Greedy Controllability of Linear Dynamical Systems by the Reduced-Basis Method

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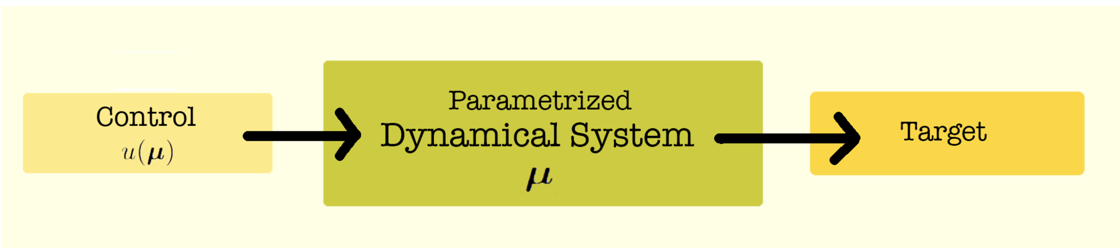
Introduction

Often a dynamical system is characterized by one or more parameters describing physical features of the problem or geometrical configurations of the computational domain. As a consequence, by assuming that the system is controllable, corresponding to different parameter values, a range of optimal controls exists. The goal of the proposed approach is to avoid the computation of a control function for any instance of the parameters. The greedy controllability consists in the selection of the most representative values of the parameter set that allows a rapid approximation of the control function for any desired new parameter value, ensuring that the system is steered to the target within a certain accuracy. By proposing the Reduced Basis method in this framework, the computational costs are drastically reduced and the efficiency of the greedy controllability approach is significantly improved.

Controllability problem



Finite-dimensional linear control system (e.g., after spatial discretization of a PDE):

$$\dot{x}(t) = \mathrm{A}_\mu x(t) + \mathrm{B}_\mu u(t) \text{ for } t \in (0,T] \quad \text{and} \quad x(0) = x^\circ \qquad (\mathbf{DS}_\mu)$$

State variable: $x = x_\mu : [0,T] \to \mathbb{R}^n$ with large n
Control variable: $u = u_\mu : [0,T] \to \mathbb{R}^m$ with $m \le n$
Parameter set: $\mathscr{D}_{\text{ad}} = \{\mu \in \mathbb{R}^{\wp} \,|\, \mu_a \le \mu \le \mu_b \text{ in } \mathbb{R}^{\wp}\}$, i.e., compact
System matrices: $\mathscr{D}_{\text{ad}} \ni \mu \mapsto \mathrm{A}_\mu \in \mathbb{R}^{n\times n}$ and $\mathscr{D}_{\text{ad}} \ni \mu \mapsto \mathrm{B}_\mu \in \mathbb{R}^{n\times m}$ Lipschitz-continuous

Goal: Given a final target $x^1 \in \mathbb{R}^n$ find a control $u = u_\mu$ such that the state $x = x_\mu$ of $(\mathbf{DS}_\mu)$ satisfies

$$x_\mu(T) = x^1 \quad \text{for every } \mu \in \mathscr{D}_{\text{ad}}$$

Assumption: System $(\mathbf{DS}_\mu)$ is controllable for all values of $\mu \in \mathscr{D}_{\text{ad}}$.
Solution approach: If $(\varphi_\mu, \varphi_\mu^\circ)$ solves the linear-quadratic problem

$$\begin{cases} \min J_\mu(\varphi,\varphi^\circ) = \frac{1}{2}\int_0^T \|\mathrm{B}_\mu^\top \varphi(t)\|^2\,\mathrm{d}t + \langle x^1, \varphi^\circ\rangle + \langle x^\circ, \varphi(0)\rangle \\ \text{s.t. } \dot{\varphi}(t) = -\mathrm{A}_\mu^\top \varphi(t) \text{ for } t \in [0,T) \quad \text{and} \quad \varphi(T) = \varphi^\circ \end{cases} \qquad (\mathbf{QP}_\mu)$$

then $u_\mu = \mathrm{B}_\mu^\top \varphi_\mu$ is the control that steers the solution $x = x_\mu$ of

$$\dot{x}(t) = \mathrm{A}_\mu x(t) + \mathrm{B}_\mu u_\mu(t) \text{ for } t \in (0,T] \quad \text{and} \quad x(0) = x^\circ$$

to the desired target x^1;
Numerical strategy: apply CG method to minimize $\hat{J}_\mu(\varphi^\circ) = J_\mu(\varphi_\mu(\varphi^\circ), \varphi^\circ)$
Problem: computationally too expensive

Greedy Controllability

Goal [Lazar/Zuazua'16] Given a tolerance $\varepsilon > 0$ find $N(\varepsilon)$ parameters $\mu_1, \ldots, \mu_{N(\varepsilon)}$, so that for all $\mu \in \mathscr{D}_{\text{ad}}$ the associated control $u_\mu^\star = \sum_{j=1}^{N(\varepsilon)} \alpha_j u_j$ yields

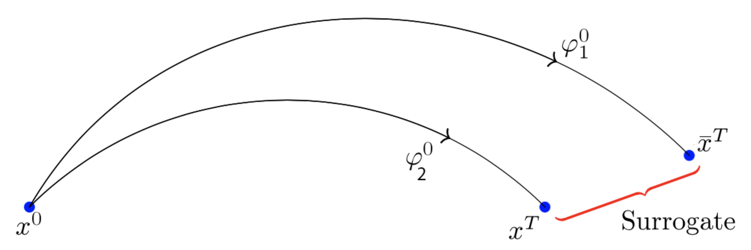
$$\|x_\mu^\star(T) - x^1\| \le \varepsilon \quad \text{for any chosen } \mu \in \mathscr{D}_{\text{ad}}$$

for the state $x = x_\mu^\star$ solving $\dot{x}(t) = \mathrm{A}_\mu x(t) + \mathrm{B}_\mu u_\mu^\star(t)$ $\quad \to N(\varepsilon)$ as small as possiblle.

Greedy controllability – Offline: For given tolerance $\varepsilon > 0$ select (with a greedy approach) $\mu_1, \ldots, \mu_{N(\varepsilon)} \in \mathscr{D}_{\text{ad}}^{\text{gr}}$ ($\mathscr{D}_{\text{ad}}^{\text{gr}} \subset \mathscr{D}_{\text{ad}}$ discrete training set) and compute associated $u_1, \ldots, u_{N(\varepsilon)}$.

Greedy algorithm (offline)
[1] Choose $\mu_1 \in \mathscr{D}_{\text{ad}}^{\text{gr}}$;
[2] Compute $(\varphi_1, \varphi_1^\circ)$ by solving $(\mathbf{QP}_\mu)$ with $\mu = \mu_1$; set $u_1 = \mathrm{B}_{\mu_1}^\top \varphi_1$, $\Phi_1^\circ = \text{span}\{\varphi_1^\circ\}$;
[3] Find $\mu_2 = \arg\max\{\text{dist}(\varphi_\mu^\circ, \Phi_1^\circ) \,|\, \mu \in \mathscr{D}_{\text{ad}}^{\text{gr}}\}$;
[4] Compute $(\varphi_2, \varphi_2^\circ)$ by solving $(\mathbf{QP}_\mu)$ with $\mu = \mu_2$; set $u_2 = \mathrm{B}_{\mu_2}^\top \varphi_2$, $\Phi_2^\circ = \text{span}\{\varphi_1^\circ, \varphi_2^\circ\}$;
⋮
[n] **until** $\text{dist}(\varphi_{\mu_N}^\circ, \Phi_N^\circ) \le \epsilon$ for all $\mu \in \mathscr{D}_{\text{ad}}^{\text{gr}}$

Remark: Step [3] is expansive.
Solution: Replace the distance $\text{dist}(\varphi_\mu^\circ, \Phi_1^\circ)$ with a surrogate distance given by $\text{dist}(x^T, \bar{x}^T)$:



$x^T \leftarrow (\mathbf{DS}_\mu)$ with $u = u_1$ and $\mu = \mu_1$.
$\bar{x}^T \leftarrow (\mathbf{DS}_\mu)$ with $u = u_1$ and $\mu \neq \mu_1$

Greedy controllability – Online: For any $\mu \in \mathscr{D}_{\text{ad}}$ set $u_\mu^\star = \sum_{i=1}^{N(\varepsilon)} \alpha_i u_i$ and solve

$$\dot{x}_\mu^\star(t) = \mathrm{A}_\mu x_\mu^\star(t) + \mathrm{B}_\mu u_\mu^\star(t) \Rightarrow \|x_\mu^\star(T) - x^1\| \le \varepsilon.$$

Reduced-Order Greedy controllability

Remark: the exploration of the parameter domain require repetitive evaluation of the system $\to$ use a reduced order state system (Reduced Basis Method).

Reduced-Order Greedy algorithm (offline)
[1] The initial basis is composed by the target function x^1.
[2] Run the Greedy algorithm (with repetitive evaluations of the reduced systems) and select μ_{next}.
[3] Find the optimal control $u_{\mu_{next}}$ and the state solution $x_{\mu_{next}}(t)$.
[4] Enrich the existing basis with the POD of $x_{\mu_{next}}(t)$.
[5] Repeat step [2] for the selection of the remaining parameter values until the surrogate distance is smaller than the desired tolerance.

Numerical Example

State equation: for $\mu \in \mathscr{D}_{\text{ad}} = [0.5, 4] \subset \mathbb{R}$

$$\begin{aligned} \dot{y}(t,\boldsymbol{x}) - \mu\Delta y(t,\boldsymbol{x}) + \beta\cdot\nabla y(t,\boldsymbol{x}) &= 0 && \text{for } (t,\boldsymbol{x}) \in Q = (0,T)\times\Omega \\ \mu\frac{\partial y}{\partial n}(t,\boldsymbol{s}) &= \sum_{i=1}^m u_i(t)\chi_i(\boldsymbol{s}) && \text{for } (t,\boldsymbol{s}) \in \Sigma = (0,T)\times\partial\Omega \\ y(0,\boldsymbol{x}) &= y^\circ(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in \Omega = (0,2)\times(0,1) \subset \mathbb{R}^2 \end{aligned} \qquad (\mathbf{S}_\mu)$$

Control shape functions: $\chi_i = \chi_{\Gamma_i}$ for $1 \le i \le m$ with $\partial\Omega = \dot{\bigcup}_{i=1}^m \Gamma_i$

Finite element (FE) Galerkin discretization of $(\mathbf{S}_\mu)$:

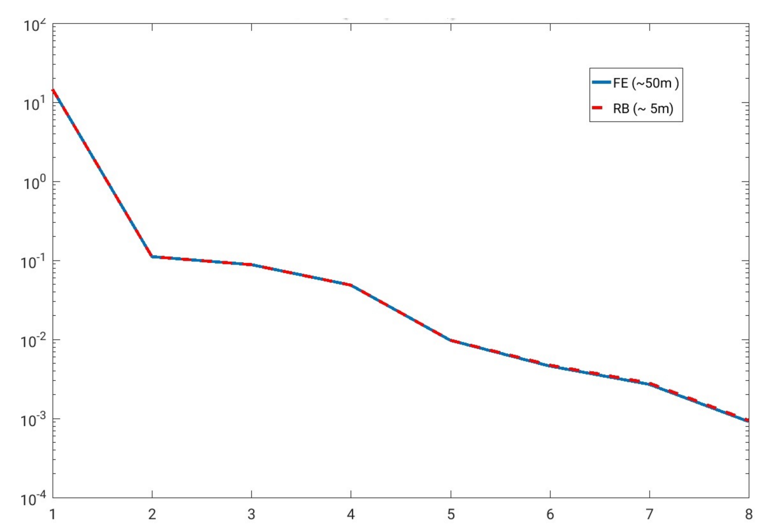
$$\dot{x}(t) = \mathrm{A}_\mu x(t) + \mathrm{B}u(t) \text{ for } t \in (0,T], \quad x(0) = x^\circ$$

Numerical Results:

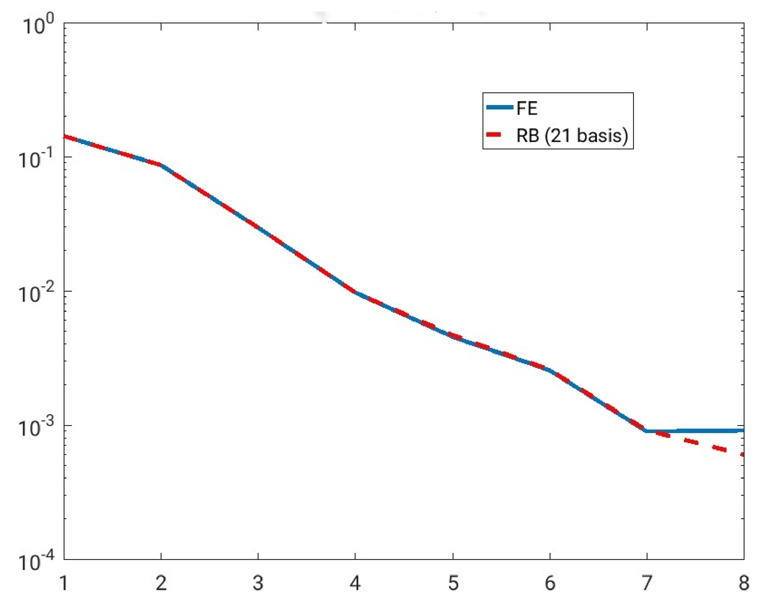
Classical approach:

μ	0.5	1	1.5	2	2.5	3	3.5	4
CG iterations	59	34	28	29	28	24	20	20
CPU time	62 s	38 s	32 s	33 s	32 s	26 s	22 s	22 s
$\|x_\mu(T) - x^1\|$	8.6e-3	2.8e-3	9.6e-3	7.4e-3	8.1e-3	6.9e-3	7.7e-3	6.5e-3

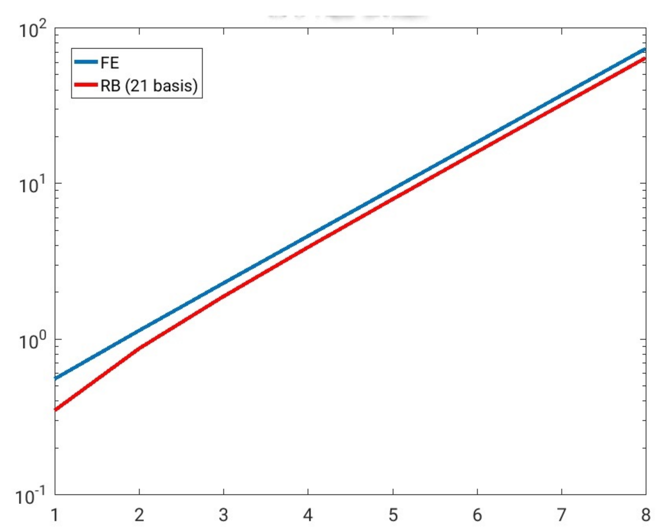
Greedy controllability (offline): 50 minutes (with $N = 18$, $|\mathscr{D}_{\text{ad}}^{\text{gr}}| = 1000$)
Reduced-order Greedy controllability (offline): 5 minutes, speed-up factor ≈ 10



Error decay greedy algorithm (offline)



Error decay greedy algorithm (online)



CPU time (online)

Conclusions

- Greedy controllability approach for parametrized linear dynamical systems.
- Significant speed-up by the reduced-order greedy controllability.

References

- [Fabrini/Iapichino/Volkwein '18]: Reduced-order greedy controllability of finite dimensional linear systems, IFAC PapersOnLine
- [Hesthaven/Rozza/Stamm '16], Certified Reduced Basis Methods for Parametrized Partial Differential Equations, SpringerBriefs in Mathematics.
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Acknowledgemnets

I gratefully acknowledge Konstanz Women in Mathematics for the financial support.