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MASTER SEMINAR ON REAL ALGEBRAIC GEOMETRY–SS15

List of possible topics

1. **Cylindrical decomposition of semialgebraic sets**
This seminar will discuss how a semialgebraic subset of R^n (where R is a real closed field) can be decomposed as a disjoint union of finitely many semialgebraic sets such that each of them is semialgebraically homeomorphic to an open hypercube of R^d.
(See [5, Lecture 16, 17], [1, Section 2.3])

2. **Topological definition of dimension of semialgebraic sets**
Using the cylindrical decomposition it is possible to give a topological definition of dimension of a semialgebriac set K. In this seminar, this notion will be introduced, discussed and compared with the dimension of K as an algebraic set.
(See [5, Lecture 22], [1, Section 2.8])

3. **Moment Problem: Proof of Haviland's Theorem**
Haviland's Theorem represents one of the first general result about the multivariate moment problem. Several are the proofs of this result available in literature. This seminar will present a proof of Haviland's Theorem based on Riesz' representation theorem.
(See [6, Lectures 16, 17, 18], [7, Section 3.2], [2], [3])

4. **Moment Problem: Schmüdgen Theorem (1991)**:
This seminar will present, in various equivalent formulations, the celebrated result of 1991 due to Schmüdgen about the characterizion of the $K-$moment sequences for compact semialgebraic sets K. The main objective is to give in details the original proof of this result, which is based on the Positivstellensatz and the spectral theorem for commuting bounded self-adjoint operators.
(See [10])

5. **Proof of Hilbert's 1888 Theorem for ternary quartics**: $\mathcal{P}_{3,4} = \Sigma_{3,4}$.
This seminar will present the original proof due to Hilbert of the fact that every psd ternary quartic is a sum of not more than three squares of quadratic forms. The central idea of Hilbert's proof is to associate to any ternary quartic a curve in the complex projective plane and then use the classical theory of algebraic curves.
(See [4] and [8, Chapter 7])

6. **The necessity part of Hilbert's 1888 Theorem**: $\Sigma_{n,m} \subsetneq \mathcal{P}_{n,m}$ for all $n \geq 3$, $m \geq 4$ and $(n,m) \neq (3,4)$ with m even.
Using algebraic geometry, Hilbert showed that there exist psd quaternary quartics and ternary sextic which are not sos forms and also that this is sufficient to get psd not sos forms in all the other cases mentioned above. This seminar will present an explicit example of psd ternary sextic which is not a sos provided by Schmüdgen in 1979 without applying the theory of algebraic curves. The connections between the existence of such examples and the multivariate moment problem might also be highlighted.
(See [9] and [4], [8, Chapter 7])

7. **Saturation of preorderings in $\mathbb{R}[\underline{X}]$.**
In this seminar the question of when a finitely generated preordering in the polynomial ring $\mathbb{R}[X]$ is saturated will be examined. It will be presented the result by Scheiderer of 2000, which shows that saturation never occurs when the associated basic closed semialgebraic set in $\mathbb{R}^n$ has dimension ≥ 3. Moreover, various examples where saturation holds or fails will be presented in dimensions smaller than two.
(See [7, Section 2.6-2.7] and [6, Lecture 11])

8. **Positive polynomials and sum of squares in formal power series rings** This seminar will discuss the question of whether a nonnegative polynomial can be written as sum of squares in the ring of formal power series $\mathbb{R}[[X]]$.
(See [7, Section 1.6], [6, Lecture 10])

References

[1] J. Bochnak, M. Coste, M. Roy. *Real Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer Vol. 36, Berlin, 1998

[2] E. K. Haviland. On the moment problem for distribution functions in more than one dimension I. *Amer. J. Math.*, 57:562–572, 1935.

[3] E. K. Haviland. On the moment problem for distribution functions in more than one dimension II. *Amer. J. Math.*, 58:164–168, 1936.

[4] D. Hilbert. Über die Darstellung definiter Formen als Summe von Formenquadraten, *Math. Ann.*, 32: 342-350, 1888; *Ges. Abh.* 2: 154-161, Springer, Berlin, reprinted by Chelsea, New York, 1981.

[5] S. Kuhlmann. Notes on Real Algebraic Geometry (WS 2009- 10) available online at: `http://www.math.uni-konstanz.de/~kuhlmann/Lehre/WS09-10-ReelleAlgGeo/Skripts.htm`

[6] S. Kuhlmann. Notes on Positive Polynomials (SS 2010) available online at: `http://www.math.uni-konstanz.de/~kuhlmann/Lehre/SS10-PosPoly/Skripts.htm`

[7] M. Marshall. *Positive polynomials and sums of squares*, volume 146 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2008.

[8] R.K. Rajwade. *Squares*, LMS Lecture Notes Series 171, Cambridge University Press, 1993.

[9] K. Schmüdgen. An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional. *Math. Nachr.* 88: 385-390, 1979.

[10] K. Schmüdgen. The $K-$moment problem for compact semi-algebraic sets. *Math. Ann.*, 289:203–206, 1991.