Equilibrium States of Interacting Particle Systems

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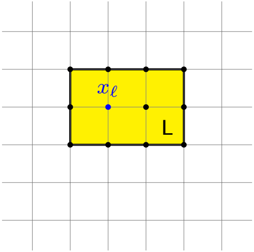
A Talk in the Framework of
"Konstanz Women in Mathematics"
February 2016

Motivation

A particular aim of statistical mechanics is to study the macroscopic behaviour of a system, knowing the behaviour of the microscopic states $x = (x_\ell)_{\ell \in \mathbb{Z}^d}$ given by collections of Ξ-valued random variables.



The equilibrium states of the system are heuristically described by probability measures of the form

$$\mu = \text{"}\frac{1}{Z} e^{-\beta H(x)} dx\text{"}.$$

However, for an infinite configuration $x = (x_\ell)_{\ell \in \mathbb{Z}^d}$, the Hamiltonian $H(x)$ is not well-defined and thus, the definition of μ makes no sense.

Dobrushin-Lanford-Ruelle Approach

Idea: construct probability measures μ on $(\Xi)^{\mathbb{Z}^d}$ with prescribed conditional probabilities given by the family of stochastic kernels

$$\pi_{\mathsf{L}}(A|y) = \frac{1}{Z_{\mathsf{L}}(y)} \int_{(\Xi)^{\mathbb{Z}^d}} \mathbb{1}_A(x) \exp\left\{ -\beta H_{\mathsf{L}}(x_{\mathsf{L}}|y)\right\} \otimes_{\ell\in\mathsf{L}} dx_\ell \otimes_{\ell'\in\mathsf{L}^c} \delta_{y_{\ell'}}.$$

Main problems (since 1970):

- existence (Dobrushin, Ruelle, ...);
- uniqueness/ non-uniqueness $\Longrightarrow$ phase transitions
  - finite or compact Ξ (Dobrushin, Ruelle, ...);
  - non-compact Ξ (Lebowitz and Presutti-for a particular model; Dobrushin and Pechersky (1983) - in the general case).

Phase Transitions - The 2D-Ising Model

- the configuration space is $X := \{-1,+1\}^{\mathbb{Z}^2}$.
- the local energy of x with boundary condition y

$$H_{\mathsf{L}}(x_{\mathsf{L}}|y) := J \sum_{\substack{\ell\sim\ell',\\ \ell,\ell'\in\mathsf{L}}} x_\ell x_{\ell'} + J \sum_{\substack{\ell\sim\ell',\ell\in\mathsf{L},\\ \ell'\notin\mathsf{L}}} x_\ell y_{\ell'} + h\sum_{\ell\in\mathsf{L}} x_\ell$$

- the system of conditional distributions $\Pi = \{\pi_{\mathsf{L}}(\cdot|y)\}_{\mathsf{L}\Subset\mathbb{Z}^2, y\in X}$, where

$$\pi_{\mathsf{L}}^{\beta}(B|y) := \frac{1}{Z_{\mathsf{L}}^{\beta}} \int_{X_{\mathsf{L}}} \mathbf{1}_B(x_{\mathsf{L}} \times y_{\mathsf{L}^c}) \exp\left\{-\beta H_{\mathsf{L}}(x_{\mathsf{L}}|y)\right\} \nu_{\mathsf{L}}(dx_{\mathsf{L}})$$

Theorem

For all β large enough and $h = 0$, there exist two pure limit Gibbs distributions for the 2D ferromagnetic($J > 0$) Ising model.

Outline

A Strategy for Solving the Existence Problem

Main question: Given a specification Π, does there exist a Gibbs measure consistent with Π?

Strategy: Introduce a topology on $\mathcal{P}(X)$, pick a boundary condition $y \in X$ and show that

(I) the net $\{\pi_\mathsf{L}(\cdot|y)\}_\mathsf{L}$ has a cluster point with respect to the chosen topology;

(II) each cluster point of $\{\pi_\mathsf{L}(\cdot|y)\}_\mathsf{L}$ is consistent with Π.

Trick: Choosing the best-suited topology on $\mathcal{P}(X)$, which in this case turns our to be the *topology of local convergence*.

Dobrushin's Existence Criterion

If the one-point specification associated to Π satisfies the condition below, then (I) is satisfied. For (II) some continuity assumption is also needed .

Compactness Condition: There exist a compact function $h : \Xi \to \overline{\mathbb{R}}_+$ and nonnegative constants C and $I_{\ell\ell'}$, $\ell \neq \ell'$ such that

(i) The matrix $I = (I_{\ell\ell'})_{\ell,\ell' \in \mathsf{V}}$ satisfies

$$\|I\|_0 := \sup_{\ell} \sum_{\ell' \neq \ell} I_{\ell\ell'} < 1.$$

(ii) For all $\ell \in \mathsf{V}$ and $y \in X$

$$\int_X h(x_\ell)\pi_\ell(dx_\ell|y) \leq C + \sum_{\ell' \neq \ell} I_{\ell\ell'} h(x_{\ell'}).$$

Outline

Approaches in Solving the Uniqueness Problem

- Dobrushin classical criterion, Dobrushin-Pechersky, Dobrushin-Shlosman;
- exponential decay of correlations;
- exponential relaxation of the corresponding Glauber dynamics, expressed by means of the log-Sobolev and Poincaré inequalities for $\pi(dx|y)$;
- Ruelle's superstability method.

Most methods work only in the case of a *compact* spin space. We will focus on the Dobrushin-Pechersky criterion, which can be applied also to more general spin spaces.

Contraction Condition

Assume that π satisfies

$$d(\pi_\ell^x, \pi_\ell^y) \le \sum_{\ell' \in \partial\ell} \kappa_{\ell\ell'} \mathbb{1}_{\neq}(x_{\ell'}, y_{\ell'}), \tag{CC}$$

for all $\ell \in \mathsf{V}$ and $x, y \in X_\ell(h, K)$, where $\kappa = (\kappa_{\ell\ell'})_{\ell,\ell' \in \mathsf{V}}$ has positive entries and null diagonal such that

$$\bar{\kappa} := \sup_{\ell \in \mathsf{V}} \sum_{\ell' \in \partial\ell} \kappa_{\ell\ell'} < 1.$$

For a constant $K > 0$, $\ell \in \mathsf{V}$ and a measurable function $h : \Xi \to \mathbb{R}_+ := [0, +\infty)$, we set

$$X_\ell(h, K) = \{x \in X : h(x_\ell) \le K \text{ for all } \ell \in \partial\ell\}.$$

Integrability Condition

Moreover, suppose h satisfies the following integrability condition

$$\pi_\ell^x(h) \leq 1 + \sum_{\ell' \in \partial\ell} c_{\ell\ell'} h(x_{\ell'}), \tag{IC}$$

for all $\ell \in \mathsf{V}$ and $x \in X$, where $c = (c_{\ell\ell'})_{\ell,\ell' \in \mathsf{V}}$ has positive entries and null diagonal such that

$$\bar{c} := \sup_{\ell \in \mathsf{V}} \sum_{\ell' \in \partial\ell} c_{\ell\ell'} < \mathfrak{C}(graph) < 1.$$

We introduce the set of *tempered measures* $\mathcal{M}(\pi, h)$ consisting of all measures $\mu \in \mathcal{M}(\pi)$ for which

$$\sup_\ell \int_X h(x_\ell)\mu(dx) < \infty.$$

The Uniqueness Result

Theorem

For each $K > K_*(graph)$ *and* $\pi \in \Pi(h, K, \kappa, c)$*, the set* $\mathcal{M}(\pi, h)$ *contains at most one element.*

The proof of the theorem follows immediately from

Lemma

Let $\mu_1, \mu_2 \in \mathcal{M}(\pi, h)$ *and* $\nu \in \mathcal{C}(\mu_1, \mu_2)$ *such that*

$$\gamma(\nu) := \sup_{\ell \in \mathsf{V}} \int_X \int_X \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell)\nu(dx^1, dx^2) = 0.$$

Then $\mu_1 = \mu_2$.

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Then $\mu_1 = \mu_2$.

Comparison with Dobrushin's Classical Criterion

An earlier uniqueness result due to Dobrushin (1968), for Ξ Polish, compact with ρ a metric that makes Ξ complete, requires that the following interdependence matrix be $\ell^\infty-$contractive, i.e.

$$D_{\ell\ell'} := \sup_{\substack{y^1, y^2 \in X \\ y^1 = y^2 \text{ off } \ell'}} \left\{ \frac{W_\rho(\pi_\ell^{y^1}, \pi_\ell^{y^2})}{\rho(y^1_{\ell'}, y^2_{\ell'})} \right\} < 1, \; \ell \neq \ell'.$$

Advantages of the DP approach:

- one needs to check the condition of weak dependence not for all boundary conditions (like here), but only for such $y \in X$ whose components y_ℓ lye in a certain ball in Ξ;
- it can also be applied for non-compact spins and for pair-potentials with more than quadratic growth.

Decay of Correlations for Gibbs measures

Theorem

Let π and K be as in the previous theorem and $\mathcal{M}(\pi, h)$ be nonempty, hence containing a single state μ. Consider bounded functions $f, g : X \to \mathbb{R}_+$, such that f is $\mathcal{B}(\Xi_{\ell_1})$-measurable and g is $\mathcal{B}(\Xi_{\ell_2})$-measurable. Then there exist positive C_K and α_K, dependent on K only, such that

$$|\mathrm{Cov}_\mu(f; g)| \leq C_K \|f\|_\infty \|g\|_\infty \exp\left[-\alpha_K \delta(\ell_1, \ell_2)\right], \quad \ell_1, \ell_2 \in \mathsf{L}$$

Outline

Configuration Spaces

- System of identical particles (or molecules of gas) in $\mathbb{R}^d$ interacting via a pair potential $V(x,y)$ with certain stability properties

$$H(\gamma) := \sum_{\{x,y\}\subset\gamma} V(x,y) \in \mathbb{R}, \quad \gamma \in \mathit{\Gamma}$$

- $\mathbb{R}^d \ni x$ – position of each particle
- $\mathcal{B}_c(\mathbb{R}^d)$ - family of all bounded Borel sets in $\mathbb{R}^d$
- $\mathit{\Gamma}$ – configuration space consisting of all locally finite subsets γ in $\mathbb{R}^d$

$$\mathit{\Gamma} := \left\{ \gamma \subset \mathbb{R}^d \,\middle|\, |\gamma_\Lambda| < \infty, \quad \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}$$

  $|\gamma_\Lambda|$ is the number of points in $\gamma_\Lambda := \gamma \cap \Lambda$

- γ is identified with the positive Radon measure $\sum_{x\in\gamma} \delta_x$

Poisson Measure

Poisson random point field $\pi_{z\sigma}$ on Γ describes the state of an ideal gas

- $z > 0$ – chemical activity
- $\sigma(\mathrm{d}x)$ – locally finite non-atomic measure on $\mathbb{R}^d$, $\sigma(\mathbb{R}^d) = \infty$,
- σ-Poisson measure $\lambda^{\Lambda}_{z\sigma}$ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$

$$\int_{\Gamma_\Lambda} F(\gamma_\Lambda)\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) := F(\{\varnothing\}) +$$

$$+ \sum_{n\in\mathbb{N}} \frac{z^n}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\})\mathrm{d}\sigma(x_1)\ldots\mathrm{d}\sigma(x_n), \quad \forall F \in L^\infty(\Gamma_\Lambda)$$

- probability measure $\pi^{\Lambda}_{z\sigma} := e^{-z\sigma(\Lambda)}\lambda^{\Lambda}_{z\sigma}$ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$
- Poisson measure $\pi_{z\sigma} \in \mathcal{P}(\Gamma)$ is the projective limit of $\pi^{\Lambda}_{z\sigma}$, i.e.

$$\pi_{z\sigma} := \mathbb{P}^{-1}_{\Lambda} \circ \pi^{\Lambda}_{z\sigma}, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$$

- *Interpretation:* for disjoint $(\Lambda_j)^N_{j=1}$, the variables $|\gamma_{\Lambda_j}|$ are mutually independent and distributed by the Poissonian law with $z\sigma(\Lambda_j)$

Local Gibbs States

- Interaction energy between $\gamma_\Lambda \in \varGamma_\Lambda$ and $\xi_{\Lambda^c} := \xi \cap \Lambda^c$
$$W(\gamma_\Lambda|\xi) := \sum\nolimits_{x\in\gamma_\Lambda, y\in\xi_{\Lambda^c}} V(x,y)$$
- Local Hamiltonians $H_\Lambda(\cdot|\xi) : \varGamma_\Lambda \to \mathbb{R}$
$$H_\Lambda(\gamma_\Lambda|\xi) := H(\gamma_\Lambda) + W(\gamma_\Lambda|\xi), \quad \gamma_\Lambda \in \varGamma_\Lambda$$
- Partition function $1 < Z_\Lambda(\xi) \le \infty$
$$Z_\Lambda(\xi) := \int_{\varGamma_\Lambda} \exp\left\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) = 1 + z +$$
$$+ \sum_{n\ge 2} \frac{z^n}{n!} \int_{\Lambda^n} \exp\left\{-\beta H_\Lambda(\{x_1,\dots,x_n\}|\xi)\right\} \mathrm{d}\sigma(x_1)\dots\mathrm{d}\sigma(x_n) \ge 1$$
- Local Gibbs states $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ with boundary conditions $\xi \in \varGamma$ $=$ probability measures on $(\varGamma_\Lambda, \mathcal{B}(\varGamma_\Lambda))$ provided $Z_\Lambda(\xi) < \infty$
$$\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi) := [Z_\Lambda(\xi)]^{-1} \exp\left\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\right\} \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda)$$

Strategies for Studying μ

- Stability Condition: allows to construct $\mu \in \mathcal{G}$ at small β and z (by *cluster expansions* or *Kirkwood-Salsburg equation*; see Ruelle '69).
- Ruelle's Superstability: proves existence at all β and z via *à-priori* bounds on *correlation functions* (i.e., certain moments) of Gibbs measures (see Ruelle '70). *Ruelle's bound* on correlation functions $\Rightarrow$ convergence $\pi_{\Lambda_N}(\mathrm{d}\gamma|\emptyset) \to \mu \in \mathcal{G}$ *locally setwise.*
  Highly nontrivial, combinatorial technique.
- Dobrushin's approach: by reduction to *lattice systems* and use of *Dobrushin's existence criterion* (1970)
- Kondratiev, Pasurek, Röckner develop an elementary technique of getting existence and *à-priori* bounds for $\mu \in \mathcal{G}^{\mathrm{t}}$; its conceptual difference is a systematic use of (infinite dimensional) Stochastic Analysis.

Uniqueness due to small z

Theorem

Under some assumptions on the pair potential W, for fixed β and small enough z, the set of Gibbs measures is a singleton.

Strategy of proof: partition $\mathbb{R}^d = \bigsqcup_{k\in\mathbb{Z}^d} Q_{gk}$ by equal cubes centred at points gk

$$Q_{gk} := \left\{ x = (x^{(i)})_{i=1}^d \,\middle|\, g\left(k^{(i)} - 1/2\right) \leq x^{(i)} < g\left(k^{(i)} + 1/2\right) \right\},$$

with edge length $g := \delta/\sqrt{d}$ and $\mathrm{diam}(Q_{gk}) = \delta$ and define an equivalent lattice model on $(\Gamma(\bar{Q_0}))^{\mathbb{Z}^d}$. Show then that in this new model, there exists at most one Gibbs measure. Then show that any Gibbs measure in the initial model corresponds to a Gibbs measure in the new lattice model.

Thank you for your attention!