On the diagonalizability of the Atkin U-operator for Drinfeld cusp forms

Maria Valentino
joint with A. Bandini

Konstanz: women in Mathematics
University of Konstanz

- The *modular group*: $SL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) | a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$;
- The *upper half plane*: $\mathcal{H} = \{z \in \mathbb{C} : Im(z) > 0\}$;
- Action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ by Möbius transformations: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot z = \frac{az+b}{cz+d}$.

Definition 1

Let k be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a *modular form of weight k* if

1. f is holomorphic on $\mathcal{H}$;
2. f is holomorphic at infinity;
3. $f(\gamma(z)) = (cz + d)^k f(z)$ for $\gamma \in SL_2(\mathbb{Z})$ and $z \in \mathcal{H}$.

- *Fourier expansion*: $f(z) = \sum_{n=0}^{\infty} a_n q^n$, $\quad q = e^{2\pi iz}$;
- $M_k(SL_2(\mathbb{Z})) :=$ {set of modular forms of weight k};
- $M_k(SL_2(\mathbb{Z}))$ is a finite dimensional vector space over $\mathbb{C}$.

Definition 2

A *cusp form of weight k* is a modular form of weight k whose Fourier expansion has leading coefficient $a_0 = 0$.

- $S_k(SL_2(\mathbb{Z})) :=$ {set of cusp forms of weight k} is finite dimensional vector space over $\mathbb{C}$.

- The *principal congruence subgroup of level $N \in \mathbb{Z}^+$* is
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Definition 3

A subgroup Γ of $SL_2(\mathbb{Z})$ is a *congruence subgroup of level N* if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{Z}^+$.

Definition 4

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let k be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a *modular form of weight k with respect to Γ* if

1. f is holomorphic;
2. $(c'z + d')^{-k} f(\gamma(z))$ is holomorphic at infinity for all $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$;
3. $f(\gamma(z)) = (cz + d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathcal{H}$.

If in addition,

4. $a_0 = 0$ in the Fourier expansion of $(c'z + d')^{-k} f(\gamma(z))$ for all $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$,

then f is a *cusp form of weight k with respect to Γ*.

- $M_k(\Gamma) :=$ {set of modular forms of weight k} with respect to Γ;
- $S_k(\Gamma) :=$ {set of cusp forms of weight k} with respect to Γ;
- $M_k(\Gamma)$ and $S_k(\Gamma)$ are finite dimensional vector space over $\mathbb{C}$.

Definition 5

Let $N \in \mathbb{Z}^+$ and p a prime number. The *Hecke operator* T_p acts on $M_k(\Gamma)$ in the following way:

$$T_p f(z) = \begin{cases} \displaystyle\sum_{j=0}^{p-1} p^{-k} f\left(\frac{z+j}{p}\right) + (Npz+p)^{-k} f\left(\frac{mpz+n}{Npz+p}\right) & p \nmid N \text{ and } mp - nN = 1 \\ \displaystyle\sum_{j=0}^{p-1} p^{-k} f\left(\frac{z+j}{p}\right) & p \mid N \end{cases}$$

- When $p \mid N$, $U_p := T_p$ is called *Atkin operator*;
- Let $\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$;
- The *Petersson inner product*: $\langle\ ,\ \rangle_\Gamma : S_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$;
- If $p \nmid N$ T_p on $S_k(\Gamma)$ is skew-Hermitian with respect to the Petersson inner product $\Longrightarrow$ If $p \nmid N$ T_p is always diagonalizable;
- U_p on $S_k(\Gamma)$ can fail to be diagonalizabile.

Question

What happens to U_p in the function field case?

- $F = \mathbb{F}_q(t)$, $q = p^r$, $p \in \mathbb{Z}$ prime, $A = \mathbb{F}_q[t]$;
- $F_\infty = \mathbb{F}_q((1/t))$, $A_\infty = \mathbb{F}_q[[1/t]]$, $\mathbb{C}_\infty = \{$completion of an algebraic closure of $F_\infty\}$;
- *Drinfeld upper half-plane*: $\Omega := \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus \mathbb{P}^1(F_\infty)$ (rigid analytic);
- Action of $GL_2(F_\infty)$ on Ω: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$;
- Let $\mathfrak{n}$ be an ideal of A, then the *principal congruence subgroup of level* $\mathfrak{n}$ is
$$\Gamma(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}} \right\} ;$$
- A subgroup Γ of $GL_2(A)$ is called an *arithmetic subgroup* if there exists an ideal $\mathfrak{n}$ of A such that Γ contains $\Gamma(\mathfrak{n})$ and such that this inclusion is of finite index;
- $\Gamma \backslash \mathbb{P}^1(F)$ has finite many elements called *cusps*;

Definition 6

A rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ is called a *Drinfeld modular form (DMF) of weight k and type m for Γ* if

1. $f(\gamma z)(\det \gamma)^m (cz+d)^{-k} = f(z) \quad \forall \gamma \in \Gamma$;
2. f is holomorphic at all cusps.

Moreover, f is called a *cusp form*, respectively *double cusp form*, if it vanishes at all cusps to the order at least 1, respectively to the order at least 2.

- $M_{k,m}(\Gamma) := \{$set of DMF of weight k and type m for $\Gamma\}$ finite dim. v.s over $\mathbb{C}_\infty$;
- $S^i_{k,m}(\Gamma) := \{$set of cusp forms (doubly) of weight k and type m for $\Gamma\}$ finite dim. v.s over $\mathbb{C}_\infty$.

Combinatorial counterpart of the Drinfeld upper half-plane

- $Z(F_\infty)$ the scalar matrices of $GL_2(F_\infty)$;
- *Iwahori subgroup*, $\mathcal{I}(F_\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A_\infty) \;\middle|\; c \equiv 0 \mod \frac{1}{t} \right\}$;
- *Bruhat-Tits tree* $\mathcal{T}$ of $GL_2(F_\infty)$:
  - $\mathcal{T}$ is a (q+1)-regual tree on which $GL_2(F_\infty)$ acts transitively;
  - Vertices $X(\mathcal{T}) = GL_2(F_\infty)/Z(F_\infty)GL_2(A_\infty)$
  - Oriented edges $Y(\mathcal{T}) = GL_2(F_\infty)/Z(F_\infty)\mathcal{I}(F_\infty)$
- The canonical map from $Y(\mathcal{T})$ to $X(\mathcal{T})$ associates with each oriented edge e its origin $o(e)$;
- The edge $\overline{e}$ is e with reversed orientation;
- A system of representatives of $X(\mathcal{T})$ and $Y(\mathcal{T})$
  - $S_X := \left\{ v_{i,u} = \begin{pmatrix} t^i & u \\ 0 & 1 \end{pmatrix} \middle| i \in \mathbb{Z}, u \in F_\infty/t^i A_\infty \right\}$;
  - $S_Y := S_X \cup S_X \begin{pmatrix} 0 & 1 \\ \frac{1}{t} & 0 \end{pmatrix}$;
- For Γ arithmetic subgroup, the quotient tree $\Gamma\backslash\mathcal{T}$ is called *fundamental domain*.

- Let Γ be a p' torsion free and $\det(\Gamma) = 1$;
- For $k \geqslant 0$ and $m \in \mathbb{Z}$, let $V(k,m)$ be the $(k-1)$-dimensional vector space over $\mathbb{C}_\infty$ with a basis $\{x^j y^{k-2-j} : 0 \leqslant j \leqslant k-2\}$;
- Action of $\gamma \in GL_2(F_\infty)$ on $V(k,m)$ is given by $\gamma(x^j y^{k-2-j}) \mapsto \det(\gamma)^{m-1}(dx - by)^j(-cx + ay)^{k-2-j} \quad \forall\, 0 \leqslant j \leqslant k-2\,;$
- For every $\omega \in \mathrm{Hom}(V(k,m), \mathbb{C}_\infty)$ we have an induced action of $GL_2(F_\infty)$: $(\gamma\omega)(x^j y^{k-2-j}) = \det(\gamma)^{1-m}\omega((ax+by)^j(cx+dy)^{k-2-j}) \quad \text{for } 0 \leqslant j \leqslant k-2\,.$

Definition 7

A *harmonic cocycle of weight k and type m for Γ* is a function $\mathbf{c}$ from the set of directed edges of $\mathcal{T}$ to $\mathrm{Hom}(V(k,m), \mathbb{C}_\infty)$ satisfying:

1. Harmonicity: for all vertices v of $\mathcal{T}$: $\sum_{e \mapsto v} \mathbf{c}(e) = 0$, where e runs over all edges in $\mathcal{T}$ with terminal vertex v;
2. For all edges e of $\mathcal{T}$, $\mathbf{c}(\overline{e}) = -\mathbf{c}(e)$;
3. Γ-equivariancy: for all edges e and elements $\gamma \in \Gamma$, $\mathbf{c}(\gamma e) = \gamma(\mathbf{c}(e))$.

- $C^{har}_{k,m}(\Gamma) :=$ space of harmonic cocycles of weight k and type m for Γ .

Theorem (Teitelbaum, 1991)

$$S^1_{k,m}(\Gamma) \simeq C^{har}_{k,m}(\Gamma)$$

- Let Γ be an arithmetic subgroup of level (t);
- Let $\mathfrak{n}$ be an ideal of A and denote by $P_{\mathfrak{n}}$ its monic generator;
- The *Hecke operator* $\mathbf{T}_{\mathfrak{n}}$ acts on $f \in M_{k,m}(\Gamma)$ in the following way:

$$\mathbf{T}_{\mathfrak{n}}(f)(z) := P_{\mathfrak{n}}^{k-m} \sum_{\substack{\alpha,\delta \text{ monic} \\ \beta\in A, \deg(\beta)<\deg(\delta) \\ \alpha\delta = P_{\mathfrak{n}}, (\alpha)+(t)=A}} f\left(\frac{\alpha z+\beta}{\delta}\right);$$

- For $\mathfrak{n}=(t)$ the Atkin U-operator in our context is:

$$U(f)(z) := \mathbf{T}_{(t)}(f)(z) = \sum_{\beta\in\mathbb{F}_q} f\left(\frac{z+\beta}{t}\right);$$

- Hecke action on harmonic cocycles in the following way:

$$U(\mathbf{c}(e)) = t^{k-m} \sum_{\beta\in\mathbb{F}_q} \begin{pmatrix} 1 & \beta \\ 0 & t \end{pmatrix}^{-1} \mathbf{c}\left(\begin{pmatrix} 1 & \beta \\ 0 & t \end{pmatrix} e\right)$$

- $\Gamma_1(t) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL_2(A) : a \equiv d \equiv 1 \text{ and } c \equiv 0 \pmod{t} \right\}$;
- $\dim_{\mathbb{C}_\infty} S^1_{k,m}(\Gamma_1(t)) = (k-1)$;
- Fundamental domain:

$$\overline{e}_{-2} = \underleftarrow{\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}} \quad \overline{e}_{-1} = \underleftarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \quad \overline{e}_{0} = \underleftarrow{\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}} \quad \overline{e}_{1} = \underleftarrow{\begin{pmatrix} 0 & t^2 \\ 1 & 0 \end{pmatrix}}$$

$$v_{-2} = \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} \quad v_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad v_{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad v_{1} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \quad v_{2} = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e_{-2} = \overrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix}} \quad e_{-1} = \overrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} \quad e_{0} = \overrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad e_{1} = \overrightarrow{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}}$$

- Stable edge: $\overline{e} := \overline{e}_{-1}$;
- For $j \in \{0, 1, \dots, k-2\}$, $\mathbf{c}_j(\overline{e})(X^i Y^{k-2-i}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$;

Theorem (Bandini, V., 2016)

The matrix associated to U in the above context is

$$U(\mathbf{c}_j(\overline{e})) = -(-t)^{j+1}\binom{k-2-j}{j}\mathbf{c}_j(\overline{e}) - t^{j+1}\sum_{h\neq 0}\left[\binom{k-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1}\binom{k-2-j-h(q-1)}{j}\right]\mathbf{c}_{j+h(q-1)}(\overline{e}) \,.$$

Theorem (Bandini, V. (2016))

With notations as above, we have:

1 If $q \geqslant k$, then U is diagonalizable and

$$U(\mathbf{c}_j(\overline{e})) = -(-t)^{j+1}\binom{k-2-j}{j}\mathbf{c}_j(\overline{e});$$

2 If $k = q+1, q+2$, then U is diagonalizable.

	Eigenvector	Eigenvalue
k=q+1	$\mathbf{c}_0(\overline{e}) + \mathbf{c}_{q-1}(\overline{e})$	t
	$\mathbf{c}_j(\overline{e})$, $1 \leqslant j \leqslant q-2$	$-(-t)^{j+1}\binom{k-2-j}{j}$
	$\mathbf{c}_{q-1}(\overline{e})$	0
$k = q+2,\ q \neq 2$	$\mathbf{c}_0(\overline{e}) + \mathbf{c}_{q-1}(\overline{e})$	t
	$\mathbf{c}_1(\overline{e})$	t^2
	$\mathbf{c}_j(\overline{e})$, $2 \leqslant j \leqslant q-2$	$-(-t)^{j+1}\binom{k-2-j}{j}$
	$\mathbf{c}_{q-1}(\overline{e})$	0
	$t^{q-1}\mathbf{c}_1(\overline{e}) + \mathbf{c}_q(\overline{e})$	0
$k = q+2,\ q = 2$	$\mathbf{c}_0(\overline{e}) + \mathbf{c}_{q-1}(\overline{e})$	t
	$\mathbf{c}_1(\overline{e})$	t^2
	$t^{q-1}\mathbf{c}_1(\overline{e}) + \mathbf{c}_q(\overline{e})$	0

Theorem *(Bandini, V. 2016)*

With notations as above, let $k = q + 3$. Then, U is diagonalizable if and only if q is odd.

$k = q + 3$	Eigenvalue	Eigenvector
q=4	t	$\mathbf{c}_0 + \mathbf{c}_3$
	t^3	$\mathbf{c}_2$
	0	$t^3\mathbf{c}_2 + \mathbf{c}_5$
		$\mathbf{c}_3$
	$t^{7/2}$	$t^{3/2}\mathbf{c}_1 + \mathbf{c}_4$
$q = 3$	t^3	$-t\mathbf{c}_1 + \mathbf{c}_3$
		$\mathbf{c}_2$
	$-t^3$	$t\mathbf{c}_1 + \mathbf{c}_3$
	t	$\mathbf{c}_0 + (t^2+1)\mathbf{c}_2 + \mathbf{c}_4$
	0	$t^2\mathbf{c}_2 + \mathbf{c}_4$
$q = 2$	t	$\mathbf{c}_0 - (t-1)^2\mathbf{c}_1 - (t-1)\mathbf{c}_2 + \mathbf{c}_3$
	0	$-t^2\mathbf{c}_1 - t\mathbf{c}_2 + \mathbf{c}_3$
	$t^{5/2}$	$t^{1/2}\mathbf{c}_1 + \mathbf{c}_2$

Conjecture (Bandini, V. 2016)

Let q be even. If $k \geqslant q+3$ and odd, then U is not diagonalizable.

- The characteristic polynomial is divisible by the factor $(x^2 + t^k)$;
- The $\mathbf{c}_j$'s can be divided in classes $\pmod{q-1}$ and every class is stable under the action of U;
- The associated matrix is divided in blocks $\pmod{q-1}$ and U is diagonalizable if and only of every block is;
- C_j the class of $\mathbf{c}_j$, i.e $C_j = \{\mathbf{c}_j, \mathbf{c}_{j+(q-1)}, \dots\}$;

Theorem (Bandini, V. 2016)

Assume q even, $k \equiv 1 \pmod 2$, with $k > q+3$, and $|C_{\frac{k-1-q}{2}}| = 2$. Then the matrix associated to $C_{\frac{k-1-q}{2}}$ is not diagonalizable.

Proof.

$$\begin{pmatrix} 0 & t^{\frac{k+q-1}{2}} \\ t^{\frac{k+1-q}{2}} & 0 \end{pmatrix} \Rightarrow \text{char. poly } X^2 - t^k = (X - t^{\frac{k}{2}})^2 \quad \Rightarrow \text{inseparable eigenvalue } t^{\frac{k}{2}}.$$

□

Theorem (Bandini, V. 2016)

Assume q even, $k \equiv 1 \pmod 2$, with $k > 3q-3$, and $|C_{\frac{k-3q+1}{2}}| = 4$. Then U is not diagonalizable.

- $\frac{k-3q+1}{2} + (q-1) = \frac{k-q-1}{2}$;
- $M(j, n, q)$ matrix associated to the block C_j of size n $(k = 2j + 2 + (n-1)(q-1))$;

Theorem (Bandini, V. 2016)

Let $n \in \mathbb{N}$ even, $q = 2^r$ and $0 \leqslant j \leqslant q-2$. Then, for all $j \geqslant n$, the matrix $M(j, n, q)$ is antidiagonal.

Corollary

With notation as in the previous theorem, $M(j, n, q)$ is not diagonalizable.

References

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