On the spectral properties of square matrices that are strictly sign-regular of some order

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Introduction

By "spectral structure of square matrices" we mean properties of the eigenvalues and eigenvectors of matrices. In view of the well-known Perron-Frobenius theory that describes the spectral structure of general entrywise nonnegative matrices, we study the spectral structure of strictly sign-regular matrices of some order. A matrix is called *strictly sign-regular of order* k, denoted by SSR_k, if all its minors of order k are nonzero and have the same sign. For example, *totally positive* matrices (TP), i.e., matrices with all minors positive, are SSR_k for all k. Another important class of matrices are those that are SSR_k for all odd k. Such matrices have interesting sign variation diminishing properties, and it has been recently shown that they play an important role in the analysis of certain nonlinear cooperative systems.

Definitions and Notations

We will introduce the following definitions.

1. A vector $x \in \mathbb{R}^n$ is called *totally nonzero* if no entry of x is zero.
2. Let v be the function from the totally nonzero vectors $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ into the nonnegative integers defined by

   $$v(x) := \#\{i : x_i x_{i+1} < 0\}, \text{for } i = 1, 2, \ldots, n-1,$$

   the total sign variation of x. For general vectors $x \in \mathbb{R}^n$, $v^-(x)$ is the minimum value of $v(y)$ among all totally nonzeros y that agree with x in its nonzero entries and $v^+(x)$ is the maximum value of $v(y)$ among all such vectors. In the case that x has zero entries, $v^-(x)$ is also $v(x')$ in which x' is simply the result of deleting the zero entries from x.
3. For a vector $x \in \mathbb{R}^n$, let

   $$v_c^-(x) := \max_i v^-(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i),$$

   the cyclic number of sign variations.
4. Consider the linear time-varying system

   $$\dot{x}(t) = A(t)x(t) \tag{1}$$

   with $A(t)$ a continuous matrix function of t. This system is called *totally positive differential system*, denoted by $TPDS$, on a time interval (a, b) if its transition matrix $\phi(t, t_0)$ is totally positive for any pair(t_0, t) with $a < t_0 < t < b$. Here the transition matrix is the matrix satisfying $x(t) = \phi(t, t_0)x(t_0)$. In particular, $\phi(t_0, t_0) = I$. In the special case where $A(t)$ is a constant matrix, i.e., $A(t) \equiv A$, then $\phi(t, t_0) = exp((t - t_0)A)$. Of course, the transition matrix is real, square, and nonsingular.
5. Let $\mathbb{M}^+ \subset \mathbb{R}^{n\times n}$ denote the subset of $n \times n$ real matrices that are tridiagonal with positive entries on the super- and sub-diagonals.
6. Let $A \in \mathbb{R}^{n\times n}$. We say that a set of complex numbers $c_1, \ldots, c_m \in \mathbb{C}^n$ matches A if $\sum_{i=1}^{m} |c_i|^2 > 0$, and for every i if the eigenvector v^i of A is real then c_i is real and, if v^i,v^{i+1} is a conjugate complex pair then $c_{i+1} = \bar{c}_i$.

   The following example is a simple illustrative example to the previous definitions.

Example

$$v^+\begin{bmatrix}1\\-1\\2\\-3\\0\end{bmatrix}=4, \qquad v^-\begin{bmatrix}1\\-1\\2\\-3\\0\end{bmatrix}=3, \qquad v_c^+\begin{bmatrix}1\\-1\\2\\-3\\0\end{bmatrix}=4, \qquad v_c^-\begin{bmatrix}1\\-1\\2\\-3\\0\end{bmatrix}=4$$

Motivation and Previous Works

Applications of strictly sign-regular matrices often come from their interpretation as variation-diminishing transformations. Strictly sign-regular matrices are characterized by the strong variation-diminishing property, i.e., $v^+(Ax) \leq v^-(x)$ for all nonzero $x \in \mathbb{R}^n$. In particular, multiplying a vector by a *TP* matrix can only decrease the number of sign variations in the vector. It was shown that the system (1) is $TPDS$ if and only if $A \in \mathbb{M}^+$. Several studies analysed certain non-linear dynamical systems by showing that the transition matrix in the variational system satisfies a variation-diminishing property with respect to the cyclic number of sign variations in a vector. Motivated by this, Theorem (A) shows that the SSR_k property is equivalent to a non-standard variation-diminishing property. Theorem (B) provides a simple necessary and sufficient condition for a nonsingular square matrix A to satisfy a $CVDP$.

Theorem A Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Pick $k \in \{1, \ldots, n\}$. Then the following two conditions are equivalent.
(i) For any vector $x \in \mathbb{R}^n \setminus \{0\}$ with $s_c^-(Ax) \leq k-1$, $v^+(Ac) \leq k-1$.
(ii) A is SSR_k.

Theorem B Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. The following two conditions are equivalent.
(i) For any vector $x \in \mathbb{R}^n \setminus \{0\}$ $v_c^+(Ax) \leq v_c^-(x)$.
(ii) The matrix A is SSR_r for all odd r in the range $r \in \{1, \ldots, n\}$.

Main Results

Our main results extend Theorem (A) to vectors $x \in \mathbb{C}^n$ and study the properties of the eigenvalues of the matrices in Theorem (B).

1. Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and SSR_k for some $k \in \{1, \ldots, n-1\}$. For any $c_1, \ldots, c_{n-1} \in \mathbb{C}$ that match A, we have

   $$v^+\left(\sum_{i=0}^{k} c_i v^i\right) \leq k-1.$$

2. Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and SSR_k for all odd $k \in \{1, 3, \ldots, n\}$. Then the following statements are true.
   1. λ_1 is a positive simple eigenvalue of A.
   2. The algebraic multiplicity of any eigenvalue of A is not greater than 2.
   3. The inequalities $|\lambda_1| > |\lambda_2| \geq |\lambda_3| > |\lambda_4| \geq |\lambda_5| \ldots$ hold.
   4. For every $i \in \{3, 5, 7, \ldots\}$, λ_{i-1}, λ_i are either a pair of complex conjugate or both are real and of the same sign.
   5. If n is even, then λ_n is real.

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