True, false, independent: how the Continuum Hypothesis can be solved (or not).

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The Continuum Hypothesis

The Continuum Hypothesis (CH), Cantor, 1878

There is no set whose cardinality is strictly between that of the natural and the real numbers: $|P(\mathbf{N})| = 2^{\aleph_0} = \aleph_1$.

- Question arises from Cantor's work on ordinals and cardinals: $|\,N\,|\,=\aleph_0,\,\text{but what is }|\,R\,|\,?$
- Cantor tried to prove the CH but did not succeed.
- Hilbert posed the CH as the first problem on his list of important open questions in 1900.

Incompleteness Theorem, Gödel, 1931

Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F.

- A statement that cannot be proved or disproved from such a system *F* is called *independent* from *F*.
- Independence is important for finding axioms.
- But: No matter how many axioms one adds, the system will never be *complete*.

Standard axiomatization of set theory ZFC:

- Extensionality.
- Pairing.
- Union.
- Infinity.
- Power Set.
- Foundation.
- Replacement.
- Comprehension.
- Choice.

To show that CH is independent from ZF(C) we have to show that:

- 1. CH can be added to ZF(C) as an axiom and the resulting theory is consistent iff ZF(C) is consistent, and
- 2. \neg CH can be added to ZF(C) as an axiom and the resulting theory is consistent iff ZF(C) is consistent.

In practice that means that we have to find models M and M' such that $M \models ZF(C) + CH$ and $M' \models ZF(C) + \neg CH$.

The constructible universe L

Definition

A set x is definable over a model (M, \in) , where M is a set, if there exists a formula φ in the set of all formulas of the language $\{\in\}$ and some $a_1, \ldots, a_n \in M$ such that $x = \{y \in M : (M, \in) \models \varphi[y, a_1, \ldots, a_n]\}.$

$$def(M) = \{x \subset M : x \text{ is definable over } (M; \in)\}$$

The Hierarchy of Constructible Sets

Define:

•
$$L_0 = \emptyset$$
, $L_{\alpha+1} = def(L_{\alpha})$,

•
$$L_{\alpha} = \bigcup_{\beta < \alpha} L_{\beta}$$
 if α is a limit ordinal, and

•
$$L = \bigcup_{\alpha \in ORD} L_{\alpha}$$
.

The class L is the class of the constructible sets.

Axiom of Constructibility

V = L, i.e. "every set is constructible".

- For every α , $\alpha \subset L_{\alpha}$ and $L_{\alpha} \cap ORD = \alpha$.
- Each L_{α} is transitive, $L_{\alpha} \subset L_{\beta}$ if $\alpha < \beta$, and L is a transitive class.
- L is a model of ZF.
- There exists a well-ordering of the class *L* i.e. the Axiom of Choice holds.
- *L* is an inner model of *ZF* (an inner model of *ZF* is a transitive class that contains all ordinals and satisfies the aioms of ZF). Indeed, *L* is the smallest inner model of *ZF*.

Theorem

The Continuum Hypothesis holds in L.

Proof Outline

- 1. Define a hierarchy for the complexity of formulas.
- 2. Show that V = L is absolute.
- 3. Prove that CH follows from V = L.

Theorem

L satisfies the Axiom of Constructibility, V = L.

Proof: To verify V = L in L, we have to prove that the property "x is constructible" is absolute for L, i.e., that for every $x \in L$ we have (x is constructible)^L.

The Levy Hierarchy

Definition

- 1. A formula of set theory is a Δ_0 -formula if
 - it has no quantifiers, or
 - it is $\varphi \land \psi$, $\varphi \lor \psi$, $\neg \varphi$, $\varphi \rightarrow \psi$ or $\varphi \leftrightarrow \psi$ where φ and ψ are Δ_0 -formulas, or
 - it is $(\exists x \in y)\varphi$ or $(\forall x \in y)\varphi$ where φ is a 0-formula.
- A formula is Σ₀ and Π₀ if its only quantifiers are bounded, i.e., a Δ₀-formula.
- 3. A formula is Σ_{n+1} if it is of the form $\exists x \varphi$ where φ is Π_n , and Π_{n+1} if it is of the form $\forall x \varphi$ where φ is Σ_n .

A property (class, relation) is Σ_n (Π_n) if it can be expressed by a Σ_n (Π_n) formula. It is Δ_n if it is both Σ_n and Π_n .

A function F is Σ_n (Π_n) if the relation y = F(x) is Σ_n (Π_n).

Definition

A formula φ is absolute for a transitive model M if for all x_1, \ldots, x_n

$$\varphi^{\mathcal{M}}(x_1,\ldots,x_n)\leftrightarrow\varphi(x_1,\ldots,x_n).$$

Lemma

 Δ_0 and Δ_1 properties are absolute for transitive models.

Example for a Δ_0 -formula:

x is empty $\leftrightarrow (\forall u \in x) u \neq u$.

Theorem

L satisfies the Axiom of Constructibility, V = L.

Proof: We can show that the function $\alpha \mapsto L_{\alpha}$ is Δ_1 . Then the property "x is constructible" is absolute for inner models of ZF and therefore:

For every $x \in L$, $(x \text{ is constructible})^L$ iff x is constructible and hence "every set is constructible" holds in L.

The Generalized Continuum Hypothesis

 $2^{\aleph_{\alpha}} = \aleph_{\alpha+1}$

for all α .

Theorem (Gödel)

If V = L then $2^{\aleph_{\alpha}} = \aleph_{\alpha+1}$ for every α .

Proof Outline: If X is a constructible subset of ω_{α} then there exists a $\gamma < \omega_{\alpha+1}$ such that $X \in L_{\gamma}$.

Therefore $P^{L}(\omega_{\alpha}) \subset L_{\omega_{\alpha+1}}$, and since $|L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$, we have $|P^{L}(\omega_{\alpha})| \leq \aleph_{\alpha+1}$.

Forcing

Negation of CH

Aim to show independence of CH

There exists a model *M* of ZFC such that it satisfies $2^{\aleph_0} > \aleph_1$.

Easy solution: Add more than \aleph_1 many new reals to a model!

We only have to make sure that:

- The new model is still a model of ZFC.
- The relevant cardinal notions mean the same in the two models.
- The reals we add are in fact new reals.
- We can see what is true or false in the new model (at least to a certain degree).



We want to show the consistency of $ZF + V \neq L$ (or any stronger theory such as $ZFC + \neg CH$). What is the model we start from?

Idea 1: We work with a ZFC-model: In ZFC define a transitive proper class **N** and prove that each axiom of $ZF + V \neq L$ is true in **N**. Then $L \neq \mathbf{N}$ but since *L* is minimal, $L \subset \mathbf{N}$. So there is a proper extension of *L*, i.e. $ZFCV \neq L$. Contradiction because ZFC + V = L is consistent.

Idea 2: We work with a *set* model: In ZFC produce a set model for ZFC. Contradiction to the Incompleteness Theorem, because it would follow that ZFC could prove its own consistency.

Idea 3: We work with a countable, transitive model M for any desired *finite* list of axioms of ZFC!

The forcing notion

Forcing schema

We extend a countable, transitive model M of ZFC, the ground model, to a model M[G] by adding a new object G that was not part of the ground model. This extension model is a model of ZFC plus some additional statement that follows from G.

Definition

- 1. Let *M* be a ctm of ZFC and let $P = (P, \leq)$ be a nonempty partially ordered set. *P* is called a notion of forcing and the elements of *P* are the forcing conditions.
- 2. If $p, q \in P$ and there exists $r \in P$ such that $r \leq p$ and $r \leq q$ then p and q are compatible.
- A set D ⊂ P is dense in P if for every p ∈ P there is q ∈ D s.t. q ≤ p.

Definition

A set $F \subset P$ is a filter on P if

- F is non-empty;
- if $p \leq q$ and $p \in F$, then $q \in F$;
- if $p, q \in F$, then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

A set of conditions $G \subset P$ is generic over M if

- G is a filter on P;
- if D is dense in P and $D \in M$, then $G \cap D \neq \emptyset$.

Let P be a set of finite 0 - 1 sequences $\langle p(0), \ldots, p(n+1) \rangle$ and a condition p is stronger than q if p extends q. Then p and q are compatible, if either $p \subset q$ or $q \subset p$.

Let *M* be the ground model and let $G \subset P$ be generic over *M*. Let $f = \bigcup G$. Since *G* is a filter, all elements in *G* are pairwise compatible and so *f* is a function. Each $p \in G$ is a finite approximation to *f* and "determines" *f*: *p* forces *f*.

Genericity: For every $n \in \omega$, the sets $D_n = \{p \in P : n \in dom(p)\}$ is dense in P, hence it meets G, and so dom $(f) = \omega$.

f is not in the ground model: For every such $g \in M$, let $D_g = \{p \in P : p \notin g\}$. Then D_g is dense, so it meets G and it follows that $f \neq g$.

The new real added is $A \subset \omega$ with characteristic function f.

Lemma

If (P, \leq) is a partially ordered set and **D** is a countable collection of dense subsets of P, then there exists a **D**-generic filter on P. In particular, for every $p \in P$ there exists a **D**-generic filter G on P such that $p \in G$.

Proof. Let D_1, D_2 , be the sets in **D**. Let $p_0 = p$ and for each n, let p_n be such that $p_n \leq p_{n-1}$ and $p_n \in D_n$. The set

$$G = \{q \in P : q \ge p_n \text{ for some } n \in \mathbf{N}\}$$

is a **D**-generic filter on P and $p \in G$.

The extension model

Theorem

Let *M* be a transitive model of ZFC and let (P, \leq) be a notion of forcing in *M*. If $G \subset P$ is generic over *P*, then there exists a transitive model M[G] such that:

- i) M[G] is a model of ZFC;
- ii) $M \subset M[G]$ and $G \in M[G]$;
- iii) $Ord^{M[G]} = Ord^{M}$;
- iv) if N is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.

M[G] is called the *generic extension* of M. The sets in M[G] are definable from G and finitely many elements of M. Each element of M[G] will have a *name* in M describing how it has been constructed. M[G] can be described in the ground model.

The forcing language: It contains a name for every element of M[G], including a constant \dot{G} , the name for a generic set. Once a G is selected then every constant of the forcing language is interpreted as an element of the model M[G].

The *forcing relation*: It is a relation between the forcing conditions and sentences of the forcing language:

 $p \Vdash \sigma$

(p forces σ).

The forcing language and the forcing relation are defined in the ground model.

Theorem

Let (P, \leq) be a notion of forcing in the ground model M. If σ is a sentence of the forcing language, then for every $G \subset P$ generic overe M,

$$M[G] \models \sigma$$
 if and only if $(\exists p \in G)p \Vdash \sigma$.

Remark: In the left-hand-side one interprets the constants of the forcing language according to G.

Working in M, we don't know what G is or any object in the extension constructed from G. But: we can comprehend the names for these objects and G. We may also be able to deduce some of the properties of G:

Adding a Cohen real: We know that f^G is a function from ω to $\{0,1\}$. We don't know what $f^G(0)$ because it depends on the choice of G. But we know that $f^G(0) = 0$ if $\{\langle 0,0\rangle\} \in G$ and $f^G(0) = 1$ if $\{\langle 0,1\rangle\} \in G$.

Theorem

There is a generic extension M[G] that satisfies $2^{\aleph_0} > \aleph_1$.

Proof: Find a forcing notion P that adjoins \aleph_2 Cohen generic reals to the ground model.

Let P the set of all functions p such that

i)
$$dom(p)$$
 is a finite subset of $\omega_2 \times \omega$,

ii) $ran(p) \subset \{0,1\}$,

and let p be stronger that q iff $p \subset q$.

If G is a generic set of conditions, we let $f = \bigcup G$. We claim that

- i) f is a function,
- ii) $dom(p) = \omega_2 \times \omega$,
- i) holds because G is a filter.

ii) holds because the sets $D_{\alpha,n} = \{p \in P : (\alpha, n) \in dom(p)\}$ are dense in *P*, hence *G* meets each of them and so $(\alpha, n) \in dom(f)$ for all $(\alpha, n) \in \omega_2 \times \omega$.

For each, $\alpha < \omega_2$, let $f_\alpha : \omega \to \{0,1\}$ be the function defined as follows:

$$f_{\alpha}(n)=f(\alpha,n).$$

If $\alpha \neq \beta$, then $f_{\alpha} \neq f_{\beta}$, because the set $D = \{p \in P : p(\alpha, n) \neq p(\beta, n) \text{ for some } n\}$ is dense in P and hence $G \cap D \neq \emptyset$. Thus we have a one-to-one mapping $\alpha \mapsto f_{\alpha}$ of ω_2 into $\{0, 1\}^{\omega}$.

Each f_{α} is the generic function of a set $a_{\alpha} \subset \omega$ and therefore P adjoins \aleph_2 many Cohen reals to the ground model.

(There is only the small question left of how to preserve cardinals...)

Outlook: CH and the multiverse

Theorem (Hamkins)

The universe V has forcing extensions

- 1. V[G], collapsing no cardinals, such that $V[G] \models \neg CH$.
- 2. V[H], adding no new reals, such that $V[H] \models CH$.

"On the multiverse view, consequently, CH is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extend to which it holds and fails in the multiverse....." Thank You!