Uniform denominators in Hilbert's 17th Problem Theorems by Pólya and Reznick

Robin Schabert

July 16, 2019

Abstract

Hilbert's 17th Problem was solved by Artin, who showed that every real positive semidefinite polynomial can be written as a sum of squares of rational functions. In this handout, we are going to present a related result due to Pólya: For every homogenous, positive definite and even polynomial f there is a $k \in \mathbb{N}$ such that $(\sum X_i^2)^k f$ has only positive coefficients and is therefore a sum of squares of monomials, i.e. we can uniformly choose the denominators in Artin's solution. A similar result was proven by Reznick for polynomials not necessarily even together with a bound for the exponent k. We will prove the two above mentioned theorems and their non-homogenous versions. We will also discuss these results by giving examples in the positive definite case and a counterexample in the positive semidefinite one, and sketching some related open questions.

1 Introduction

In 1888 Hilbert [4] showed the existence of positive semidefinite polynomials which are not sums of squares of polynomials. In 1900 he posed his famous 23 Problems and, in particular, the 17th can be stated as follows:

Is it true that every positive semidefinite polynomial is a sum of squares of rational functions? Or equivalently: If f is a positive semidefinte polynomial, is there a polynomial q such that $q^2 f$ is a sum of squares?

A positive answer to Hilbert's 17th Problem was given in 1927 by Artin [1]. However, his proof is not constructive, so it is natural to look for explicit solutions to Hilbert's 17th problem. In particular, an explicit solution with uniform denominators would be even more desirable, because it would provide a uniform certificate for the positive semidefiniteness of a polynomial. The aim of this handout is to present the following results due to Pólya [9] and Reznick [13], which exactly deal with this problem. In 1928, shortly after Artin's solution to Hilbert's 17th Problem, Pólya found an explicit solution to Hilbert's 17th Problem in one special case:

If f is a positive definite and even form, then $(\sum X_i^2)^k f$ has positive coefficients and is therefore a sum of squares of monomials for $k \in \mathbb{N}_0$ large enough.

In 1939 Habicht [3] used Pólya's Theorem to construct explicit solutions to Hilbert's 17th Problem for any positive definite form, but without getting uniform denominators. In 1993 Reznick could extend Pólya's result to any positive definite form:

If f is a positive definite form, then $(\sum X_i^2)^k f$ is a sum of squares for $k \in \mathbb{N}_0$ large enough.

Let us briefly describe the structure of this handout. In Section 2 we will introduce some preliminary notions and results needed to prove the theorems by Pólya and Reznick mentioned above. In particular, we will define (Archimedean) preprimes and quadratic modules and state the Representation Theorem. We will also recall the definition of homogenization and dehomogenization of a polynomial as well as some basic related properties, which we will need to conclude non-homogenous versions of the theorems of Pólya and Reznick. Most of the material in this section is taken from [8]. In Section 3 we will prove the theorems of Pólya and Reznick and their non-homogenous versions. Here we will also state bounds for the exponents due to Powers and Reznick (see [13] and [10]). In Section 4 we will provide examples of positive semidefinite polynomials for which Reznick's Theorem still works. However, Reznick's Theorem does not hold for all positive semidefinite forms because of the existence of so called *bad points* for some polynomials. We will present a counterexample due to Delzell [2]. These examples will motivate some concluding open questions.

2 Preliminaries

In the following, let $n \in \mathbb{N}$ and denote by $\mathbb{R}[X_0, \dots, X_n]$ the algebra of polynomials in $n+1$ variables. We will always assume that A is a commutative ring with 1 and $\mathbb{Q} \subseteq A$ (e.g. $A = \mathbb{R}[X_0, \dots, X_n]$). For any subset S of $\mathbb{R}[X_0, \dots, X_n]$, we set

$$K_S := \{x \in \mathbb{R}^{n+1} \mid \forall f \in S : f(x) \geq 0\}.$$

Definition 2.1. Let $T \subseteq A$. T is called a *preprime* of A if

$$T + T \subseteq T,\ TT \subseteq T \text{ and } \mathbb{Q}_{\geq 0} \subseteq T.$$

A preprime T of A is called

(i) *Archimedean* if $\forall a \in A\ \exists N \in \mathbb{N}_0 : N + a \in T$.

(ii) *quasi-preordering* if $\forall a \in A\ \exists r \in \mathbb{Q}_{>0} : (r+a)^2 \in T$.

(iii) *preorder* if $A^2 \subseteq T$.

Remark 2.2. Any Archimedean preprime is a quasi-preordering.

Proof. Suppose T is an Archimedean preprime and let $a \in A$. Because T is Archmidean, there are $N_1, N_2 \in \mathbb{N}_0$ s.t. $a + N_1, a^2 + N_2 \in T$. Set $N := \max\{N_1, N_2\}$ and $r := N + 1 \in \mathbb{Q}_{>0}$. Then

$$(r+a)^2 = (N+1+a)^2 = (N+1)^2 + (N+1)a + a^2 = (N+1)(N+1+a) + a^2 = (\underbrace{N+1}_{\in T})(\underbrace{N+a}_{\in T}) + \underbrace{N+1+a^2}_{\in T} \in T$$

□

Definition 2.3. Let $T \subseteq A$ be a preprime. Then

$$H_T := \{a \in A \mid \exists N \in \mathbb{N}_0 : N \pm a \in T\}$$

is called *the ring of bounded elements of A with respect to T*.

Proposition 2.4. *Let $T \subseteq A$ be a preprime. Then*

(i) H_T is a subring of A.

(ii) T is Archimedean iff $H_T = A$.

Proof. See Proposition 5.1.3 in [8]. □

>

Definition 2.5. Let $S \subseteq A$. The *preprime generated* by S is defined as

$$T(S) := \left\{ \sum_{e=(e_1,\dots,e_n)\in\mathbb{N}_0^n} q_e s_1^{e_1} \cdots s_n^{e_n} \mid n \in \mathbb{N}, s_1, \dots, s_n \in S, q_e \in \mathbb{Q}_{\geq 0} \right\}.$$

Note that $T(a_1, \dots, a_n)$ is the smallest preprime of A containing $a_1, \dots, a_n$.

Definition 2.6. Let T be a preprime of A.

(i) A *T-module* of A is a subset M of A such that $M + M \subseteq M$, $TM \subseteq M$ and $1 \in M$.

(ii) A T-module M of A is said to be *Archimedean* if for each $a \in A$ there exists $N \in \mathbb{N}$ such that $N + a \in M$.

(iii) A quadratic module M of A is a $\sum A^2$-module.

Proposition 2.7. *Let M be a quadratic module of $\mathbb{R}[X_0, \dots, X_n]$. Then M is Archimedean iff there is $N \in \mathbb{N}_0$ s.t. $N - \sum_{i=0}^n X_i^2 \in M$.*

Proof. See Corollary 5.2.4 in [8]. $\square$

Theorem 2.8 (Representation Theorem)**.** *Suppose that T a quasi-preordering of $\mathbb{R}[X_0,\dots,X_n]$ and M is an Archimedean T-module of $\mathbb{R}[X_0,\dots,X_n]$. Then, for any $f \in \mathbb{R}[X_0,\dots,X_n]$, the following holds:*

$$f > 0 \text{ on } K_M \Rightarrow f \in M$$

Proof. See Theorem 5.4.4 in [8]. $\square$

Definition 2.9.

(i) $f \in \mathbb{R}[X_0,\dots,X_n]$ is called a *form* or *homogeneous*, if f is a an $\mathbb{R}$-linear combination of monomials of the same degree. Moreover, f is called a k-form ($k \in \mathbb{N}_0$) if f is an $\mathbb{R}$-linear combination of monomials of degree k.

(ii) If $f \in \mathbb{R}[X_1,\dots,X_n] \setminus \{0\}$ and $d := \deg f$, then the d-th homogeneous part of f is called the *leading form $LF(f)$ of f*. We set $LF(0) := 0$.

(iii) If $f \in \mathbb{R}[X_1,\dots,X_n] \setminus \{0\}$, $d := \deg f$, and $f = \sum_{k=0}^{d} f_k$ with f_k k-form for all $k \in \{0,\dots,d\}$, then the *homogenization* $f^* \in \mathbb{R}[X_0,\dots,X_n]$ of f (with respect to X_0) is given by

$$f^* := \sum_{k=0}^{d} X_0^{d-k} f_k = X_0^d f\left(\frac{X_1}{X_0},\dots,\frac{X_n}{X_0}\right) \quad \text{and} \quad 0^* = 0.$$

(iv) For homogeneous $f \in \mathbb{R}[X_0,\dots,X_n]$, we call $\widetilde{f} := f(1, X_1,\dots,X_n)$ the *dehomogenization* of f (with respect to X_0).

Definition 2.10. Let $f \in \mathbb{R}[X_1,\dots,X_n]$. If $f \geq 0$ on $\mathbb{R}^n$, f is called *positive semidefinite.* If $f > 0$ on $\mathbb{R}^n$ or f is homogenous and $f > 0$ on $\mathbb{R}^n \setminus \{0\}$, f is called *positive definite.*

Proposition 2.11.

(i) $LF(f) = f^*(0, X_1,\dots,X_n)$ for all $f \in \mathbb{R}[X_1,\dots,X_n]$.

(ii) $\widetilde{f^*} = f$ for all $f \in \mathbb{R}[X_1,\dots,X_n]$.

(iii) Let $f \in \mathbb{R}[X_1,\dots,X_n]$. f^* is positive semidefinite iff f is positive semidefinite.

(iv) Let $f \in \mathbb{R}[X_1,\dots,X_n]$. f^* is positive definite iff f and $LF(f)$ are positive definite.

Proof.

(i) Let $f \in \mathbb{R}[X_1,\dots,X_n]$ of degree $d \in \mathbb{N}_0$ and $f = \sum_{k=0}^{d} f_k$ with f_k k-form for all $k \in \{0,\dots,d\}$. Then

$$LF(f) = f_d = f_d + \sum_{k=0}^{d-1} 0^{d-k} f_k = f^*(0, X_1,\dots,X_n).$$

(ii) Let $f \in \mathbb{R}[X_1,\dots,X_n]$. Then

$$f^* = X_0^d f\left(\frac{X_1}{X_0},\dots,\frac{X_n}{X_0}\right) \text{ and, therefore, } \widetilde{f^*} = 1^d f\left(\frac{X_1}{1},\dots,\frac{X_n}{1}\right) = f$$

(iii) See Proposition 2.2.6 in [14].

(iv) "$\Rightarrow$": If $f^* > 0$ on $\mathbb{R}^{n+1} \setminus \{0\}$, then also on $\{1\} \times \mathbb{R}^n$ and on $\{0\} \times \mathbb{R}^n \setminus \{0\}$, i.e. f and $LF(f)$ are positive definite.
"$\Leftarrow$": Let f and $LF(f)$ be positive definite. Clearly, we can write $\deg f = 2d$ with $d \in \mathbb{N}_0$ and let $x = (x_0,\dots,x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. If $x_0 = 0$, then

$$f^*(x) = f^*(0, x_1,\dots,x_n) \overbrace{=}^{2.11(i)} = LF(f)(x_1,\dots,x_n) > 0.$$

If $x_0 \neq 0$, then

$$f^*(x) = \underbrace{x_0^{2d}}_{>0} f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) > 0.$$

Hence, f is positive definite.

$\square$

3 Theorems of Pólya and Reznick

We have now introduced everything needed to prove the theorems by Pólya and Reznick. For the proofs we follow [8]. We will also show the non-homogenous versions of these theorems and state bounds for the exponent k.

Theorem 3.1 (Pólya 1928)**.** *Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be homogenous of degree $d \in \mathbb{N}$. If*

$$f > 0 \text{ on } K := \{x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \forall i \in \{0, \dots, n\} : x_i \geq 0,\ x \neq 0\},$$

then there is $k \in \mathbb{N}_0$ s.t. $\left(\sum_{i=0}^n X_i\right)^k f$ has positive coefficients.

Proof. Let $f > 0$ on K. Consider the set T of polynomials in $\mathbb{R}[X_0, \dots, X_n]$ with positive coefficients. We want to show $\left(\sum_{i=0}^n X_i\right)^k f \in T$ for some $k \in \mathbb{N}$. It is easy to see that T is the preprime generated by $X_0, \dots, X_n$ and $\mathbb{R}_{\geq 0}$.
Consider now the ideal I generated by $1 - \sum_{i=0}^n X_i$ and the preprime $T' = T + I$. Then

$$K_{T'} = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^n x_i = 1,\ \forall i \in \{0, \dots, n\} : x_i \geq 0 \right\} \subseteq K,$$

so $f > 0$ on $K_{T'}$. Since for any $j \in \{0, \dots, n\}$, we have $X_j \in T'$ and $1 - X_j = \sum_{\substack{i=0 \\ j \neq i}}^n X_i + 1 - \sum_{i=0}^n X_i \in T'$, we get $X_0, \dots, X_n \in H_{T'}$. Since $\mathbb{R}_{\geq 0} \subseteq T'$, we have $\mathbb{R} \subseteq H_{T'}$. By Proposition 2.4, $H_{T'}$ is a subring of $\mathbb{R}[X_0, \dots, X_n]$ and hence $H_{T'} = \mathbb{R}[X_0, \dots, X_n]$. Then T' is Archimedean and, therefore, a quasi-preordering by Remark 2.2. We can now use the Representation Theorem 2.8 and get $f \in T'$, i.e.

$$f = g + h\left(1 - \sum_{i=0}^n X_i\right)$$

for some $g \in T$, $h \in \mathbb{R}[X_0, \dots, X_n]$. Substituting $\frac{X_j}{\sum_{i=0}^n X_i}$ for X_j $(j \in \{0, \dots, n\})$, we get

$$\begin{aligned}
\frac{1}{\left(\sum_{i=0}^n X_i\right)^d} f &= f\left(\frac{X_0}{\sum_{i=0}^n X_i}, \dots, \frac{X_n}{\sum_{i=0}^n X_i}\right) \\
&= g\left(\frac{X_0}{\sum_{i=0}^n X_i}, \dots, \frac{X_n}{\sum_{i=0}^n X_i}\right) + h\left(\frac{X_0}{\sum_{i=0}^n X_i}, \dots, \frac{X_n}{\sum_{i=0}^n X_i}\right)\left(1 - \underbrace{\sum_{j=0}^n \frac{X_j}{\sum_{i=0}^n X_i}}_{=1}\right) \\
&= g\left(\frac{X_0}{\sum_{i=0}^n X_i}, \dots, \frac{X_n}{\sum_{i=0}^n X_i}\right)
\end{aligned}$$

By multiplying by $\left(\sum_{i=0}^n X_i\right)^k$ for $k \in \mathbb{N}_0$ large enough to clear denominators, we get $\left(\sum_{i=0}^n X_i\right)^{k-d} f \in T$, so $\left(\sum_{i=0}^n X_i\right)^{k-d} f$ has positive coefficients by the definition of T. $\square$

Remark 3.2. For $f \in \mathbb{R}[X_0, \dots, X_n]$ homogenous of degree d, say $f = \sum_{|\alpha|=d} a_\alpha X^\alpha$, Powers und Reznick proved in [10] a bound for k in Theorem 3.1, i.e. Pólya's Theorem holds for

$$k > \frac{d(d-1)L}{2\lambda} - d,$$

where $L = L(f) := \max\{\frac{|a_\alpha|}{|c_\alpha|} \mid |\alpha| = d\}$, $c_\alpha = \binom{d}{\alpha}$ and $\lambda = \min\{f(x) \mid x \in \mathbb{R}^{n+1}, |x| = 1\}$.

We will now proof the version of Pólya's Theorem 3.1, that was stated in the introduction. To this end we need the notion of even polynomials.

Definition 3.3. Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be homogenous. f is called *even*, if every monomial of f has even degree in each variable or equivalently if $f(X_0, \dots, X_{i-1}, -X_i, X_{i+1}, \dots, X_n) = f$ for all $i \in \{0, \dots, n\}$.

Corollary 3.4. *Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be homogenous. If f is positive definite and even, then there is $k \in \mathbb{N}_0$ s.t.*

$$\left(\sum_{i=0}^{n} X_i^2\right)^k f$$

has positive coefficients and is therefore a sum of squares of monomials.

Proof. Let f be positive definite and even. As f is even, we have $\hat{f} := f(\sqrt{X_0}, \dots, \sqrt{X_n}) \in \mathbb{R}[X_0, \dots, X_n]$. Due to the positive definiteness of f and the fact that $\mathbb{R}_{\geq 0} = \mathbb{R}^2$, we have that

$$\hat{f} > 0 \text{ on } K := \{x \in \mathbb{R}^n \mid \forall i \in \{0, \dots, n\} : x_i \geq 0,\ x \neq 0\}.$$

By Pólya's Theorem 3.1 there is $k \in \mathbb{N}_0$ s.t. $\left(\sum_{i=0}^n X_i\right)^k \hat{f}$ has positive coefficients and so $\left(\sum_{i=0}^n X_i^2\right)^k f$ has positive coefficients. Since $\left(\sum_{i=0}^n X_i^2\right)^k f$ is even and $\mathbb{R}_{\geq 0} = \mathbb{R}^2$, it is a sum of squares of monomials. □

Remark 3.5. We can also conclude Pólya's Theorem 3.1 from Corollary 3.4: Let f be as in Theorem 3.1 and consider the homogenous, even and positive definite polynomial $\hat{f} := f(X_0^2, \dots, X_n^2) \in \mathbb{R}[X_0, \dots, X_n]$. By Corollary 3.4, there is $k \in \mathbb{N}_0$ s.t. $\left(\sum_{i=0}^n X_i^2\right)^k \hat{f}$ has positive coefficients and so $\left(\sum_{i=0}^n X_i\right)^k f$ has positive coefficients.

We will now proof a non-homogenous version of Pólya's Theorem 3.1.

Corollary 3.6. *Let $f \in \mathbb{R}[X_1, \dots, X_n]$. If*

$$f > 0,\ LF(f) > 0 \text{ on } K := \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\} : x_i \geq 0,\ x \neq 0\},$$

then there is $k \in \mathbb{N}_0$ s.t.

$$\left(1 + \sum_{i=1}^{n} X_i\right)^k f$$

has positive coefficients.

Proof. Let $f > 0$, $LF(f) > 0$ on K. Set $d := \deg f$, $K^* := \{x \in \mathbb{R}^{n+1} \mid \forall i \in \{0, \dots, n\} : x_i \geq 0,\ x \neq 0\}$ and consider the homogenization $f^* = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$ of f. As $f > 0$ and $LF(f) > 0$ on K, we get $f^* > 0$ on K^* by proposition 2.11 (iv). By Pólya's Theorem 3.1, $g := \left(\sum_{i=0}^n X_i\right)^k f^*$ has positive coefficients for some $k \in \mathbb{N}_0$. Then the dehomogenization $\widetilde{g}$ of g is

$$\widetilde{g} = g(1, X_1, \dots, X_n) = \left(1 + \sum_{i=1}^{n} X_i\right)^k 1^d f\left(\frac{X_1}{1}, \dots, \frac{X_n}{1}\right) = \left(1 + \sum_{i=1}^{n} X_i\right)^k f$$

and has positive coefficients. □

Theorem 3.7 (Reznick 1995)**.** *Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be homogenous. If f is positive definite, then there is $k \in \mathbb{N}$ s.t.*

$$\left(\sum_{i=0}^{n} X_i^2\right)^k f$$

is a sum of squares.

Proof. Let f be positive definite and consider $T = \sum \mathbb{R}[X_0, \ldots, X_n]^2 + I$, where I is the ideal generated by $1 - \sum_{i=0}^n X_i^2$. Then

$$K_T = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\},$$

where $||\cdot||$ is the euclidean norm on $\mathbb{R}^{n+1}$. Since $1 - \sum_{i=0}^n X_i^2 \in T$, T is Archimedean by Proposition 2.7. Moreover, f is positive definite, so $f \in T$ by the Representation Theorem, i.e.

$$f = \sum_{j=1}^t g_j^2 + h\left(1 - \sum_{i=0}^n X_i^2\right)$$

for some $t \in \mathbb{N}$, $g_1 \ldots, g_t, h \in \mathbb{R}[X_0, \ldots, X_n]$. Substituting $\frac{X_i}{||X||}$ for X_i $(i \in \{0, \ldots, n\})$, where $||X|| := \sqrt{\sum_{i=0}^n X_i^2}$, we get

$$\begin{aligned}
\frac{1}{(\sum_{i=0}^n X_i^2)^{\frac{d}{2}}} f = \frac{1}{||X||^d} f = f\left(\frac{X_0}{||X||}, \ldots, \frac{X_n}{||X||}\right) \\
&= \sum_{j=1}^t g_j\left(\frac{X_0}{||X||}, \ldots, \frac{X_n}{||X||}\right)^2 + h\left(\frac{X_0}{||X||}, \ldots, \frac{X_n}{||X||}\right)\left(\underbrace{1 - \sum_{j=0}^n \frac{X_j^2}{||X||^2}}_{=1}\right) \\
&= \sum_{j=1}^t g_j\left(\frac{X_0}{||X||}, \ldots, \frac{X_n}{||X||}\right)^2
\end{aligned}$$

Since $||X||^2 \in \mathbb{R}[X_0, \ldots, X_n]$ we can write any $a \in \mathbb{R}[X_0, \ldots, X_n][||X||]$ uniquely as $a = b + c||X||$ for some $b, c \in \mathbb{R}[X_0, \ldots, X_n]$. So by multiplying by $||X||^{2N}$ for $N \in \mathbb{N}$ large enough to clear the denominators, we get

$$\left(\sum_{i=0}^n X_i^2\right)^{N - \frac{d}{2}} f = \sum_{j=1}^t (p_j + q_j||X||)^2$$

for some $p_1, \ldots, p_t, q_1, \ldots, q_t \in \mathbb{R}[X_0, \ldots, X_n]$. By expanding and comparing coefficients, we get

$$f = \sum_{j=1}^t p_j^2 + q_j^2||X||^2 \in \sum \mathbb{R}[X_0, \ldots, X_n]^2 \quad \left(\text{and} \sum_{j=1}^t 2p_jq_j = 0\right)$$

as required. □

Remark 3.8. In [13] Reznick proves even more. Indeed, he shows that if $L \subseteq \mathbb{R}$ is an ordered field, $f \in L[X_0, \ldots, X_n]$ is homogenous of degree $d \in \mathbb{N}_0$, positive definite and $k \geq \frac{nd(d-1)}{(4\log 2)\epsilon(f))} - \frac{n+d}{2}$, then $\left(\sum_{i=0}^n X_i^2\right)^k f$ is a non-negative $L-$linear combination of $(d+2k)$-th powers of linear forms in $\mathbb{Q}[X_0, \ldots, X_n]$. Here

$$\epsilon(f) := \frac{\inf\{f(x) \mid x \in \mathbb{R}^{n+1},\ |x| = 1\}}{\sup\{f(x) \mid x \in \mathbb{R}^{n+1},\ |x| = 1\}}$$

We will now proof a non-homogenous version of Reznick's Theorem 3.7.

Corollary 3.9. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be positive definite with positive definite leading form. Then there is $k \in \mathbb{N}$ s.t.*

$$\left(1 + \sum_{i=1}^n X_i^2\right)^k f$$

is a sum of squares.

Proof. Set $d := \deg f$. Since f and $LF(f)$ are both positive definite, the homogenization $f^* := X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$ of f is also positive definite by Proposition 2.11 (iv). By Reznick's Theorem 3.7, $g := \left(\sum_{i=0}^n X_i^2\right)^k f^*$ is a sum of squares. Then the dehomogenization

$$\widetilde{g} = g(1, X_1, \ldots, X_n) = \left(1 + \sum_{i=1}^n X_i^2\right)^k 1^d f\left(\frac{X_1}{1}, \ldots, \frac{X_n}{1}\right) = \left(1 + \sum_{i=1}^n X_i^2\right)^k f$$

is also a sum of squares. □

4 Examples and open questions

We will now consider two famous examples by Motzkin and Lax of polynomials which are positive semidefinite, but not sums of squares. We will notice that for these polynomials Reznick's Theorem still holds although the polynomials are not positive definite. A counterexample by Delzell shows however that Reznick's Theorem does not hold for any positive semidefinite form.

Example 4.1 (Motzkin Polynomial). Consider the Motzkin Polynomial

$$M := X^4Y^2 + X^2Y^4 - 3X^2Y^2Z^2 + Z^6.$$

It is well known that M is positive semidefinte and not a sum of squares, but

$$(X^2+Z^2)M = (Z^44 - X^42Y^2)2 + (XYZ^2 - X^3Y)^2 + (XZ^3 - XY^2Z)^2 \in \sum \mathbb{R}[X,Y,Z]^2.$$

and also $(X^2+Y^2+Z^2)M$ is a sum of squares.
Since M is even, we can add ϵZ^6 to M for a small $\epsilon > 0$ to get an even positive definite form. Consider the following positive definite modifications of the Motzkin polynomial

$$g := M + 9 \quad \text{and} \quad h := f(\sqrt{X}, \sqrt{Y}, \sqrt{Z}) + 9$$

As g is even and positive definite, we can even use Pólya's Theorem 3.1 or Corollary 3.4. Indeed $(X^2+Y^2+Z^2)^7g$ and $(X^2+Y^2+Z^2)^7h$ have positive coefficients and g is therefore a sum of squares of monomials.

Example 4.2 (Lax-Lax Polynomial). Consider the Lax-Lax Polynomial

$$L = X_1X_2X_3X_4 + \sum_{i=1}^{4}(-X_i)\prod_{\substack{j=1\\ j\neq i}}^{4}(X_j - X_i) \in \mathbb{R}[X_1,X_2,X_3,X_4]$$

It is possible to show that L is positive semidefinite and not a sum of squares, but $(X_1^2+X_2^2+X_3^2+X_4^2)L$ is a sum of squares. Indeed,

$$(X_1^2+X_2^2+X_3^2+X_4^2)L = f_1^2 + \frac{4}{7}f_2^2 + \frac{7}{6}f_3^2 + \frac{8}{11}f_4^2 + \frac{11}{9}f_5^2 + f_6^2 + 4f_7^2,$$

where

$$\begin{aligned}
f_1 =& -\frac{1}{2}X_1^2X_4 + X_1X_2X_4 + X_1X_3X_4 - \frac{1}{2}X_1X_4^2 - \frac{1}{2}X_2^2X_4 + X_2X_3X_4 - \frac{1}{2}X_2X_4^2 - \frac{1}{2}X_3^2X_4 - \frac{1}{2}X_3X_4^2 + X_4^3\\
f_2 =& -\frac{1}{2}X_1^2X_3 + \frac{3}{4}X_1^2X_4 - \frac{1}{2}X_1X_2X_3 + X_1X_3^2 - \frac{1}{2}X_1X_3X_4 - \frac{3}{4}X_1X_4^2 - \frac{1}{2}X_2^2X_3 + \frac{3}{4}X_2^2X_4 + X_2X_3^2\\
&- \frac{1}{2}X_2X_3X_4 - \frac{3}{4}X_2X_4^2 - \frac{1}{2}X_3^3 - \frac{5}{4}X_3^2X_4 + \frac{7}{4}X_3X_4^2\\
f_3 =& \frac{9}{14}X_1^2X_3 - \frac{3}{14}X_1^2X_4 - \frac{6}{7}X_1X_2X_3 + \frac{3}{14}X_1X_3^2 - \frac{6}{7}X_1X_3X_4 + \frac{3}{14}X_1X_4^2 + \frac{9}{14}X_2^2X_3 - \frac{3}{14}X_2^2X_4 + \frac{3}{14}X_2X_3^2\\
&- \frac{6}{7}X_2X_3X_4 + \frac{3}{14}X_2X_4^2 - \frac{6}{7}X_3^3 + \frac{6}{7}X_3^2X_4\\
f4 =& -\frac{1}{2}X_1^2X_2 - \frac{3}{8}X_1^2X_3 + \frac{9}{8}X_1^2X_4 + X_1X_2^2 - \frac{1}{2}X_1X_2X_3 - \frac{1}{2}X_1X_2X_4 + \frac{3}{8}X_1X_3^2 - \frac{9}{8}X_1X_4^2 - \frac{1}{2}X_2^3 + \frac{5}{8}X_2^2X_3\\
&- \frac{7}{8}X_2^2X_4 - \frac{1}{8}X_2X_3^2 - \frac{1}{2}X_2X_3X_4 + \frac{11}{8}X_2X_4^2\\
f_5 =& -\frac{15}{22}X_1^2X_2 - \frac{3}{22}X_1^2X_3 + \frac{9}{22}X_1^2X_4 - \frac{3}{22}X_1X_2^2 + \frac{9}{11}X_1X_2X_3 + \frac{9}{11}X_1X_2X_4 + \frac{3}{22}X_1X_3^2 - \frac{9}{22}X_1X_4^2\\
&+ \frac{9}{11}X_2^3 - \frac{9}{11}X_2^2X_3 - \frac{9}{11}X_2^2X_4 - \frac{6}{11}X_2X_3^2 + \frac{9}{11}X_2X_3X_4\\
f_6 =& -X_1^2X_2 + X_1^2X_3 + X_1X_2^2 - X_1X_3^2 - X_2^2X_3 + X_2X_3^2\\
f_7 =& -\frac{1}{2}X_1^3 + \frac{1}{4}X_1^2X_2 + \frac{1}{4}X_1^2X_3 + \frac{1}{4}X_1^2X_4 + \frac{1}{4}X_1X_2^2 - \frac{1}{2}X_1X_2X_3 - \frac{1}{2}X_1X_2X_4 + \frac{1}{4}X_1X_3^2 - \frac{1}{2}X_1X_3X_4\\
&+ \frac{1}{4}X_1X_4^2.
\end{aligned}$$

Let us now introduce an example of a positive semidefinite form for which Reznick's Theorem does not hold.

Definition 4.3. Let $f \in \mathbb{R}[X_0, \ldots, X_n]$ be positive semidefinite. A point $x \in \mathbb{R}^{n+1}$ is called *bad point* for f if for every polynomial $q \in \mathbb{R}[X_0, \ldots, X_n]$ with $q(x) \neq 0$, we have that qf is not a sum of squares, i.e. every common denominator in a solution of Hilbert's 17th problem has to be zero at x.

Example 4.4 (Dellzell). The polynomial

$$D = X_1^4 X_2^2 X_4^2 + X_2^4 X_3^2 X_4^2 + X_1^2 X_3^4 X_4^2 - 3X_1^2 X_2^2 X_3^2 X_4^2 + X_3^8 \in \mathbb{R}[X_1, X_2, X_3, X_4]$$

is positive semidefinite but has a bad point at $(0, 0, 0, 1)$. Therefore,

$$\left(\sum_{i=0}^{n} X_i^2 \right)^k D$$

is not a sum of squares for any $k \in \mathbb{N}$. Hence, Reznick's Theorem does not hold in general for positive semidefinite forms.
However, $(X_1^2 + X_2^2 + X_3^2)D$ is a sum of squares. Indeed,

$$(X_1^2 + X_2^2 + X_3^2)D = \frac{1}{4}f_1^2 + \frac{1}{4}f_2^2 + \frac{1}{4}f_3^2 + \frac{1}{4}f_4^2 + \frac{1}{4}f_5^2 + \frac{1}{4}f_6^2 + \frac{1}{4}f_7^2 + \frac{1}{3}f_8^2,$$

where

$$\begin{aligned}
f_1 &= 2X_3^5 \\
f_2 &= 2X_2 X_3^4 \\
f_3 &= -X_1^2 X_2^2 X_4 - X_1^2 X_3^2 X_4 + 2X_2^2 X_3^2 X_4 \\
f_4 &= -2X_1^2 X_2 X_3 X_4 + 2X_2^3 X_3 X_4 \\
f_5 &= -2X_1 X_2^2 X_3 X_4 + 2X_1 X_3^3 X_4 \\
f_6 &= 2X_1 X_3^4 \\
f_7 &= -2X_1^3 X_2 X_4 + 2X_1 X_2 X_3^2 X_4 \\
f_8 &= -\frac{3}{2} X_1^2 X_2^2 X_4 + \frac{3}{2} X_1^2 X_3^2 X_4.
\end{aligned}$$

These examples lead us to the following questions:

(i) It is well known that for any positive semidefinite form $f \in \mathbb{R}[X_0, \ldots, X_n]$, there exists $p_1, \ldots, p_t \in \mathbb{R}[X_0, \ldots, X_n]$ and $q \in \mathbb{R}[X_1, \ldots, X_n]$ s.t. $f = \sum_{i=0}^{t} \left(\frac{p_i}{q}\right)^2$. Does this still hold in Reznick's Theorem, i.e. is there for $f \in \mathbb{R}[X_0, \ldots, X_n]$ homogenous and positive definite a $k \in \mathbb{N}_0$ s.t.

$$\left(X_1^2 + \cdots + X_n^2\right)^k f$$

is a sum of squares?

(ii) Are there similar results to the one of Pólya for different K? For example, one should be able to derive from the non-homogenous version of Pólya's Theorem a similar result for $K = \mathbb{R}_{>0}^s \times (\mathbb{R} \setminus \{0\})^t$.

(iii) Let S_n denote the symmetric group. There are positive semidefinite symmetric polynomials $f_1, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_n]^{S_n}$, i.e. positive semidefinite polynomials that are invariant under permutation of variables, s.t. for any positive semidefinite symmetric polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]^{S_n}$

$$f = \sum_{e=(e_1,\ldots,e_m)\in\{0,1\}^m} p_e f_1^{e_1}, \ldots, f_m^{e_m}$$

for some $p_e \in \mathbb{R}(X_1, \ldots, X_n)^{S_n}$. This result holds for arbitrary finite groups - or even compact lie groups - instead of S_n (see [11] for more details). For the trivial group the statement is just Hilbert's 17th Problem.
Are there similar results to the one of Pólya or Reznick for positive definite symmetric polynomials?

(iv) Is there a uniform choice in Hilbert's 17th Problem for positive semidefinite forms with respect to bad points?

References

[1] Emil Artin. "Über die zerlegung definiter funktionen in quadrate". In: *Abhandlungen aus dem mathematischen Seminar der Universität Hamburg*. Vol. 5. 1. Springer. 1927, pp. 100–115.

[2] Charles Neal Delzell. "A constructive, continuous solution to Hilberts 17 th problem, and other results in semi-algebraic geometry". PhD thesis. 1980.

[3] Walter Habicht. "Über die Zerlegung strikte definiter Formen in Quadrate". In: *Commentarii Mathematici Helvetici* 12.1 (1939), pp. 317–322.

[4] David Hilbert. "Über die darstellung definiter formen als summe von formenquadraten". In: *Mathematische Annalen* 32.3 (1888), pp. 342–350.

[5] Erich Kaltofen et al. "Exact certification of global optimality of approximate factorizations via rationalizing sums-of-squares with floating point scalars". In: *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*. ACM. 2008, pp. 155–164.

[6] Anneli Lax and Peter D Lax. "On sums of squares". In: *Selected Papers Volume II* (2005), pp. 570–574.

[7] Murray Marshall. "A general representation theorem for partially ordered commutative rings". In: *Mathematische Zeitschrift* 242.2 (2002), pp. 217–225.

[8] Murray Marshall. *Positive polynomials and sums of squares*. American Mathematical Soc., 2008.

[9] George Pólya. "Über positive darstellung von polynomen". In: *Vierteljschr. Naturforsch. Ges. Zürich* 73 (1928), pp. 141–145.

[10] Victoria Powers and Bruce Reznick. "A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra". In: *Journal of pure and applied algebra* 164.1-2 (2001), pp. 221–229.

[11] Claudio Procesi and Gerald Schwarz. "Inequalities defining orbit spaces". In: *Inventiones mathematicae* 81.3 (1985), pp. 539–554.

[12] Bruce Reznick. "Some concrete aspects of Hilbert's 17th problem". In: *Contemporary mathematics* 253 (2000), pp. 251–272.

[13] Bruce Reznick. "Uniform denominators in Hilbert's seventeenth problem". In: *Mathematische Zeitschrift* 220.1 (1995), pp. 75–97.

[14] Markus Schweighofer. *Real Algebraic Geometry I*. WS 2016/17. URL: `http://www.math.uni-konstanz.de/~schweigh/17/real-alg-geo-16-17.pdf`.