Response Theory for non-smooth observables and Women in mathematics in the UK

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Proportion of females in mathematics

School level

Subject	2015			2015	
	Female	Male	Total	Female	Male
Mathematics	35.937	56.774	97.7111	39%	61%
Further Math.	4.177	10.8016	14.993	28%	72%

University level

Dom.	Level of Study	2014/15			2014/15	
		Female	Male	Total	Female	Male
UK	First degree	2.400	3.795	6.200	39%	61%
	Masters	90	250	340	27%	73%
	Doctorate	60	185	245	24%	76%
Non UK	First degree	655	715	1.370	48%	52%
	Masters	290	445	735	40%	60%
	Doctorate	70	175	240	28%	72%

Same as in the previous years

Academic Staff

Comparison with other subjects

Dom.	Subject	2014/15			2014/15	
		Female	Male	Total	Female	Male
UK	Math	365	1.535	1.895	19%	81%
	Other	48.085	61.715	109.800	44%	56%
Non-UK	Math	400	1.415	1.815	22%	78%
	Other	20.290	26.105	46.395	44%	56%

Comparison by level

Subject	Type	2014/15			2014/15	
		Female	Male	Total	Female	Male
Math	Lect.	520	1.700	2.220	23%	77%
	Prof.	60	645	705	9%	91%
	Researchers	195	660	850	23%	77%
Other	Lect.	4.025	12.670	16.695	46%	54%
	Prof.	44.800	52.530	97.325	24%	76%
	Researchers	20.255	23.535	43.790	46%	54%

Proportion of females in mathematics

University level

Dom.	Level of Study	2014/15 Total	2014/15 Female	2014/15 Female	2014/15 Male
UK	First degree	6.200	2.400	39%	61%
	Masters	340	90	27%	73%
	Doctorate	245	60	24%	76%
	Fix term researchers	850	195	23%	77%
	Permanent staff	2.925	580	19%	81%

German situation (2014)

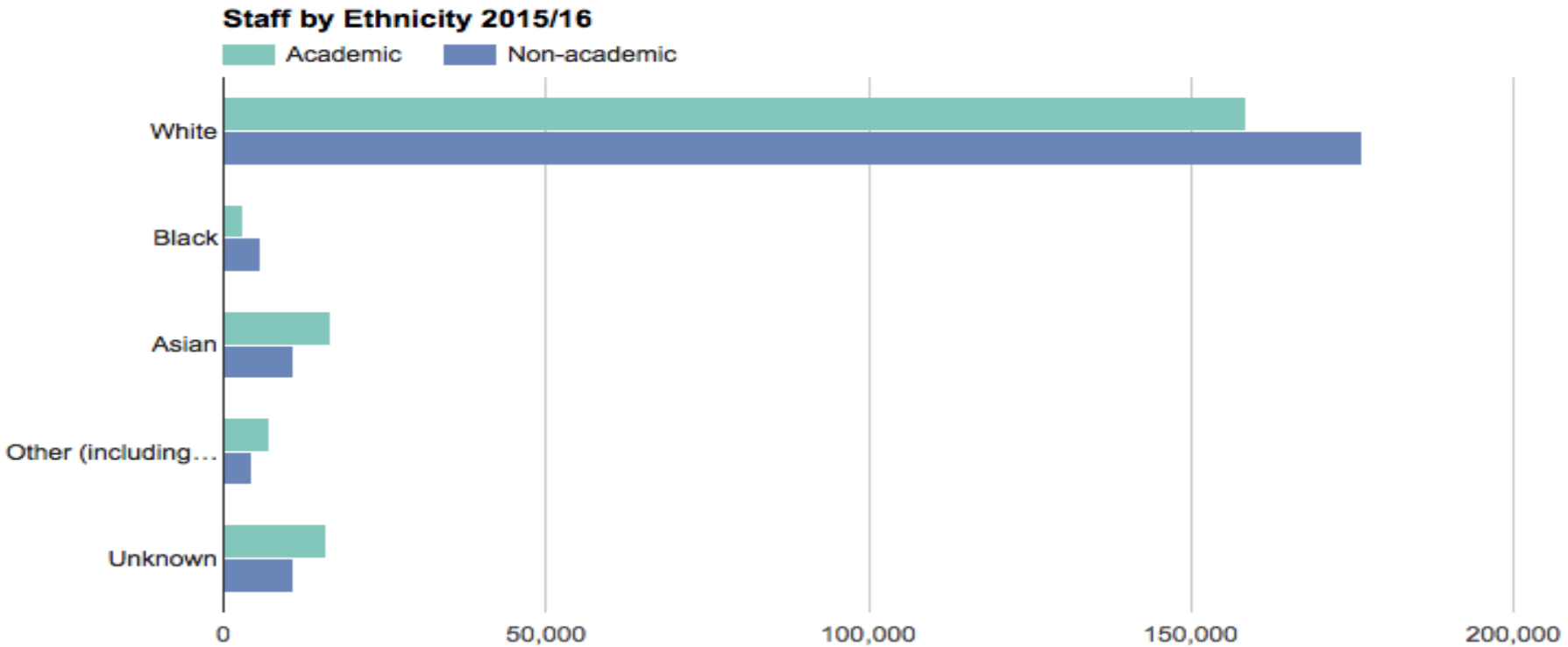
	Total	**Women**	**Percentage**
Students enrolled	72391	33728	46.6
Bachelors completed	2665	1020	38.2
Masters completed	1117	395	35.4
PhD completed	562	132	23.5
Fixed-term researchers (e.g. postdocs, fixed-term lecturer)	3697	905	24.5
Professors (tenured and non)	1247	185	14.8

Academic Staff

On a good way?

Subject	Type	2011/12 Fem.	2012/13 Fem.	2013/14 Fem.	2014/15 Fem.
Math	Lect.	23%	23%	23%	23%
	Prof.	7%	7%	9%	9%
Other	Lect.	45%	45%	45%	46%
	Prof.	21%	22%	23%	24%

Other areas?



Athena SWAN

- ECU's Athena SWAN *(Scientific Women's Academic Network)* Charter was established in 2005 to encourage and recognise commitment to advancing the careers of women in science, technology, engineering, maths and medicine (STEMM) employment in higher education and research.
- Extended in 2015
- Covers women (and men where appropriate) in:
  - academic roles in STEMM
  - professional and support staff
  - trans staff and students
- In relation to their:
  - representation
  - progression of students into academia
  - journey through career milestones
  - working environment for all staff
- Mechanism: Awards (Bronze, Silver, Gold)
- 669 total awards, Success rate 65%

Development of Athena Swan

- Athena SWAN was extended to include these groups of staff, covering aspects including:
  - career development (promotions, appraisals and training)
  - flexible working
  - contract type
  - recruitment and turnover
  - workload modelling
  - maternity and paternity and cover arrangements for when staff take family-related career breaks
  - requirement for founding (discussion)
- There will be an expectation that institutions will acknowledge the different needs of different men and women, rather than seeing them as homogeneous groups, by considering the interplay of gender with other equality characteristics (for example race, age and disability).

Athena Swan in Reading

- Application 2009, in 2010 Silver awarded, renewal 2013.
- current application 2017 for Gold
- Mathematics department with Silver Award (after LMS) Leeds, Loughborough, Oxford, UCL
- Actions Reading took:
  - Management of parental leave and return to work after leave
  - Flexible work arrangement
  - Research/early career staff forum
  - Clear promotion procedure
  - Monitoring of visibility opportunity at all levels
  - Flexible working page on website
  - workshop on unconscious gender bias

2.5 Recruitment of staff

Issue identified	Proposed action
Proportion of female staff applying for posts needs to be improved.	1. Assessment of job adverts to ensure gender balance. 2. Arrange training courses for department staff in recruitment practice, particularly new staff and those on panels. 3. Include a paragraph on all job advert on flexible working possibilities in the School with link to HR's flexible working policies on all job adverts. 4. Advertise posts on Daphnet to encourage women applicants 5. Seek out potential female applicants to posts and invite them to apply. 6. Link to Athena SWAN award on all job adverts. 7. Monitor trends using the TRENT HR systems. 8. Investigate statistics and share good practice with 1994 group universities.

Dynamical system: statistical approach

- Let M be a Riemannian manifold.
- $f_\alpha : M \to M$ be a diffeomorphism
- Dynamical system (M, f_α)
- Trajectory:

$$x, \quad f_\alpha(x), \quad f_\alpha(f_\alpha(x)), \quad \dots \quad f_\alpha^{\circ n}(x)$$

  n can be seen as time.
- For short: $f_\alpha^{\circ n} = f_\alpha^n$.
- Interested in long time statistics

$$\mu_{x,\alpha}(A) := \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n} \mathbb{1}_A(f_\alpha^t(x))$$

  Fraction of time the trajectory spend in x (started in x).
- Ergodicity: independent of x.

Response theory

- μ_α is an invariant measure by construction, that is

$$\mu_\alpha(f_\alpha^{-1}(A)) = \mu_\alpha(A) \qquad \forall A \subset M$$

- Regularity in parameter: continuity, Hölder, differentiable, real-analytic

$$\alpha \mapsto \int_M \varphi(y)\mu_\alpha(dy)$$

  for φ smooth enough.
- If differentiable then formally

$$\frac{d}{d\alpha}\int_M \varphi(y)\mu_\alpha(dy) = \sum_{n=0}^{\infty}\int_M \varphi\circ f_\alpha^n \operatorname{div}\left((\frac{d}{d\alpha}f_\alpha)\circ f_\alpha^{-1}\mu_\alpha\right)$$

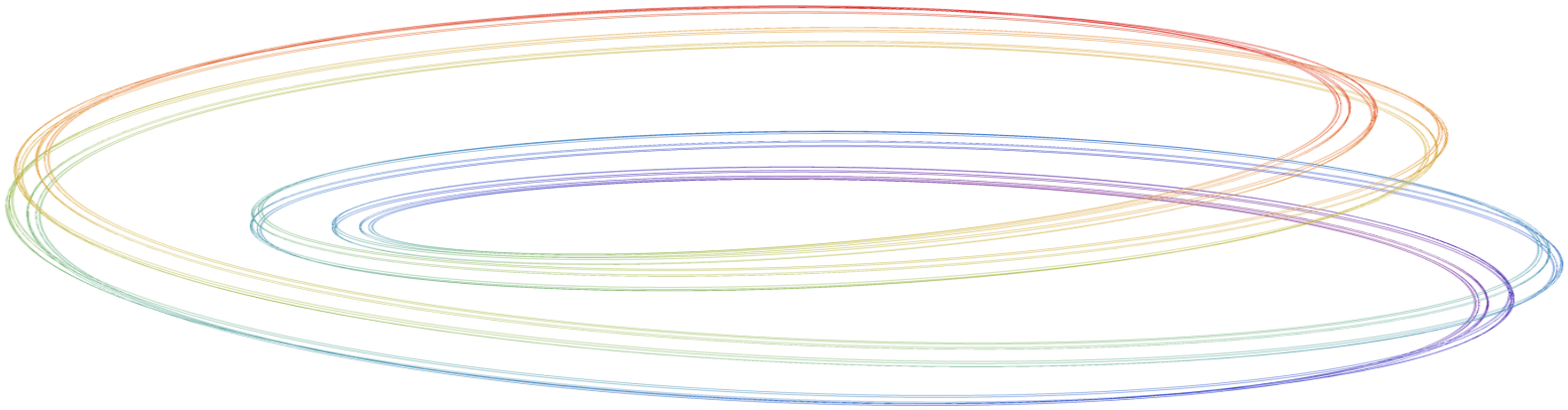
  r.h.s only derivative of f_α.
- Idea R. Kubo 1966, first rigorous result D. Ruelle 1998

- A non-empty set $\Lambda_\alpha \subset M$ is called a *transitive attractor* iff
  - $f_\alpha : \Lambda_\alpha \to \Lambda_\alpha$.
  - There exists an open $V_\alpha \supset \overline{\Lambda_\alpha}$.
  - $\bigcap_{n\geq 0} f_\alpha^n(V_\alpha) = \Lambda_\alpha$.
  - there exists a point in Λ_α which as a dense orbit.
- It is called a *uniform hyperbolic attractor* iff there exists constants $C > 1$ and $\nu < 1$ with
  - $TM \restriction_{\Lambda_\alpha}$ decomposes in two bundels $TM \restriction_{\Lambda_\alpha} = E^u \oplus E^s$.
  - E^u and E^s are Tf_α invariant.
  - for all x and n holds

$$\| Tx f_\alpha^n \restriction_{E^s_x \to E^s_{f_\alpha^n(x)}} \| \leq C\nu^n \quad \| Tx f_\alpha^n \restriction_{E^u_x \to E^u_{f_\alpha^{-n}(x)}} \| \leq C\nu^n$$

  - Then there exists a unique SRB measure μ_α
    - $\mu_x = \mu_\alpha$ for a set of positive Lebesgue measure
    - density w.r.t. Lebesgue in the unstable leaves
    - zero noise limit

- Attractor of Smale-Williams Solenoid



  Local structure:
  - Fractal
  - product of smooth manifold in unstable direction
  - Cantor set like in stable direction
- Unstable leaves: lines in picture
- SRB: density along lines
- SRB: singular not point measure orthogonal to line.

Transfer operator

- Gouzel S and Liverani C 2006
- Transfer operator: $\mathcal{L}_\alpha : \mathcal{C}^r(M) \to \mathcal{C}^r(M)$

$$\int \varphi \ (\mathcal{L}_\alpha \rho)\, dm := \int \varphi \circ f_\alpha \ \rho dm$$

  for all $\varphi \in \mathcal{C}_c^\infty(M)$.
- In terms of density

$$(\mathcal{L}_\alpha \rho) := \frac{\rho \circ f_\alpha^{-1}}{|\det Df_\alpha \circ f_\alpha^{-1}|}$$

- Hence for an invariant measure $\mu_\alpha = \rho_\alpha m$ holds

$$\mathcal{L}_\alpha \rho_\alpha = \rho_\alpha$$

- Choose Banach space such that $\mathcal{L}_\alpha$ has good spectral property (Frobenius-Perron type result)
  - ρ_α is eigenvector to the eigenvalue 1.
  - Spectral radius 1
  - essential spectral radius $r_{ess} < 1$
  - Eigenvalue 1 is isolated
- This implies exponential convergence of $\mathcal{L}_\alpha^n \rho$ to ρ_α.
- Then response theory follows easily (at least formally)

$$\begin{aligned}\int \varphi \frac{d}{d\alpha}\rho_\alpha dm &= \frac{d}{d\alpha}\int \varphi \mathcal{L}_\alpha\, \rho_\alpha dm = \frac{d}{d\alpha}\int \varphi \circ f_\alpha \rho_\alpha dm \\ &= \int \varphi \circ f_\alpha \frac{d}{d\alpha}\rho_\alpha dm + \int \nabla(\varphi) \circ f_\alpha\; Tf_\alpha \rho_\alpha dm\end{aligned}$$

and

$$\begin{aligned}\int \nabla(\varphi) \circ f_\alpha\; Tf_\alpha \rho_\alpha dm &= \int \nabla(\varphi)\; Tf_\alpha \circ f_\alpha^{-1}\; \rho_\alpha dm \\ &= \int \varphi\, \mathrm{div}\left(Tf_\alpha \circ f_\alpha^{-1} \rho_\alpha\right) dm\end{aligned}$$

all together $(1 - \mathcal{L}_\alpha)\frac{d}{d\alpha}\rho_\alpha = \mathrm{div}\left(Tf_\alpha \circ f_\alpha^{-1}\; \rho_\alpha\right)$

- Difficulty ρ_α does not exists as a function
- Consider Banach space of generalized functions $\mathcal{B}$ on M.
- Note that any Radon measure μ can be considered as a generalized function ρ with $\mu" = "\rho m$.
- In unstable direction ρ_α is a smooth function
- In stable direction ρ_α is singular to Lebesgue measure.
- Mixed Sobolev spaces $B^{-s,t}$ with $s, t > 0$: that is $\rho \in B^{-s,t}$ if Fourier transform in stable direction $O(|\xi|^{s-1})$ and in the unstable $O(|\eta|^{-t-1})$.
- Toy example:
  - x direction expanding y direction contracting
  - $\varphi(x, y) = \theta(x)h(y)$, where θ Heaviside function
  - Fourier transform $\hat{\varphi}(\xi, \eta) = \left(\mathcal{P}\frac{1}{\xi}\right)\hat{h}(\eta)$ where $\mathcal{P}\frac{1}{\xi} = O(|\xi|^{-1})$ and $\hat{h}(\eta) = O(|\eta|^{-N})$ for all N.
  - In stable cone $|\xi| \leq c|\eta|$ we have $O(|\xi|^{-s-1}|\eta|^{-N}) = O(|\xi|^{-s-1}|\eta|^{s-N})$ so regular enough for large N.

Main theorem

Theorem

Let $\alpha \mapsto f_\alpha$ be a $\mathcal{C}^3$ maps of $\mathcal{C}^4$-diffeomorphism with a compact hyperbolic attractor. Let

$$\varphi(x) = h(x)\theta(g(x) - a)$$

with $h, g : M \to \mathbb{R}$ in $\mathcal{C}^4$ and $a \in \mathbb{R}$ not a critical value of g and assume that

$$\{x \in M : g(x) = a\} \cap \operatorname{supp}(h)$$

admits a $\mathcal{C}^4$-foliation of admissible stable pseudo leaves.
Then the map $\alpha \mapsto \rho_\alpha$ is differentiable in the weak sense.

Extreme value theory

Classical probabilistic theory:

- Consider random sequence $(X_n)_n$
- Assymptotic distribution of $\max_{n \leq N} X_n$
- (Gnedenko) Universal distribution like in CLT: only three free parameter
- Good starting point for statistics.

Dynamical system:

- Analogous structure
- Basic building block: over-threshold even

$$\mu(f_\alpha^n > r)$$

- Recent monograph
- Response theory means stability of parameters.

Open questions

- Not the generic situation for extreme values
- In generic case only very weak regularity under unrealistic strong assumption
- How to exclude non-generic situations which destroy regularity.
- Final aim: prove heuristic formula for the derivative.
- Infinite dimensional dynamical systems are better?