$ variables with coefficients in $k$. We denote elements of $k^n$ by $\underline{x} := (x_1, \ldots, x_n)$. For $d \in \mathbb{N}_0$ we set $k[\underline{X}]_d$ to be the subring of $k[\underline{X}]$ consisting of polynomials in $n$ variables with coefficients in $k$ and total degree at most $d$.

The following is one of the most important concept in this report (see e.g. [7]).

**Definition 2.1.** Let $R$ be a real closed field and $n \in \mathbb{N}$. For $s \in \mathbb{N}$, and $f_1, \ldots, f_s \in R[\underline{X}]$ we define the associated **algebraic (zero) set** to be

$$\mathcal{Z}(f_1, \ldots, f_s) := \{\underline{a} \in R^n \mid \forall i \in \{1, \ldots, s\} : f_i(\underline{a}) = 0\}.$$

*Remark* 2.2. If we drop the assumption of $R$ being real closed, we call this set an *affine variety* of $\mathbb{R}^n$ and denote it by $\mathcal{V}(f_1, \ldots, f_s)$ (see e.g. [8]). Recall that

$$\mathcal{V}(f_1, \ldots, f_s) = \mathcal{V}(\langle f_1, \ldots, f_s \rangle),$$

where $\langle f_1, \ldots, f_s \rangle$ is the ideal generated by $f_1, \ldots, f_s$ over $R$.

In the following we always assume the underlying field to be $\mathbb{R}$, which is real closed. Therefore, both notions coincide and are equivalently applicable for defining the core variety.

A fundamental object in the study of the classical $K$-Moment Problem is the cone (i.e. a set closed under non-negative linear combinations of its elements) $\operatorname{Psd}(K)$ of all polynomials in $n$-variables which are non-negative on a fixed closed subset $K$ of $\mathbb{R}^n$.

**Definition 2.3.** Let $S$ be a set and $V$ a set of real valued functions on $S$. For $K \subseteq S$ we define

$$\operatorname{Psd}_V(K) := \{f \in V \mid \forall x \in K : f(x) \geq 0\}.$$

*Remark* 2.4. If $V$ is clear from the context, we simplify to $\operatorname{Psd}(K) := \operatorname{Psd}_V(K)$.

Recall also the following notion.

**Definition 2.5.** Let $n \in \mathbb{N}$ and $S \subseteq \mathbb{R}^n$. The **convex cone generated by** $S$ is defined as

$$\operatorname{cone}(S) := \left\{ \sum_{i=1}^{m} \lambda_i f_i \mid m \in \mathbb{N}, \lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}, f_1, \ldots, f_m \in S \right\}.$$

*Remark* 2.6. The $\operatorname{cone}(S)$ is the smallest convex cone in $\mathbb{R}^n$ containing $S$.

As the Moment Problem simultaneously evolved in algebra and functional analysis, it is indispensable to recall some basic definitions of measure theory and functional analysis.

A non-empty topological space $(S, \tau)$ induces a measurable space $(S, \mathcal{B}(S))$, where $\mathcal{B}(S)$ is the Borel $\sigma$-algebra induced by $\tau$ on $S$. In particular, for $S = \mathbb{R}^n$ we have $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^{n} \mathcal{B}(\mathbb{R})$.

Throughout this script we are working over a non-empty topological T1 space.

**Definition 2.7.** A non-empty topological space $(S, \tau)$ is a **T1 space** if for any distinct $x, y \in S$ there exist two neighborhoods $N_x$ of $x$ and $N_y$ of $y$ such that $x \notin N_y$ and $y \notin N_x$.

*Remark* 2.8. Requiring the T1 property on $(S, \tau)$ is equivalent to require any singleton of $S$ to be closed w.r.t. $\tau$. Note that any Hausdorff space is a T1 space.

To attack the truncated Moment Problem, we need to understand the notion of representing measures for a linear functional.

**Definition 2.9.** Let $(S,\tau)$ be a non-empty topological space, $V$ a vector space of Borel measurable real valued functions over $(S,\mathcal{B}(S))$ and $V^* := \{L\colon V \to \mathbb{R} \mid L \text{ is linear}\}$. A non-negative Borel measure $\mu$ on $(S,\mathcal{B}(S))$ is a **representing measure** for $L \in V^*$, if for any $f \in V$ we have

$$L(f) = \int_S f \,\mathrm{d}\mu.$$

The main focus of the classical Moment Problem are representing non-negative Radon measures.

**Definition 2.10.** Let $(S,\tau)$ be a non-empty topological Hausdorff space. A **Radon measure** $\mu$ on $(S,\mathcal{B}(S))$ is a locally finite (i.e. for all $x \in S$, there exists some neighborhood $N$ of $x$ such that $\mu(N) < \infty$) and inner regular (i.e. for any $B \in \mathcal{B}(S)$, $\mu(B) = \sup(\{\mu(K) \mid K \subseteq B, K \text{ is compact}\})$) measure.

Let us recall two further basic definitions from measure theory.

**Definition 2.11.** Let $(S,\Sigma)$ be a measurable space. A measure $\mu$ on $(S,\Sigma)$ is **$\sigma$-finite** if there exists at most countably many $A_i \in \Sigma$ such that $\mu(A_i) < \infty$ and $\bigcup_i A_i = S$.

**Definition 2.12.** Let $(S,\Sigma)$ be a measurable space, $A \in \Sigma$ and $\mu$ a measure on $(S,\Sigma)$. $A$ is called an **atom** of $\mu$ if $\mu(A) > 0$ and for any $B \in \Sigma$ with $B \subseteq A$ either $\mu(B) = \mu(A)$ or $\mu(B) = 0$.

For more details on the notions introduced above see e.g. [5], [6], [7] and [8].

## 3. The Core Variety

We are now ready to introduce the core variety and observe some of its properties. From now on let $(S,\tau)$ be a non-empty topological T1 space inducing the measure space $(S,\mathcal{B}(S))$. Furthermore, let $V$ be a finite dimensional $\mathbb{R}$-vector space in $\{f\colon (S,\mathcal{B}(S)) \to (\mathbb{R},\mathcal{B}(\mathbb{R})) \mid f \text{ is measurable}\}$ and let $V^* := \{L\colon V \to \mathbb{R} \mid L \text{ is linear}\}$, i.e. $V^*$ is the (algebraic) dual space of $V$. We set $\mathrm{Psd}(\cdot) := \mathrm{Psd}_V(\cdot)$ as defined in Definition 2.3.

**Definition 3.1.** For $L \in V^*$, set

$$\begin{aligned} S_0 &:= S_0(L) &&:= S \\ S_1 &:= S_1(L) &&:= \mathcal{Z}(\{p \in \mathrm{Psd}(S) \mid L(p) = 0\}) \\ S_{i+1} &:= S_{i+1}(L) &&:= \mathcal{Z}(\{p \in \mathrm{Psd}(S_i) \mid L(p) = 0\}) \quad \text{for any } i \in \mathbb{N}. \end{aligned}$$

**Definition 3.2.** For $L \in V^*$, the set

$$\mathcal{CV}(L) := \bigcap_{i=0}^{\infty} S_i$$

is called the **core variety** corresponding to $L$.

*Remark* 3.3. It can be easily observed that the sequence $\{S_i\}_{i\in\mathbb{N}_0}$ is decreasing w.r.t. $\subseteq$, i.e. for all $i \in \mathbb{N}_0$ we observe $S_{i+1} \subseteq S_i$.

Indeed, this can be proven by induction over $i \in \mathbb{N}_0$.
<u>Base case $i = 0$</u> Clear.
<u>Inductive assumption</u> Suppose that $S_{j+1} \subseteq S_j$ holds for all $j \leq i$.
<u>Step of induction $i \mapsto i+1$</u> Let us fix $x \in S_{i+2}$. Since $i + 2 \geq 2$ $(i \in \mathbb{N}_0)$, according to Definition 3.1, we have $p(x) = 0$ for any $p \in \mathrm{Psd}(S_{i+1})$ with $L(p) = 0$. Let $q \in \mathrm{Psd}(S_i)$ with $L(q) = 0$. The inductive assumption $S_{i+1} \subseteq S_i$ implies $q \in \mathrm{Psd}(S_{i+1})$. Hence, we can conclude $q(x) = 0$. Since $q$ was arbitrarily chosen, we get $x \in S_{i+1}$.

The following observation reveals that the sequence $\{S_i\}_{i\in\mathbb{N}_0}$ might stabilize.

**Lemma 3.4.** *If $S_k = S_{k+1}$ for some $k \in \mathbb{N}_0$, then for any $l \geq k$ $S_l = S_k$.*

*Proof.* If $\emptyset = S_k = S_{k+1}$, then the claim trivially holds by Remark 3.3.
Therefore, w.l.o.g. assume $\emptyset \neq S_k = S_{k+1}$. Let $x \in S_k = S_{k+1}$, then for any $g \in \mathrm{Psd}(S_k) = \mathrm{Psd}(S_{k+1})$ with $L(g) = 0$, $g(x) = 0$. Hence, $x \in S_{k+2}$. Since $x$ was arbitrarily chosen in $S_k$, we conclude $\emptyset \neq S_k = S_{k+1} = S_{k+2}$ using Remark 3.3. Iteratively the claim follows. ∎

From now on we will keep the following assumption.

$$\forall L \in V^* \, \exists p_L \in V \text{ s.t. } L(p_L) > 0 \text{ and } p_L \text{ is strictly positive on } S. \tag{3.1}$$

This will be used to prove the next lemma.

**Lemma 3.5.** *Let $f \in V$ and $S_{k+1} \neq \emptyset$ with $k \in \mathbb{N}_0$. If $f \restriction_{S_k} \equiv 0$, then $L(f) = 0$.*

*Proof.* Assume that there exists some $f \in V$ such that $f \restriction_{S_k} \equiv 0$ but $L(f) \neq 0$. W.l.o.g. $L(f) < 0$ (else use $-f$ and exploit the linearity of $L$). Set $g := p_L - \frac{L(p_L)}{L(f)} f \in V$ and observe for any $x \in S_k$ that

$$g(x) = \underbrace{p_L(x)}_{>0} - \frac{L(p_L)}{L(f)} \underbrace{f(x)}_{=0} > 0.$$

Therefore, $g$ is strictly positive on $S_k$. By the linearity of $L$, we have

$$L(g) = L(p_L) - \frac{L(p_L)}{L(f)} L(f) = 0.$$

Altogether we conclude $\emptyset = S_{k+1}$. ↯ ■

We are now ready to prove the main property of the sequence $\{S_i\}_{i \in \mathbb{N}_0}$.

**Proposition 3.6.** *The sequence $\{S_i\}_{i \in \mathbb{N}_0}$ stabilizes, i.e. there exists some $k \in \mathbb{N}_0$ sufficiently large such that, for all $l \geq k$, $S_l = S_k$.*

*Proof.* Let $i \in \mathbb{N}_0$ and set

$$W_i := \{f \restriction_{S_i} \mid f \in V\}.$$

W.l.o.g. $\emptyset \neq S_{i+1}$. Define $\psi\colon W_i \to W_{i+1}$ by $\psi(f \restriction_{S_i}) := f \restriction_{S_{i+1}}$. According to Remark 3.3, $S_{i+1} \subseteq S_i$ and so $\psi$ is well-defined and trivially surjective.

Recalling that $V \supseteq W_i$ is finite dimensional, we deduce

$$\dim W_{i+1} = \dim(\operatorname{im}(\psi)) = \dim W_i - \dim(\ker(\psi)) = \dim W_i - \dim(\ker(\psi)). \tag{3.2}$$

Now assume that the sequence has not yet stabilized, i.e. $S_{i+1} \subsetneq S_i$, and so we can fix some $x \in S_i \backslash S_{i+1}$. For such an $x$ there exists some $f \in \operatorname{Psd}(S_i)$ such that $L(f) = 0$ but $f(x) \neq 0$ by the definition of $S_{i+1}$. We conclude $f \restriction_{S_i} \not\equiv 0$.

Now let $y \in S_{i+1}$. Since $f \in \operatorname{Psd}(S_i)$ with $L(f) = 0$, we know $f(y) = 0$. Moreover, $y$ was arbitrarily chosen in $S_{i+1}$ and so

$$f \restriction_{S_{i+1}} \equiv 0.$$

Hence, we found a non-trivial element - given by $f$ - in the kernel of $\psi$. Therefore, $\dim(\ker(\psi)) > 0$. Plugging this observation into (3.2) we deduce

$$\dim W_{i+1} < \dim W_i.$$

The dimension of the $W_i$ drops in such a case.

Now we put it all together by examining what cases might happen for the sequence $\{S_i\}_{i \in \mathbb{N}_0}$.
Case 1 Assume $\emptyset = S_i$. Then according to Remark 3.3 the sequence stabilizes for $k = i$.
Case 2 Assume $\emptyset = S_{i+1} \subsetneq S_i \neq \emptyset$. Then according to Remark 3.3 the sequence stabilizes for $k = i+1$.
Case 3 Assume $\emptyset \neq S_i = S_{i+1}$. Then according to Lemma 3.4 the sequence stabilizes for $k = i$.
Case 4 Assume $\emptyset \neq S_{i+1} \subsetneq S_i \neq \emptyset$. Then by $\dim W_{i+1} < \dim W_i$, the drop of dimension can only take place finitely often, since $V = W_0$ is finite dimensional. Therefore, one of *Case 1* to *Case 3* is fulfilled after a finite time. This leads to a stabilization of the sequence in at most one more step (*Case 2*). ■

We can now immediately derive a more explicit description of the core variety and an upper bound for the stabilization index of the sequence $\{S_i\}_{i \in \mathbb{N}_0}$.

**Lemma 3.7.** *The sequence $\{S_i\}_{i \in \mathbb{N}_0}$ stabilizes in at most $(\dim V) - 1$ steps. Hence, for all $k \geq (\dim V) - 1$ we have $S_k = S_{(\dim V)-1}$.*

*Proof.* Let us construct the longest possible chain, which does not stabilize by exploiting the proof of Proposition 3.6. To do that *Case 4* has to be repeatedly fulfilled for the longest time possible. Clearly $W_0 = V$. Therefore, we start with full dimension $\dim V$. The least drop of dimension possible in each step is one.

If the stabilization has not taken place yet at step $i$, we have $S_{i+1} \neq \emptyset$ and so $p_L \restriction_{S_{i+1}} \in W_{i+1}$. Clearly $L(p_L) > 0$ by the choice of $p_L \in V$. Moreover, $S_{i+1} \neq S$ as we have not yet stabilized.

Suppose that $\dim W_{i+1} = 1$ and so $W_{i+1} = \langle p_L \restriction_{S_{i+1}} \rangle$, i.e. for any $f \in V$ there exists some $c_f \in \mathbb{R}$ such that $f \equiv c_f p_L$ on $S_{i+1}$. Hence,

$$L(f) = c_f \underbrace{L(p_L)}_{>0}$$

by exploiting Remark 3.5. Clearly $L(f) = 0$ if and only if $c_f = 0$ and so if and only if $f \equiv 0$. Therefore, we conclude

$$S_{i+2} := \mathcal{Z}(\{c_f p_L \in \mathrm{Psd}(S_{i+1}) \mid c_f = 0\}) = \mathcal{Z}(0) = S. \lightning$$

Hence, $\dim W_{i+1} \geq 2$ and so the dimension of $W_i$ can drop down at most to two. This leaves us with at most $\dim V - 2$ drop of dimensions in total.

In the next step one of *Case 1* to *Case 3* is fulfilled and so the stabilization occurs in at most one more step (*Case 2*). Altogether the sequence terminates after at most $\dim V - 1$ steps. ∎

The properties introduced so far enable us to give a more explicit description of the core variety.

*Remark* 3.8. Let $k$ be a stabilizing index as in Proposition 3.6. Then we obtain

$$\mathcal{CV}(L) = \bigcap_{i=0}^{\infty} S_i = \bigcap_{i=0}^{k} S_i = S_k = S_{(\dim V)-1}$$

by exploiting Remark 3.3 and Lemma 3.7. Note that $k < (\dim V) - 1$ might hold.

As already emphasized $(\dim V) - 1$ is only an upper bound for the stabilization index, which does not necessarily need to be reached. Yet it is the least upper bound we can find, as showed by the next example.

**Example 3.9.** Set $S := [0, \infty) \subseteq \mathbb{R}$ and endow it with the Borel $\sigma$-algebra $\mathcal{B}(S)$. Moreover, set

$$\begin{aligned} f_0(x) &:= -\frac{1}{2}x + \mathbb{1}_{[0,2]}(x) \\ f_1(x) &:= \frac{(x-2)(4-x)}{x-1}\mathbb{1}_{[1,4]}(x) \\ f_2(x) &:= \frac{(x-4)(6-x)}{x-3}\mathbb{1}_{[3,6]}(x) \end{aligned}$$

and

$$p :\equiv 1.$$

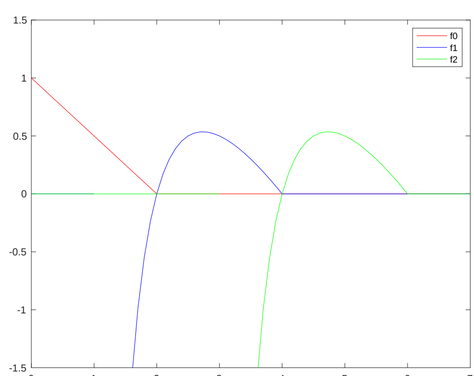


Fig 1: Visualization of $f_0$, $f_1$ and $f_2$

The behavior of $f_0$, $f_1$ and $f_2$ on $S := [0, \infty)$ is visualized in *Fig. 1*. Obviously $p$, $f_0$, $f_1$ and $f_2$ are linear independent $\mathcal{B}(S)$-measurable functions over $\mathbb{R}$ as *Fig. 1* emphasizes. Now set

$$V := \langle f_0, f_1, f_2, p\rangle_{\mathbb{R}},$$

which is a finite dimensional vector space over $\mathbb{R}$. More precisely, $\dim V = 4$. Define $L \in V^*$ by

$$L(p) := 1$$

and

$$L(f_i) := 0$$

for $i = 0, \dots, 2$. By setting $p_L := p$ the standing assumption (3.1) is fulfilled. Now observe

$$S_0 := S$$

and

$$S_1 := \mathcal{Z}(\{f \in \operatorname{Psd}(S) \mid L(f) = 0\}).$$

Any $f \in V$ with $L(f) = 0$ must be of the form $f = a_0 f_0 + a_1 f_1 + a_2 f_2$ with $a_0, a_1, a_2 \in \mathbb{R}$ for obvious reasons. Let $f \in \operatorname{Psd}(S)$. Then obviously $a_2 \geq 0$ and

$$0 \leq \lim_{x \searrow 3} f(x) = \lim_{x \searrow 3} a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) = \frac{1}{2} + a_2 \lim_{x \searrow 3} \underbrace{\frac{(x-4)(6-x)}{x-3}}_{\to -\infty}. \tag{3.3}$$

This immediately yields $a_2 = 0$. This behavior can also be deduced by carefully considering *Fig.1*, where it gets obvious that $f$ goes to $-\infty$ as $x \searrow 3$ if and only if $a_2 \neq 0$.
Analogously $a_1 = 0$ can be easily seen by letting $x \searrow 1$. Therefore, we conclude $f = a_0 f_0 \in \langle f_0 \rangle$ with $a_0 \geq 0$ and hence

$$S_1 = \mathcal{Z}(f_0) = [2, \infty).$$

Similarly, we determine

$$S_2 := \mathcal{Z}(\{f \in \operatorname{Psd}(S_1) = \operatorname{Psd}([2, \infty)) \mid L(f) = 0\}).$$

Again any $f \in V$ is of the form $f = a_0 f_0 + a_1 f_1 + a_2 f_2$ with $a_0, a_1, a_2 \in \mathbb{R}$ and if $f$ is also non-negative on $[2, \infty)$, then (3.3) shows that $a_2 = 0$ and $a_1 \geq 0$. Hence,

$$S_2 := \mathcal{Z}(f_0, f_1) = [4, \infty).$$

Lastly we observe that

$$S_3 := \mathcal{Z}(\{f \in \operatorname{Psd}(S_2) = \operatorname{Psd}([4, \infty)) \mid L(f) = 0\}) = \mathcal{Z}(f_0, f_1, f_2) = [6, \infty).$$

So we found a strictly decreasing sequence

$$S = S_0 \supsetneq S_1 \supsetneq S_2 \supsetneq S_3 = S_k$$

where $k := 3 = 4 - 1 = \dim V - 1$ is the highest index possible for which the sequence $\{S_k\}_{k \in \mathbb{N}_0}$ might has not yet stabilized according to Lemma 3.7.

## 4. An existence criterion for representing measures

The core variety will now be used to state and prove a solvability criterion for the existence of a representing measure for a fixed linear functional $L$ on $V$ as in Section 3. Recall that $(S, \tau)$ is assumed to be a T1 space, $V$ a finite dimensional $\mathbb{R}$-vector space consisting of real valued measurable functions over $(S, \mathcal{B}(S))$ s.t. the standing assumption (3.1) holds.

A very simple type of linear functionals on $V$ having a representing measure are the point evaluations.

**Definition 4.1.** For any fixed $s \in S$, we define the **point evaluation at** $s$ to be the linear functional on $V$ s.t.

$$L_s(f) := f(s), \ \forall f \in V.$$

Obviously for any $s \in S$, $L_s$ has a representing measure given by $\delta_s$, where $\delta_s$ denotes the Dirac measure with mass at $s$. Point evaluations are in fact representable by a finitely atomic measure.

**Definition 4.2.** A $\sigma$-finite measure $\mu$ on a measurable space $(A, \Sigma)$ is called **finitely atomic**, if there exists a partition of $A$ consisting of countably many measurable $U_i$ such that each $U_i$ is either an atom of $\mu$ or a zero set.

Since $(S, \tau)$ is assumed to fulfill the T1 property in our setting, we know by Remark 2.8 that any singleton is closed and so measurable. Therefore, any non-zero non-negative finitely atomic measure $\mu$ on $(S, \mathcal{B}(S))$ is necessarily a non-negative linear combination of Dirac measures i.e.

$$\mu = \sum_{i=1}^{m} \lambda_i \delta_{s_i}$$

for some $m \in \mathbb{N}$, $s_1, \ldots, s_m \in S$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}$. Note that if $(S, \tau)$ is a Hausdorff space, then any finitely atomic measure is especially a Radon measure.

Recall that a corollary to the *Bayer-Teichmann Theorem* (see [1, Theorem 2]) guarantees that if $L \in V^*$ has a representing measure $\mu$, then $L$ has a finitely atomic representing measure.

**Definition 4.3.** Set

$$\mathcal{C} := \mathrm{Cone}(\{L_s \mid s \in S\})$$

and

$$\mathcal{M} := \{L \in V^* \mid L \text{ has a representing measure}\},$$

i.e. $\mathcal{C}$ is the convex cone of all linear functionals on $V$ coming from finitely atomic measures on $(S, \mathcal{B}(S))$ and $\mathcal{M}$ is the set of all linear functionals on $V$ having a representing measure on $(S, \mathcal{B}(S))$.

Note that $\overline{\mathcal{C}} = \mathrm{Psd}(S)^*$ w.r.t. the product topology on $\mathbb{R}^S$, where $\mathbb{R}$ is endowed with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ (see [2, Proposition 2.1] for a proof).

Moreover, there exists a close relation between the linear functionals representable by finitely atomic measures and the ones representable by a not necessarily finitely atomic measure.

**Proposition 4.4.** *Let $\mathbb{R}$ be endowed with the Euclidean topology and let $\mathcal{B}(\mathbb{R})$ be the corresponding Borel $\sigma$-algebra. Then*

$$int(\mathcal{C}) = int(\mathcal{M}) = int(Psd(S)^*) = \{L \in V^* \mid \forall f \in Psd(S)\backslash\{0\} : \; L(f) > 0\} =: \mathcal{SP}$$

*w.r.t. the product topology on $\mathbb{R}^S$. Any $L \in \mathcal{SP}$ is said to be* **strictly positive** over $\mathrm{Psd}(S)$.

*Proof.* See [2, Corollary 2.2]. ■

As already mentioned in the introduction, we are interested in $K$-representing measures. Therefore, let us recall the definition of the support of a measure.

**Definition 4.5.** Let $\mu$ be a Borel measure on $(S, \mathcal{B}(S))$ induced by $(S, \tau)$. Then the **support of** $\mu$ is defined as

$$\mathrm{supp}(\mu) := \{s \in S \mid \mu(U) > 0 \text{ for all open sets } U \ni s\}.$$

In particular, based on [2, Corollary 2.2] and Proposition 4.4 we observe the following.

**Corollary 4.6.** *If $L \in \mathcal{SP}$, then $L \in \mathcal{C}$ and*

$$\bigcup_{\mu \in \mathcal{A}(L)} supp(\mu) = S$$

*where $\mathcal{A}(L) := \{\mu \text{ measure on } (S, \mathcal{B}(S)) \mid \mu \text{ finitely atomic representing measure for } L\}$.*

The next proposition will be crucial in the proof of the main theorem of this report. For $k \in \mathbb{N}_0$ set

$$W_k := \{f \restriction_{S_k} \; \mid f \in V\}$$

and for $L \in V^*$ define $\tilde{L} \colon W_k \to \mathbb{R}$ $\tilde{L}(f \restriction_{S_k}) := L(f)$ for all $f \in V$.

**Proposition 4.7.** *Let $L \in V^*$.*

*(1) If $\emptyset \neq S_k = S_{k+1}$ for some $k \in \mathbb{N}_0$, then*

$$\tilde{L} \in \mathcal{SP}_k := \{l \in W_k^* \mid \forall f \in Psd(S_k)\backslash\{0\} : \, l(f \restriction_{S_k}) > 0\}$$

*and $\tilde{L}$ has a finitely atomic representing measure. Moreover,*

$$\bigcup_{\mu \in \mathcal{A}(\tilde{L})} supp(\mu) = S_k,$$

*where $\mathcal{A}(\tilde{L}) := \{\mu \text{ measure on } (S_k, \mathcal{B}(S_k)) \mid \mu \text{ finitely atomic representing measure for } \tilde{L}\}$.*

*(2) If any $f$ in $V$ is continuous on $S$, then $supp(\mu) \subseteq S_1$ for any representing measure $\mu$ of $L$.*

*Proof.*

(1) Let us observe that $\tilde{L} \in \mathcal{SP}_k$. It can be easily seen that $\tilde{L}$ is a well-defined linear functional on $W_k$. Indeed if $f \restriction_{S_k} = g \restriction_{S_k}$ then $f - g \equiv 0$ on $S_k$ and so by Lemma 3.5

$$0 = L(f - g) = L(f) - L(g),$$

where the last equation follows by the linearity of $L$. Altogether $\tilde{L}(f \restriction_{S_k}) = L(f) = L(g) = \tilde{L}(g \restriction_{S_k})$. Note that $\tilde{L}$ is clearly linear, since $L$ is linear. Putting it all together, we conclude $\tilde{L} \in W_k^*$.

So assume that $\tilde{L} \notin \mathcal{SP}_k$, due to the existence of some $q \in \mathrm{Psd}(S_k)\backslash\{0\}$ such that $\tilde{L}(q \restriction_{S_k}) \leq 0$.

<u>Case 1</u> Assume that $0 = \tilde{L}(q \restriction_{S_k}) := L(q)$. Since $q$ is an element of $\mathrm{Psd}(S_k)\backslash\{0\}$, we know that $q(s) > 0$ for some $s \in S_k$. Therefore, such $s$ is not an element of $S_{k+1}$, which yields a contradiction since $s \in S_k = S_{k+1}$. ↯

<u>Case 2</u> Assume that $b := \tilde{L}(q \restriction_{S_k}) < 0$ and set $a := \tilde{L}(p_L \restriction_{S_k}) := L(p_L) > 0$. Define $\overline{p} := p_L - \frac{a}{b}q$. Now fix $s \in S_k$ and observe $\overline{p}(s) = \underbrace{p_L(s)}_{>0} - \underbrace{\underbrace{\frac{a}{b}}_{<0}\underbrace{q(s)}_{\geq 0}}_{\leq 0} > 0$ ($p_L(s) > 0$ since $p_L$ is strictly positive on $S$ according to the standing assumption (3.1)). So $\overline{p} > 0$ on $S_k$, since $s \in S_k$ was arbitrarily chosen. Moreover,

$$L(\overline{p}) =: \tilde{L}(\overline{p} \restriction_{S_k}) = \tilde{L}(p_L \restriction_{S_k}) - \frac{a}{b}\tilde{L}(q \restriction_{S_k}) = 0$$

by exploiting the linearity of $\tilde{L}$. We conclude $S_{k+1} = \emptyset$, due to the existence of a strictly positive element over $S_k$ on which $L$ vanishes. This yields a contradiction as $S_{k+1} = S_k \neq \emptyset$.

Since $\tilde{L} \in \mathcal{SP}_k$, Corollary 4.6 (by setting $S = S_k$ and $\mathcal{SP} = \mathcal{SP}_k$) ensures that $\tilde{L}$ is an element of $\mathcal{C}_k := \mathrm{cone}(\{\tilde{L}_s \mid s \in S_k\})$, where $\tilde{L}_s$ is the restriction of $L_s$ on $S_k$. Moreover, $\tilde{L}$ has a finitely atomic representing measure on $(S_k, \mathcal{B}(S_k))$ and

$$\bigcup_{\mu \in \mathcal{A}_k(\tilde{L})} \mathrm{supp}(\mu) = S_k.$$

(2) Assume that $\mu$ is a representing measure for $L$ such that $\mathrm{supp}(\mu) \not\subseteq S_1$. Then there exists some $s \in \mathrm{supp}(\mu)\backslash S_1 \subseteq S\backslash S_1 = S_0\backslash S_1$. As an immediate consequence there exists some $p \in \mathrm{Psd}(S)$ such that $L(p) = 0$ and $p(s) > 0$. Moreover, since any $f$ in $V$ is assumed to be continuous, especially the considered $p \in V$ is continuous. Hence, there exists some open neighborhood $U$ of $s$ such that $p(U) > 0$. On top of that $s$ was chosen to be an element of the support of $\mu$ and so we deduce $\mu(U) > 0$. Exploiting $\mu$ being a representing measure of $L$, we have

$$0 = L(p) = \int_S p \, \mathrm{d}\mu \; \geq \int_U p \, \mathrm{d}\mu \; > 0. \text{ ↯}$$

■

*Remark* 4.8. In particular if $\emptyset \neq S_1 = S_0 = S$, then $k = 0$ and so $W_k = W_0 = V$. Hence, $\tilde{L} = L$ and by (1) $L$ is strictly positive over $\mathrm{Psd}(S)$ and has a finitely atomic representing measure.

We can now state and prove the main result of this report, which gives a necessary and sufficient condition for the existence of a representing measure for a fixed $L \in V^*$ using the core variety.

**Theorem 4.9.** *$L \in V^*$ fulfilling the standing assumption (3.1) has a representing measure if and only if $\mathcal{CV}(L) \neq \emptyset$. In particular, if $\mathcal{CV}(L) \neq \emptyset$, then*

$$\bigcup_{\mu \in \mathcal{A}(L)} supp(\mu) = \mathcal{CV}(L),$$

*where $\mathcal{A}(L) := \{\mu$ measure on $(S, \mathcal{B}(S)) \mid \mu$ finitely atomic measure representing $L\}$.*

*Proof.* Recall the standing assumption (3.1)

$$\forall L \in V^* \; \exists p_L \in V \text{ s.t. } L(p_L) > 0 \text{ and } p_L \text{ is strictly positive on } S.$$

$\Rightarrow$
Assume that $L$ has a representing measure $\mu$. By Proposition 3.6, there exists a $k \in \mathbb{N}_0$ sufficiently large such that $\{S_i\}_{i \in \mathbb{N}_0}$ stabilizes at index $k$. So by Remark 3.8 we have $\mathcal{CV}(L) = S_k$. Let us show by induction over $i \in \mathbb{N}_0$ that the following claim holds.
Claim For all $i \in \mathbb{N}_0$, $S_i \in \mathcal{B}(S)$ and

$$\mu(S_i) = \mu(S_0) = \mu(S) > 0. \tag{4.1}$$

Base case i=0 Clear from the standing assumption (3.1).
Inductive assumption Assume that for any $j \leq i$, $S_j \in \mathcal{B}(S)$ and $\mu(S_j) = \mu(S_0) = \mu(S) > 0$.
Step of induction $i \mapsto i+1$ Set $T := \langle \{f \in \ker L \mid f \in \mathrm{Psd}(S_i)\} \rangle$ and note that $S_{i+1} = \mathcal{Z}(T)$.

Case 1 Assume that $T = \{0\}$. Then $S_{i+1} = \mathcal{Z}(T) = \mathcal{Z}(0) = S \in \mathcal{B}(S)$ and clearly

$$\mu(S_{i+1}) = \mu(S_0) = \mu(S) > 0.$$

Case 2 Assume that $T \neq \{0\}$. Then there exist $m \in \mathbb{N}$ and $f_1, \ldots, f_m \in T \subseteq V$ such that $T = \langle f_1, \ldots, f_m \rangle$, since $V$ is a finite dimensional vector space and $T$ a subspace of $V$. Observe that

$$S_i \supseteq S_{i+1} = \mathcal{Z}(T) = \mathcal{Z}(\langle f_1, \ldots, f_m \rangle) = \mathcal{Z}\left(\sum_{i=1}^{m} f_i\right),$$

where the last equality holds, because $f_i \in \mathrm{Psd}(S_i)$ for $i \in \{1, \ldots, m\}$. Clearly, $\sum\limits_{i=0}^{m} f_i$ is Borel measurable and so

$$S_{i+1} = \mathcal{Z}\left(\sum_{i=1}^{m} f_i\right) = \left(\sum_{i=1}^{m} f_i\right)^{-1}(\{0\}) \in \mathcal{B}(S)$$

as $\{0\} \in \mathcal{B}(\mathbb{R})$. Now set $C := S_i \cap S_{i+1}^c \in \mathcal{B}(S)$ and observe that

$$\sum_{i=1}^{m} f_i \restriction_{S_{i+1}} \equiv 0 \text{ and } \sum_{i=1}^{m} f_i \restriction_C > 0.$$

Since $\mu$ is assumed to be a representing measure of $L$, exploiting Remark 3.5, we get

$$\begin{aligned} 0 &= L\left(\sum_{i=1}^{m} f_i\right) = \int_S \sum_{i=1}^{m} f_i \,\mathrm{d}\mu \underset{\mu(S)=\mu(S_i)}{=} \int_{S_i} \sum_{i=1}^{m} f_i \,\mathrm{d}\mu = \int_{C \dot\cup S_{i+1}} \sum_{i=1}^{m} f_i \,\mathrm{d}\mu \\ &= \int_C \sum_{i=1}^{m} f_i \,\mathrm{d}\mu + \int_{S_{i+1}} \underbrace{\sum_{i=1}^{m} f_i}_{=0} \,\mathrm{d}\mu = \int_C \underbrace{\sum_{i=1}^{m} f_i}_{>0} \,\mathrm{d}\mu. \end{aligned}$$

Hence, $\mu(C) = 0$ and so

$$\mu(S_{i+1}) = \mu(S_i \backslash C) = \mu(S_i) - \mu(C) = \mu(S_i) = \mu(S_0) = \mu(S) > 0.$$

By induction we can conclude that (4.1) holds. This implies that

$$\mu(\mathcal{CV}(L)) = \mu(S_k) > 0$$

and so that $\mathcal{CV}(L) \neq \emptyset$.

$\Leftarrow$
Assume that $\mathcal{CV}(L) \neq \emptyset$ and let $k \in \mathbb{N}_0$ be such that $\mathcal{CV}(L) = S_k$. Set

$$W_k := \{f \restriction_{S_k} \mid f \in V\}$$

and define $\tilde{L}\colon W_k \to \mathbb{R}$ via $L(f \restriction_{S_k}) := L(f)$. Applying Proposition 4.7, we deduce that $\tilde{L}$ is strictly positive over $\mathrm{Psd}(S_k)\backslash\{0\}$ and has a finitely atomic representing measure. Moreover, the union of the supports of all such measures is exactly $S_k$.

Now set $\mathcal{C}_k := \mathrm{Cone}(\{\tilde{L}_s \mid s \in S_k\})$, where $\tilde{L}_s$ is the restriction of $L_s$ on $S_k$. Then we just observed that $\tilde{L} \in \mathcal{C}_k$, i.e.

$$\tilde{L} = \sum_{i=1}^{m} a_i \tilde{L}_{r_i}$$

for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{R}_{>0}$ and $r_1, \ldots, r_m \in S_k$. Therefore, for any $f \in V$

$$L(f) =: \tilde{L}(f \restriction_{S_k}) = \sum_{i=1}^{m} a_i \tilde{L}_{r_i}(f \restriction_{S_k}) \underset{r_i \in S_k}{=} \sum_{i=1}^{m} a_i f(r_i) = \sum_{i=1}^{m} a_i L_{r_i}(f)$$

and so $L$ has a finitely atomic representing measure. More precisely, any finitely atomic measure representing $\tilde{L}$ is also a finitely atomic measure representing $L$ without the restriction on $S_k$.

Therefore, $S_k$ is clearly a subset of the union of the supports of all finitely atomic representing measures for $L$.

For the other inclusion, let $\mu$ be a finitely atomic measure representing $L$. The Claim ensures that $\mu(S_k) = \mu(S)$ and so atoms of $\mu$ must lie in $S_k$. Hence, the union of the supports of all finitely atomic measures representing $L$ must be a subset of $S_k$. Altogether we obtained that

$$\bigcup_{\mu \in \mathcal{A}(L)} \mathrm{supp}(\mu) = S_k = \mathcal{CV}(L).$$

■

*Remark* 4.10. If $L \equiv 0$ (so the standing assumption (3.1) is dropped), then clearly $\mathcal{CV}(L) = \emptyset$. Nevertheless, $L$ is representable by the zero measure.

To clarify the importance of this main result, we will apply it to Example 3.9.

**Example 4.11.** Recalling Example 3.9 and Lemma 3.7, we deduce

$$\mathcal{CV}(L) = S_{\dim V - 1} = S_3 = [6, \infty) \neq \emptyset.$$

Therefore, $L$ has a finitely atomic representing measure and

$$\bigcup_{\mu \in \mathcal{A}(L)} \mathrm{supp}(\mu) = [6, \infty),$$

where $\mathcal{A}(L) := \{\mu \text{ measure on } ([0,\infty), \mathcal{B}([0,\infty))) \mid \mu \text{ finitely atomic measure representing } L\}$.

In the classical truncated Moment Problem we are working over the finite dimensional $\mathbb{R}$-vector space $\mathbb{R}[\underline{X}]_d$ and are interested in the existence of appropriate non-negative Radon measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Since $\mathbb{R}^n$ endowed with the Euclidean topology is a Hausdorff topological space, let us see what happens, if $(S, \tau)$ is assumed to be Hausdorff.

**Corollary 4.12.** *Let $(S, \tau)$ be a Hausdorff space. If any $f \in V$ is continuous and $L \in V^*$, then*

$$\mathcal{CV}(L) = \bigcup_{\mu \in \mathcal{R}(L)} supp(\mu),$$

*where $\mathcal{R}(L) := \{\mu$ measure on $(S, \mathcal{B}(S)) \mid \mu$ representing non-negative Radon measure for $L\}$.*

*Proof.* Let $\mu \in \mathcal{R}(L)$. Then we claim that for any $i \in \mathbb{N}_0$ we have

$$\operatorname{supp}(\mu) \subseteq S_i.$$

Indeed, this can be proven by induction over $i \in \mathbb{N}_0$.
Base case $i = 0$ Clear.
Inductive assumption Assume that for any $j \leq i$ $\operatorname{supp}(\mu) \subseteq S_i$.
Step of induction $i \mapsto i+1$ Assume $\operatorname{supp}(\mu) \not\subseteq S_{i+1}$. Hence, it exists some $x \in \operatorname{supp}(\mu) \backslash S_{i+1} \subseteq S_i \backslash\ S_{i+1}$, i.e. for some $f \in \operatorname{Psd}(S_k)$ with $L(f) = 0$ we have $f(x) > 0$. Since any $f \in V$ is assumed to be continuous, there exists some open neighborhood $U$ of $x$ such that $f(U) > 0$. Also, $\mu(U) > 0$, because $x \in \operatorname{supp}(\mu)$ and $U$ is an open neighborhood of $x$. Putting all together, we get

$$\int_U f \,\mathrm{d}\mu \ > 0.$$

Moreover, $f \in \operatorname{Psd}(S_i)$ and $\operatorname{supp}(\mu) \subseteq S_i$ yields that $f \restriction_{\operatorname{supp}(\mu)} \geq 0$. Since $\mu$ is a representing non-negative Radon measure for $L$, we deduce that

$$0 = L(f) = \int_S f \,\mathrm{d}\mu \ = \int\limits_{\operatorname{supp}(\mu)} \underbrace{f}_{\geq 0} \,\mathrm{d}\mu \ \geq \int_U f \,\mathrm{d}\mu \ > 0. \ \lightning$$

In conclusion for $k \in \mathbb{N}_0$ stabilizing $\{S_i\}_{i \in \mathbb{N}_0}$, we have that

$$\bigcup_{\mu \in \mathcal{R}(L)} \operatorname{supp}(\mu) \subseteq S_k = \mathcal{CV}(L) \underset{(*)}{=} \bigcup_{\mu \in \mathcal{A}(L)} \operatorname{supp}(\mu) \subseteq \bigcup_{\mu \in \mathcal{R}(L)} \operatorname{supp}(\mu),$$

where $\mathcal{A}(L) := \{\mu \text{ measure on } (S, \mathcal{B}(S)) \mid \mu \text{ finitely atomic measure representing } L\}$ and $(*)$ holds according to Theorem 4.9. ∎

## 5. Application to the classical truncated Moment Problem

In this section we will see how the observed results can be applied to the truncated Moment Problem. For $d \in \mathbb{N}$ and $n \in \mathbb{N}$, we take $V := \mathbb{R}[\underline{X}]_d$ and set $(S, \mathcal{B}(S)) := (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) = \left(\mathbb{R}, \bigotimes_{i=1}^{n} \mathcal{B}(\mathbb{R})\right)$. Clearly, $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is a non-empty Hausdorff topological space and so the T1 property is fulfilled. Let $\vec{m} := (m_\alpha)_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq d}$ be a sequence of real numbers.

We now want to give a necessary and sufficient condition for $\vec{m}$ to be a truncated Moment Sequence, i.e. for the existence of a non-negative Radon measure $\mu$ such that for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq d$

$$m_\alpha = \int_{\mathbb{R}} \underline{X}^\alpha \,\mathrm{d}\mu.$$

To do that, we consider the *Riesz functional* $L_{\vec{m}} \colon \mathbb{R}[\underline{X}]_d \to \mathbb{R}$ associated to the sequence $\vec{m}$ defined by

$$L_{\vec{m}}(\underline{X}^\alpha) := m_\alpha \text{ for any } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq d.$$

Clearly, any $p \in \mathbb{R}[\underline{X}]_d$ is continuous and Borel measurable. Furthermore, we assume that $m_{\underline{0}} > 0$ and so $L(1) = L(\underline{X}^{\underline{0}}) = m_{\underline{0}} > 0$. This means that there exists some strictly positive $p_L \in V$, namely $p_L \equiv 1$, such that the standing assumption (3.1) is fulfilled.

Altogether we established the required setting in Section 4 and so using the results obtained so far, we get the following theorem.

**Theorem 5.1.** *Let $n, d \in \mathbb{N}$, $\vec{m} := (m_\alpha)_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq d} \subseteq \mathbb{R}$ with $m_{\underline{0}} > 0$ and $L_{\vec{m}}$ be the associated Riesz functional. Then $\vec{m}$ is a truncated Moment Sequence if and only if $\mathcal{CV}(L_{\vec{m}}) \neq \emptyset$. Moreover, any representing Radon measure of $L_{\vec{m}}$ is necessarily supported on a subset of the core variety $\mathcal{CV}(L_{\vec{m}})$.*

## 6. Conclusion

According to J. Stochel in [10] a given full $K$-Moment Problem is solvable if and only if any corresponding truncated version is solvable. Therefore, the new insights on the truncated Moment Problem provided by the core variety approach here presented, also shed light on the full Moment Problem. However, there are still many questions to be answered. For example:

(1) We do not have an explicit description of the core variety which allows an effective membership verification. Instead, to compute the core variety we have to determine the whole sequence $\{S_i\}_{i\in\mathbb{N}_0}$ iteratively, which can be rather hard to do.
(2) Having an easy criterion for checking whether the core variety is empty or not would be extremely useful because of Theorem 5.1 and would provide an effective solution to the truncated Moment Problem.
(3) It is also not clear if the core variety approach could be directly applied to an infinite dimensional $V$, i.e. if it could be directly used for getting new results on the full Moment Problem without passing to the truncated version.
(4) Applying the core variety approach to any finite dimensional subspace of an infinitely generated commutative unital $\mathbb{R}$-algebra would lead to new insight into the general infinite dimensional Moment Problem, for which few results are known in literature.

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