Hilbert's 17th problem for ordered fields

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ABSTRACT.

In 1975 K. McKenna investigates the properties of ordered fields K where the following holds:

For every rational function f in n variables with coefficients from K such that f is non-negative everywhere it is defined on K, f can be written as a sum of squares of rational functions in n variables with coefficients from K.

Artin proved that this holds for Archimedean ordered fields, but in his paper McKenna goes further and gives a characterization of all fields having the desired property. This characterization includes the relation of K to its real closure, an axiomatization of K as well as a property he calls the *Weak Hilbert Property (WHP).*
In this talk we will give a detailed proof of this characterization. In the end we will introduce both an example and a counterexample of such a field as well as some open related questions.

1 Introduction

In 1888 Hilbert asked the question: Is a rational function in n variables with rational coefficients which is everywhere non-negative on the rationals necessarily a sum of squares of rational functions with rational coefficients? Artin proved that this is true for an arbitrary Archimedean field. In 1975 McKenna investigated the necessary and sufficient properties of fields where Hilbert's question has a positive answer.
In this report we focus on two of the results he proved in his paper. In the first one (Theorem 3.3) the necessary and sufficient conditions mentioned above are given. In the second one (Theorem 3.7) we concentrate on a characterization of being dense in the real closure which is not a trivial property of a field as we will see in the examples. We show that being dense in the real closure is a first order property of an ordered field and further is equivalent to the so-called *Weak Hilbert Property (WHP)*. The WHP has been further investigated by Prestel in [5], where he further generalizes McKenna's results. In

this report we also give a glimpse into the results in [5].

Let us briefly describe the structure of this report. In Section 2 we give some preliminaries - in real algebraic geometry (cf. [2]) and model theory (cf. [4]). In Section 3 we prove the two main results as well as some needed lemmata. In the last two sections we give some examples and describe the work we are planning to do in the future on this topic.

2 Preliminaries

In this section we will introduce some notations, definitions and results which are needed to state and prove McKenna's characterization. Let us start by introducing basic notions and results concerning orderings on fields.

Definition 2.1. *Let K be a field and $T \subseteq K$ such that*

(i) $T + T \subseteq T$,

(ii) $TT \subseteq T$,

(iii) $a^2 \in T$ *for every* $a \in K$,

(iv) $-1 \notin T$,

(v) $-T \cup T = K$.

Then T is called a positive cone *(or* ordering*) of K.*

Definition 2.2. *Let K be a field. If there exists $P \subseteq K$ such that P is an ordering on K, we will say K is* (formally) real.

Notation 2.3. *Let K be an ordered field and $f \in K(X_1, ..., X_n) =: K(\underline{X})$. If f is a sum of squares of elements in $K(\underline{X})$, i.e. $f = f_1^2 + ... + f_n^2$ for some $n \in \mathbb{N}$ and $f_i \in K(\underline{X})$ for all $1 \leq i \leq n$, we write*

$$f \in \sum K(\underline{X})^2.$$

In the same style we write $\sum K^2$ for the elements in K which can be represented by a sum of squares of elements in K.

Remark 2.4 (cf. lecture 3, Exercise 1.6 and Corollary 3.4 in [2])**.**

1. *For a field K, $\sum K^2$ always satisfies properties* (i)-(iii) *in Definition 2.1, so $\sum K^2$ is always a preordering of K.*
2. *A field K is real if and only if $\sum K^2$ is a proper preordering, i.e. $\sum K^2$ satisfies additionally* (iv) *in Definition 2.1.*

Definition 2.5. *Let K be an ordered field and $f \in K(\underline{X})$. We will say that f is* definite *on K (or simply* definite*) if f is non-negative everywhere it is defined on K^n.*

By the definition of an ordering on a field K we immediately obtain the following.

Corollary 2.6. *Let K be an ordered field. If $f \in K(\underline{X})$ is a sum of squares, then f is definite on K.*

The next definition is quite important, as it appears in one of our main results.

Definition 2.7. *We say that a field K* is rigid *if K is uniquely orderable.*

Theorem 2.8 (*Artin (1927)*, cf. lecture 3, Corollary 3.5 in [2])**.** *Let K be a real field. Then*

$$\sum K^2 = \bigcap \{P : \ P \text{ is an ordering of } K\}.$$

In other words, if K is a real field and $a \in K$, then

$$a \geq_P 0 \text{ for every ordering } P \Leftrightarrow a \in \sum K^2.$$

Putting the property of rigidness of a field and *Artin's* theorem together, we obtain the following.

Corollary 2.9. *Let K be a field. K is rigid if and only if $\sum K^2$ is an ordering of K.*

Now we will introduce the notion of real closed fields by a characterization due to *Artin* and *Schreier*.

Propostion 2.10 (*Artin-Schreier (1926)*, cf. lecture 5, Propostion 1.1 in [2])**.** *Let K be a field. The following are equivalent:*

(i) K is real closed,

(ii) K has an ordering P which does not extend to any proper algebraic extension,

(iii) K is real, has no proper algebraic extension of odd degree, and

$$K = K^2 \cup -(K^2).$$

Another characterization of real closed fields, which is quite useful, is given by the following result.

Corollary 2.11 (cf. lecture 5, Corollary 1.4 in [2])**.** *Let (K, P) be an ordered field. Then K is real closed if and only if*

(a) every positive element in K has a square root in K, and

(b) every polynomial of odd degree with coefficients from K has a root in K.

Let us now recall the definition of a real closure of an ordered field.

Definition 2.12. *Let (K, P) be an ordered field. R is a real closure of (K, P) if*

(i) R is real closed,

(ii) $K \subseteq R$, $R|K$ is algebraic,

(iii) $P = \sum R^2 \cap K$.

Using Zorn's Lemma, we obtain the next theorem about the existence of a real closure for any ordered field.

Theorem 2.13 (cf. lecture 8, Theorem 1.2 and Corollary 2.3 in [2])**.** *Every ordered field (K, P) has a real closure which is unique up to isomorphism.*

In this report, we will denote the real closure of an ordered field K by $\overline{K}$ - if not further specified.
Another construction we will need is related to ordering extensions.

Definition 2.14. *Let (K, P) be an ordered field and $L \mid K$ a field extension. We define*

$$T_L(P) := \left\{ \sum_{i=0}^{n} p_i y_i^2 \mid n \in \mathbb{N}, p_i \in P, y_i \in L \right\}.$$

Remark 2.15 (cf. lecture 4, Remark 1.3 in [2])**.** *Let (K, P) be an ordered field and $L \mid K$ a field extension. Then $T_L(P)$ is the smallest preordering of L containing P.*

Corollary 2.16 (cf. lecture 4, Corollary 1.4 in [2])**.** *Let (K, P) be an ordered field and $L \mid K$ a field extension. Then P has an extension to an ordering Q of L if and only if $T_L(P)$ is a proper preordering (i.e. $-1 \notin T_L(P)$).*

The rest of this section is dedicated to some notions from Model Theory. First we will introduce basic notions as languages, structures, substructers and models, then the theory of real closed fields and in the end the notion of model completeness.

Definition 2.17. *We call a tuple $\mathcal{L} =< \mu, \lambda, K >$ a* language *if*

- *$\mu : I \to \mathbb{N}$, f_i function symbol with arity $\mu(i)$ and I an index set,*
- *$\lambda : J \to \mathbb{N}$, R_j relation symbol with arity $\lambda(j)$ and J an index set,*
- *for every $k \in K$: c_k is a constant symbol and K an index set.*

If not further specified, let $\mathcal{L}$ be a language and $\sum$ a set of $\mathcal{L}$-sentences. We call $\sum$ a theory.

Definition 2.18. *We call a tuple $\mathcal{A} = (A, (f_i^A)_{i \in I}, (R_j^A)_{j \in J}, (c_k^A)_{k \in K})$ an* $\mathcal{L}$-structure *if*

- *A is a set, called* domain *of $\mathcal{A}$,*

- *for every* $i \in I$, $f_i^A : A^{\mu(i)} \to A$,
- *for every* $j \in J$, $R_j^A \subseteq A^{\lambda(j)}$,
- *for every* $k \in K$, $c_k^A \in A$.

Definition 2.19. *Let* $\mathcal{A}$, $\mathcal{B}$ *be* $\mathcal{L}$*-structures. We call* $\mathcal{A}$ *a* substructure *of* $\mathcal{B}$ *if*

$$id: \mathcal{A} \hookrightarrow \mathcal{B}$$

is an $\mathcal{L}$*-embedding. We write* $\mathcal{A} \subseteq \mathcal{B}$.

Definition 2.20. *A* $\mathcal{L}$*-structure* $\mathcal{A}$ *is called* model of $\sum$ *if*

$$\mathcal{A} \models \sigma \text{ for every } \sigma \in \sum .$$

We write $\mathcal{A} \models \sum$ *and* $\mathcal{A} \in Mod\,(\sum)$.

The language we use for the theory of real closed fields is given by

$$\mathcal{L}_{or} =: < +, -, \cdot, <, 0, 1 > .$$

Using $\mathcal{L}_{or}$, we can now give an axiomatization of real closed fields.

Definition 2.21. *The theory of real closed fields, in symbols* $\sum_{rcf}$, *is given by*

(i) *the axioms of an ordered field*

(ii) $\forall x\, (0 < x \;\to\; \exists y\; x \doteq y^2)$,

(iii) $\{\forall x_0, ..., x_{2n}\, \exists z\;\; z^{2n+1} + x_{2n} z^{2n} + ... + x_0 \doteq 0 \mid n \in \mathbb{N}\}$.

Indeed, we observe that $\sum_{rcf}$ exactly formalizes the characterization of real closed fields we have seen in Corollary 2.11 - *(ii)* formalizes *(a)* and *(iii)* formalizes *(b)*.

Definition 2.22. *Let* $\mathcal{A}$, $\mathcal{B}$ *be* $\mathcal{L}$*-structures such that* $\mathcal{A} \subseteq \mathcal{B}$. *We will say that* $\mathcal{A}$ *is* existencially closed *in* $\mathcal{B}$ *if every* $\exists - \mathcal{L}_A$*-sentence which is true in* $\mathcal{B}$ *is true in* $\mathcal{A}$. *We write* $\mathcal{A} \overset{\exists}{\subseteq} \mathcal{B}$.

For the definition of model completeness we will use part of a characterization known as *Robinson's Test.*

Definition 2.23. *We say that a theory* $\sum$ *is* model complete *if for every* $\mathcal{A}, \mathcal{B} \in Mod\,(\sum)$ *such that* $\mathcal{A} \subseteq \mathcal{B}$ *we have that* $\mathcal{A} \overset{\exists}{\subseteq} \mathcal{B}$.

Theorem 2.24. $\sum_{rcf}$ *is model-complete.*

3 Hilbert's seventeenth problem for ordered fields

The main aim of this report is to characterize the ordered fields where a definite function in n variables can be written as sum of squares. Let us first give a name to this property.

Definition 3.1. *Let K be an ordered field. We say that K* has the Hilbert Property (HP) *if K satisfies the following:*

$$f \in K(\underline{X}) \text{ definite} \Rightarrow f \in \sum K(\underline{X})^2.$$

Another crucial property of fields we will need is given in the next definition.

Definition 3.2. *Let K be an ordered field with real closure $\overline{K}$. We say K is* definite preserving *if every function $f \in K(\underline{X})$ which is definite on K is definite on $\overline{K}$.*

Now we are ready to prove the first main result by McKenna.

Theorem 3.3 (cf. Theorem 1 in [3])**.** *Let K be an ordered field. Then K has HP if and only if K is rigid and definite preserving.*

Proof. "$\Leftarrow$" Suppose K is rigid and and definite preserving. Let P_K denote the unique ordering of K. Let $f \in K(\underline{X})$ be definite on K. Assume for a contradiction that f is not a sum of squares in $K(\underline{X})$. Then, by Theorem 2.8, there is an ordering P of $K(\underline{X})$ that puts f negative[1], i.e.

$$f <_P 0. \tag{1}$$

Since $P \cap K$ is an ordering on K and P_K is the only ordering on K, we have $P \cap K = P_K$, i.e. P extends P_K to $K(\underline{X})$.
Since $(K(\underline{X}), P)$ is real, it has a real closure, say L. Observing that $K \subseteq K(\underline{X}) \subseteq L$ as ordered fields, we denote the relative algebraic closure of K in L by $\overline{K}$. Note that $\overline{K}$ is real closed and we have $\overline{K} \subseteq L$ as ordered fields.
By (1), $(K(\underline{X}), P)$ models the sentence $\alpha :\equiv \exists\, \underline{x}\;\; f(\underline{x}) < 0$. Since α is a $\mathcal{L}_K$-sentence, in particular it is a $\mathcal{L}_{\overline{K}}$ -sentence. As $K(\underline{X}) \subseteq L$ and existencial sentences lift, we have that L models α. Using that L and $\overline{K}$ are both real closed and that the theory of real closed fields is model complete, we get that $\overline{K}$ models α. This contradicts the fact that K is definite preserving.

"$\Rightarrow$" Let K have HP. Then every positive element of K is expressible as a sum of squares in $K(\underline{X})$ and, therefore, as sum of squares in K by degree arguments. Hence, K is rigid. Further, every definite $f \in K(\underline{X})$ is a sum of squares in $K(\underline{X})$ and so it is definite on $\overline{K}$, i.e. K is definite preserving. □

Our remaining goal is to develop a characterization of the property of being definite preserving. We will now give the necessary definitions to state this characterization.

[1] Note that $K(\underline{X})$ is a real field.

Definition 3.4. *Let K and L be ordered fields such that $K \subseteq L$ as ordered fields. Then we say K is* dense in L *if for every two distinct elements of L there is an element of K lying between them, i.e.*

$$\forall x, y \in L,\ x < y,\ \exists k \in K : \ x < k < y.$$

Definition 3.5. *Let K be an ordered field. We say that K* has the Weak Hilbert Property (WHP) *if K satisfies the following:*

$$f \in K(\underline{X}) \text{ definite} \Rightarrow f = a_1 g_1^2 + ... + a_m g_m^2,$$

for some $a_i \in K^{>0}$ and $g_i \in K(\underline{X})$, $i \in \{1, ..., m\}$. We call a sum of this form WHP-sum.

Definition 3.6. *We define the set S of $\mathcal{L}_{or}$ -sentences as follows:*
For each natural number n, we write:

$$\forall x_0, ..., x_n, x, y, z \, \exists w \, [(x < y \ \wedge x_n x^n + ... + x_0 > 0 \ \wedge \ x_n y^n + ... + x_0 < 0 \ \wedge z \neq 0) \\ \rightarrow (x < w < y \ \wedge \ (x_n w^n + ... + x_0)^2 < z^2)].$$

These sentences express the fact that a polynomial with coefficients from K that changes sign in an interval in K must come arbitrarily close to 0 in that interval in K.

The following is the main theorem in this report.

Theorem 3.7 (cf. Corollary, p.228 in [3])**.** *Let K be an ordered field. Then the following are equivalent.*

(i) K is dense in $\overline{K}$.

(ii) K is definite preserving.

(iii) K has the WHP.

(iv) $K \models S$.

We first prove $(i) \Leftrightarrow (ii)$. To this aim, let us give some more definitions as well as proving some lemmata.

Lemma 3.8. *Let K be an ordered field and $\overline{K}$ its real closure. Then K is cofinal in $\overline{K}$, i.e. for every element $a \in \overline{K}$ there exists an element $k \in K$ such that $a <_{\overline{K}} k$. In particular, $\overline{K}$ contains no infinitesimals with respect to K.*

Proof. Let $r \in \overline{K}$. Since $\overline{K}|K$ is algebraic, r is algebraic over K. W.l.o.g. we take $|r| > 1$. Let $g(X) = X^n + a_{n-1}X^{n-1} + ... + a_0 \in K[X]$ be the minimal polynomial of r over K, i.e. $g(r) = 0$. Then

$$0 = g(r) \ \Rightarrow \ -r^n = a_{n-1}r^{n-1} + ... + a_0 \ \Rightarrow \ |r^n| = |a_{n-1}r^{n-1} + ... + a_0|.$$

By dividing both sides by r^{n-1} and using the triangle inequality, we obtain

$$|r| \leq |a_{n-1}| + ... + |\frac{a_0}{r^{n-1}}| \leq |a_{n-1}| + ... + |a_0| < 1 + |a_{n-1}| + ... + |a_0| \in K.$$

□

Lemma 3.9 (cf. Lemma 1 in [3]). *Let K be an ordered field which is not dense in its real closure $\overline{K}$. Then there is an element $p \in \overline{K}$ such that K is not dense in $K(p)$. Furthermore, there is an element $h \in K^{>0}$ so that the interval $(p-h, p+h) \subseteq \overline{K}$ is disjoint from K.*

Proof. Since K is not dense in $\overline{K}$, there are elements $p, q \in \overline{K}$ such that $(p,q) \cap K = \emptyset$. Since $\overline{K}$ contains no infinitesimals with respect to K, we cand find $h \in K^{>0}$ such that $h < q - p$. Thus, the interval $(p, p+h)$ is disjoint from K. But also $(p-h, p)$ must be disjoint from K. Indeed, if there exists a $k \in K$ such that $k \in (p-h, p)$, then also $k + h \in (p, p+h)$. This gives us that K is not dense in $K(p)$. □

Now we give a precise name to elements of $\overline{K}$ fulfilling a certain density-criterion in K.

Definition 3.10. *Let K be an ordered field. We say that $k \in \overline{K}$ is a* limit point of K *if for every $h \in K^{>0}$ we have that $(k-h, k+h) \cap K \neq \emptyset$.*

Notation 3.11. *We denote the set of all limit points of K by $K!$.*

Remark 3.12.

1. *For all $h \in \overline{K}^{>0}$ and k a limit point of K, we have that $(k-h, k+h) \cap K \neq \emptyset$.*
2. *All limit points of K are algebraic over K.*
3. *The set $K!$ is closed under addition, substraction, multiplication and inverses and, hence, $K!$ is a field. Clearly $K \subseteq K!$.*

Lemma 3.13 (cf. Lemma 2 in [3]). *Let K be an ordered field and let $f \in K[X]$. If $p \in K!$ is a root of f, then the polynomial $\frac{f(X)}{X-p}$ has coefficients in $K!$.*

Proof. We have that $K \subseteq K! \ni p$ and $K!$ is a field. □

The next lemma gives a connection between the "density"-notion we introduced in Definition 3.10 and the continuity with respect to coefficients, i.e. roots of a polynomial are continuous functions of the coefficients.

Lemma 3.14 (cf. Lemma 3 in [3]). *Let $f \in K![X]$ and $p \in \overline{K}$ a root of f. Then there exists a polynomial $m \in K[X]$ such that for $h \in K^{>0}$ we have:*

(i) $deg(f) = deg(m)$ and

(ii) m has a root in the interval $(p-h, p+h)$.

Proof. Let $f(X) = X^n + a_{n-1}X^{n-1} + ... + a_0$, where $a_0, a_1, ..., a_{n-1} \in K!$. It is enough to show the claim for the irreducible polynomials, so let us assume that f is irreducible over $K!$ and, hence, has only simple roots. Thus, f is monotonic in a neighbourhood of all its roots, say for all $x \in K!$ such that $|p_i - x| \leq d$ with $d \in K^{>0}$ sufficiently small and $p_1, ..., p_t$ are all the roots of f in $\overline{K}$. Hence, $f(p_i - d)$ and $f(p_i + d)$ have different signs

for $i = 1, ..., t$.
Choose $e \in K^{>0}$ such that $e < min\{|f(p_i + d)|\}$ and $k \in K^{>0}$ such that $|p_i + d|^j < k$ for $k = 0, ..., n$. Both e and k exist since K is cofinal in $\overline{K}$.
Since every a_j is an element of $K!$ ($a_n := 1$), we can pick $b_j \in K$ so that $|a_j - b_j| < \frac{e}{nk}$ for all $j = 0, .., n$. If $m(X) := X^n + b_{n-1}X^{n-1} + ...b_0$, then

$$|f(p_i \pm d) - m(p_i \pm d)| \leq \sum_{j=0}^{n} |a_j - b_j| \, |p_i \pm d|^j \leq \sum_{j=0}^{n} \frac{e}{nk} \cdot k = e, \text{ for all } i = 1, ...t.$$

Since $f(p_i - d)$ and $f(p_i + d)$ have different signs for $i = 1, ..., t$, $m(p_i - d)$ and $m(p_i - d)$ behave in the same manner. By the Intermediate Value Theorem for real closed fields, this yields that m has a root in $(p_i - d, p_i + d)$ for all $i = 1, ..., t$. Since we have shown this for sufficiently small d, we get the desired conclusion. □

We are ready to prove *(i)* $\Leftrightarrow$ *(ii)*.

Proof of (i) $\Leftrightarrow$ (ii). "$\Leftarrow$". Let K be definite preserving and suppose for a contradiction that K is not dense in $\overline{K}$. Then, by Lemma 3.9, we know there is $p \in \overline{K}$ and $h \in K^{>0}$ such that the interval $(p - h, p + h)$ is disjoint from K. Since p is algebraic over K, we can assume that p has the smallest degree for which the above property occurs.
Let f be the minimal polynomial of p over K.

Claim 1. *Every real root of f is isolated from K, i.e. every real root of f is not a limit point of K.*

Proof of Claim 1. Suppose this claim is not true. Then there exists $p' \in K!$ such that $f(p') = 0$. By Lemma 3.13, $g(X) := \frac{f(X)}{X-p'}$ has coefficients in $K!$. Clearly $p \neq p'$, so $g(p) = 0$. By Lemma 3.14, we can choose a polynomial $m \in K[X]$ so that m and g have the same degree and m has a root, r, that lies in the interval $(p - \frac{h}{2}, p + \frac{h}{2})$, which is disjoint from K. The degree of r is at most the degree of m, which is one less than the degree of f. The degree of f is equal to the degree of p. Since r is isolated from K, we get a contradiction to the choice of p. □

We can assume we have chosen $h \in K^{>0}$ so small that the intervals $(p_1 - h, p_1 + h), ..., (p_t - h, p_t + h)$ are all disjoint from K, where $p_1, ...p_t$ are all the roots of f in $\overline{K}$ in the order in which they occur in $\overline{K}$.
Consider now the element $p_1 + \frac{h}{2} \in \overline{K} \setminus K$. Let $s(X)$ be its minimal polynomial over K. Observe that $s(X + \frac{h}{2})$ has a root at p_1. Since $p_1 + \frac{h}{2}$ is an element of $K(p_1)$, we have that $s(X + \frac{h}{2}) = f(X)$ by definition of minimal polynomials.

Claim 2. *The sign of $s(k)$ is the same as the sign of $f(k)$, for all $k \in K$.*

Proof of Claim 2. We assume both $s(X)$ and $f(X)$ have leading coefficient 1. The sign of $f(k)$ is the same as the sign of $s(k)$ for all $k \in K$ less than p_1. The same holds for all $k \in K$ bigger than $p_t + \frac{h}{2}$.

Since both $s(X)$ and $f(X)$ are irreducible, they have only simple roots. Hence, they must change sign at their roots and only at their roots. The roots of $f(X)$ are $p_1, ...p_t \in \overline{K}$. From our above observation that $s(X + \frac{h}{2}) = f(X)$, we see the roots of $s(X)$ are $p_1 + \frac{h}{2}, ...p_t + \frac{h}{2}$. Thus, $f(X)$ changes sign at each root and stays of constant sign on (p_i, p_{i+1}) for $i \in 1, ..., t-1$. Likewise, $s(X)$ changes sign at each one of its roots and stays of constant sign on $(p_i + \frac{h}{2}, p_{i+1} + \frac{h}{2})$ for $i \in 1, ..., t-1$. But p_i and $p_i + \frac{h}{2}$ are both in the interval $(p_i - h, p_i + h)$, which is disjoint from K. This and the fact that $f(X)$ and $s(X)$ have the same sign on $(-\infty, p_1)$ and $(p_t + \frac{h}{2}, \infty)$ imply that the sign of $f(k)$ is the same as the sign of $s(k)$ for all $k \in K$. □

We now consider the polynomial $F(X) := f(X)s(X)$. $F(k)$ is positive for all $k \in K$, i.e. F is definite on K. It is clear from the choice of f and s that if h is taken small enough then $F(X)$ will only have simple roots. It follows that $F(X)$ changes sign at p_1 and, hence, is somewhere negative on $\overline{K}$. This contradicts the assumption that K is definite preserving. Hence, if K is definite preserving, then K is dense in $\overline{K}$.

"⇒" Suppose K is dense in $\overline{K}$. Let $f \in K(\underline{X})$. Assume that there exists a $\underline{k'} \in \overline{K}^n$ such that $f(\underline{k'}) < 0$. Since f is defined everywhere up to finitely many points in $\overline{K}^n$, we find an open neighbourhood $U \subseteq \overline{K}^n$ of $\underline{k'}$ such that f is defined on U, hence, f is continuous on U. We consider now the set $N := \{\underline{x} \in U | f(\underline{x}) < 0\}$.
As N is non-empty, it is an open neighbourhood in $\overline{K}^n$. The density of K in $\overline{K}$ yields that $K^n \cap N$ is non-empty, i.e. there exists $\underline{k} \in K^n$ such that $f(\underline{k}) < 0$. Hence, K is definite preserving.

□

Before proving *(ii)* ⇔ *(iii)* in Theorem 3.7, we first observe that we have characterized being definite preserving in the previous result as a property of K and $\overline{K}$ together. Now we will characterize it in terms of K alone.

Proof of (ii) ⇔ (iii). "⇐" Let K have the WHP and $f \in K(\underline{X})$ definite. Then we can write $f = a_1 g_1^2 + ... + a_m g_m^2$ where $a_i \in K^{>0}$. Thus, f is definite on $\overline{K}$, i.e. K is definite preserving.

"⇒" Let P be an ordering of K. Suppose K is definite preserving and let $f \in K(\underline{X})$ definite. Assume for a contradiction that f is not a WHP-sum, so in particular $f \notin T_{K(\underline{X})}(P)$. Since $T_{K(\underline{X})}(P)$ is a preordering of $K(\underline{X})$ and that it is proper (otherwise $-1 \in P$), by arguments similar to the ones in lecture 3 in [2], we obtain an ordering Q on $K(\underline{X})$ which extends P and puts f negative. As in the proof of Theorem 3.3, we denote the real closure of $(K(\underline{X}), Q)$ by L and observe that $K \subseteq \overline{K} \subseteq L$ as ordered fields. By the same arguments as in proof of Theorem 3.3, we get a contradiction in the fact that $\overline{K}$ must model the sentence $\exists\, \underline{x}\; f(\underline{x}) < 0$ by model completeness of real closed fields.

□

The last result we are going to prove is that being dense in its real closure is a first order property of ordered fields, i.e. we can express it by a set of first order sentences.

Proof of (i) $\Leftrightarrow$ (iv). "$\Rightarrow$" Suppose K is dense in $\overline{K}$. Let $f \in K[X]$. If there exist $x, y \in K$ such that $f(x) < 0 < f(y)$, then f has a root in the interval (x, y) in $\overline{K}$. By the density of K in $\overline{K}$, this root is a limit point of K. Since f is continuous, it must get arbitrarily small as the root is approached from the right.

"$\Leftarrow$" Suppose K models S. Assume for a contradiction that K is not dense in $\overline{K}$. By Lemma 3.9, we can then choose $p \in \overline{K}$ of minimal degree so that the interval $(p-h, p+h)$ is disjoint from K for some $h \in K^{>0}$. Let f be the minimal polynomial of p over K. Since $\deg(f') < \deg(f)$, f' has no isolated roots. Hence, we can choose $k, m \in K$ so that f' has no roots in (k, m) and so does not change sign in (k, m) and $p \in (k, m)$. Thus, by the Mean Value Theorem, f must be monotonic in (k, m). Assume $f(k)$ positive. Since K models S, f must get arbitrarily small on $(k, m) \cap K$, but, since f is monotonic, the only place this can happen is arbitrarily close to p. This is a contradiction to the fact that $(p - h, p + h)$ is disjoint from K.

$\square$

4 Examples

Let us introduce now an example of a field having HP.

Example 1. *Let K be an Archimedean ordered field, then it can be embedded in $\mathbb{R}$ as an ordered field by Hölder's Theorem (see lecture 2, Theorem 4.2 in [2]). Since we have $\mathbb{Q} \subseteq K$ and $\mathbb{Q}$ dense in $\mathbb{R}$, we have that K is dense in $\mathbb{R}$. In particular, K is dense in its real closure $\overline{K}$. Furthermore, K inherits the unique ordering of $\mathbb{R}$, i.e. it is rigid. Then K has HP due to Theorem 3.3. Note that Artin had already shown that Archimedean fields have HP.*

Now we give an example for a field K which is not dense in its real closure.

Example 2. *Set $K := \mathbb{Q}(X)$, where X is placed larger than all the rationals, i.e. X is infinitely large w.r.t. $\mathbb{Q}$. Then $(\sqrt{X}, 2\sqrt{X}) \cap K = \emptyset$. Hence, by Theorem 3.7, K does not have WHP.*

5 Future work

Let us collect here some directions we intend to develop in future in relation to this topic.

- We want to find an explicit example of a non-Archimedean rigid field, which lies dense and proper in its real closure as well as examples of fields which satisfy one of the two conditions (rigidness and density in the real closure) but not the other.
- Since in [1] an alternative proof for Theorem 3.7 is given, we want to investigate the techniques used there.
- In [5] Prestel showed the following.

Let K be a formally real field which has only a finite number of orderings. Then K has WHP if and only if K is dense in every real closure of K.

Then having the WHP is independent of the (one out of finitely many) ordering we are considering, i.e. it cannot happen that (K, P_1) is dense in its real closure R_1, whereas (K, P_2) does not lie dense in its real closure R_2. We would like study in details this work and [6] to better understand the relation to McKenna's results.

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