Numerical approximation for optimal control problems via MPC and HJB

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Outline

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Introduction and Motivations

Some history

The method is largely due to the work of Lev Pontryagin and Richard Bellman in the 1950s.
The theory of control analyzes the properties of controlled systems, *i.e.* dynamical systems on which we can act through a control.

Aim

Bring the system from an initial state to a certain final state satisfying some criteria.

Several applications:

Robotics, aeronautics, electrical and aerospace engineering, biological and medical field.

Controlled dynamical system

$$\begin{cases} \dot{y}(t) = F(y(t), u(t)) & (t > 0) \\ y(0) = x & (x \in \mathbb{R}^n) \end{cases}$$

Assumptions on the data

- $u(\cdot) \in \mathcal{U}$: control, where

$$\mathcal{U} := \{u(\cdot) : [0, +\infty[\to U \text{ measurable}\}$$

- $F : \mathbb{R}^n \times U \to \mathbb{R}^n$ is the dynamics, which satisfies:
  - F is continuous respect to (y, u)
  - F is local bounded
  - F is Lipschitz respect to y

$\Rightarrow \exists! y_x(t, u)$, solution of the problem (Caratheodory Theorem).

The infinite horizon problem

Cost functional

$$J(x, u) := \int_0^\infty L(y_x(t, u), u(t))e^{-\lambda t} dt.$$

where $\lambda > 0$ is the *interest rate* and L is the *running cost*.

Goal

We are interested in minimizing this cost functional. We want to find an optimal pair (y^*, u^*) which minimizes the cost functional.

Two possible approaches

Open-loop control

Control expressed as functions of time t (necessary condition - Pontryagin Minimum Principle or direct methods, *e.g.* **gradient method**).
Remark: it cannot take into account errors in the real state of the system due to model errors or external disturbances.

Feedback control

Control expressed as functions of the state system (Dynamic Programming, Model Predictive Control)
Remark: robust to external perturbations.

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Value Function

$$v(y_0) := \inf_{u(\cdot)\in\mathcal{U}} J(y_0, u(\cdot))$$

The **value function** is the unique **viscosity solution** of the Bellman equation associated to the control problem via Dynamic Programming.

Dynamic Programming Principle

$$v(x) = \min_{u\in\mathcal{U}} \left\{ \int_{t_0}^{t} L(y_x(s), u(s))e^{-\lambda s}\, ds + v(y_x(t))e^{-\lambda t} \right\}$$

Hamilton-Jacobi-Bellman

$$\lambda v(x) + \max_{u\in U}\{-f(x,u)\nabla v(x) - L(x,u)\} = 0, x \in \Omega, u \in U$$

Feedback Control

Given $v(x)$ for any $x \in \mathbb{R}^n$, we define

$$u(y_x^*(t))) = arg \min_{u \in U} [F(x,u) \nabla v(x) + L(x,u)]$$

where $y^*(t)$ is the solution of

$$\begin{cases} \dot{y}^*(t) = F(y^*(t), u^*(t)), & t \in (t_0, \infty] \\ y(t_0) = x, \end{cases}$$

Technical difficulties

The bottleneck is the approximation of the value function v, however this remains the main goal since v allows to get back to feedback controls in a rather simple way.

Numerical computation of the value function

The bottleneck of the DP approach is the computation of the value function, since this requires to solve a non linear PDE in high-dimension.

This is a challenging problem due to the huge number of nodes involved and to the singularities of the solution.

Another important issue is the choice of the domain.

Several numerical schemes: finite difference, finite volume, **Semi-Lagrangian** (obtained by a discrete version of the Dynamic Programming Principle).

Semi-Lagrangian discretization of HJB

These schemes are based on the direct discretization of the directional derivative $f(x,u)\nabla v(x)$.

Continuous Version

$$\lambda v(x) = -\max_{u\in U}\{-F(x,u)\cdot Dv(x) - L(x,u)\}$$

Semi-Discrete Approximation (Value Iteration)

Making a discretization in time of the continuous control problem (e.g. using the Euler method):

$$V^{k+1} = \min_{u\in U}\{e^{-\lambda\Delta t}V^k\left(x+\Delta t\,F\left(x,u\right)\right)+\Delta t\,L\left(x_i,u\right)\}$$

Semi-Lagrangian discretization of HJB

Fully discrete SL Value Iteration (VI) scheme

$$\begin{aligned} V^{k+1} &= T(V^k)\,, \quad \text{for } i = 1, \ldots, N\,, \text{with} \\ \left(T(V^k)\right)_i &\equiv \min_{u\in U}\{\beta I_1[V^k](x_i + \Delta t\, F(x_i, u)) + \mathrm{L}(x_i, u)\}. \end{aligned}$$

- Fixed grid in $\Omega \subset \mathbb{R}^n$ bounded, Steps Δx. Nodes: $\{x_1, \ldots, x_N\}$
- Stability for large time steps Δt.
- Error estimation: (Falcone/Ferretti '97)
- Slow convergence, since $\beta = e^{-\lambda \Delta t} \to 1$ when $\Delta t \to 0$

PROBLEM: Find a reasonable computational domain.

Value Iteration for infinite horizon optimal control (**VI**)

Require: Mesh G, Δt, initial guess V^0, tolerance ϵ.

1: **while** $\|V^{k+1} - V^k\| \geq \epsilon$ **do**

2: **for** $x_i \in G$ **do**

3:

$$V_i^{k+1} = \min_{u \in U}\{e^{-\lambda \Delta t} I\left[V^k\right](x_i + \Delta t\, F(x_i, u)) + \Delta t\, L(x_i, u)\}$$

4: $k = k + 1$

5: **end for**

6: **end while**

Remarks

- VI algorithm converges (slowly) for any initial guess V^0.
- We can provide an error estimate,

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Model Predictive Control

Dynamics:

$$\begin{cases} \dot{y}(t) = F(y(t), u(t)) \\ y(0) = y_0 \quad t > 0 \end{cases}$$

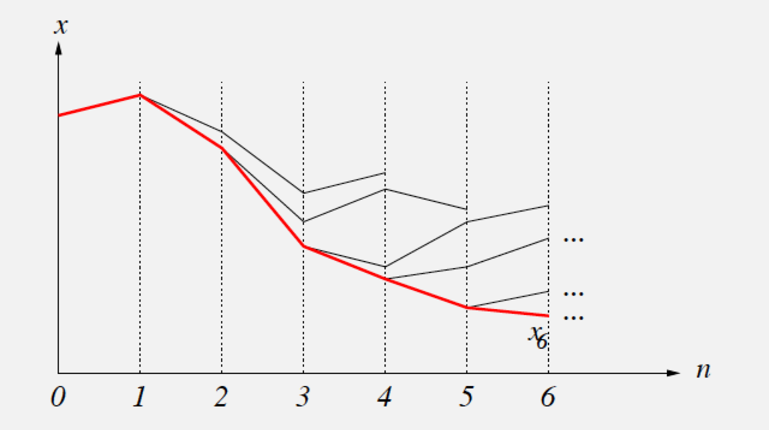
Infinite horizon cost functional:

$$J^{\infty}(u(\cdot)) = \int_0^{\infty} L(y(t; u, y_0))e^{-\lambda t} dt$$

Finite horizon cost functional:

$$J^N(u(\cdot)) = \int_{t_0}^{t_0^N} L(y(t; u, y_0))e^{-\lambda t} dt, \;\; t_0^N = t_0 + N\Delta t, \; N \in \mathbb{N}$$

MPC trajectories (in L. Grüne, J. Pannek, *NMPC*)



black=prediction (obtained with an open loop optimization)

red= MPC closed loop, $\mathrm{y}(t_n) = \mathrm{y}_{\mu^N}(t_n)$

MPC METHOD

solves infinite time horizon problem by means of iterative solution of finite horizon (**N**$\geq$ 2) optimal control problems.

$$\min J^N(u) \;\; \text{s.t.} \;\; u \in \mathcal{U}^N$$

FEEDBACK CONTROL: $\mu^N(y(t)) = u^*(t)$

We obtain a closed loop representation by applying the map μ^N

$$\dot{y} = F\left(y(t), \mu^N(y(t))\right)$$

OPTIMAL...

TRAJECTORIES: $y^*(t_0), \ldots, y^*(t_0^N)$
CONTROLS: $u^*(t_0), \ldots, u^*(t_0^N)$

Advantages and disadvantages

HJB PRO

1. Valid for all problems in any dimension, a-priori error estimates in L^∞.
2. SL schemes can work on structured and unstructured grids.
3. The computation of feedbacks is almost built in.

HJB CONS

1. "Curse of dimensionality"
2. Computational cost and huge memory allocations.

MPC PRO

1. Easy to implement,short computational time.
2. It can be applied to high dimensional problems.
3. Feedback controls.

MPC CONS

1. Approximate feedback just along one trajectory.
3. With a short horizon $\mapsto$ sub-optimal trajectory.

IDEA

Try to combine the advantages of the two methods in order to obtain an efficient algorithm.

The approximation of the HJB equation needs to restrict the computational domain Ω to a subset of $\mathbb{R}^n$. The choice of the domain is totally arbitrary.

GOAL:

To find a reasonable way to compute Ω

QUESTION:

How to compute the computational domain?

SOLUTION:

An inexact MPC solver may provide a reference trajectory for our optimal control problem.

Localized DP Algorithm

Algorithm

Start: Inizialization
Step 1: Solve MPC and compute $y_{y_\circ}^{MPC}$ for a given initial condition y_0
Step 2: Compute the distance from $y_{y_0}^{MPC}$ via the Eikonal equation
Step 3: Select the tube Ω_ρ with distance ρ with respect to $y_{y_0}^{MPC}$
Step 4: Compute the constrained value function v^{tube} in Ω_ρ via HJB
Step 5: Compute the optimal feedbacks and trajectory using v^{tube}.
End

A posteriori error-estimate for the choice of ρ

$$\|y^*(t) - y^{\mathrm{MPC}}(t)\| \leq \frac{C\frac{e^{\lambda T}}{\sigma}\|\zeta\|}{\|y^{\mathrm{MPC}}(t)\|} \qquad \forall t$$

where ζ is a perturbation which is independent of y^*, C is a computable constant.

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Heat equation

$$\begin{cases} y_t(t,\mathbf{x}) - \nu \Delta y(t,\mathbf{x}) = 0, & \text{a.e. in } Q, \\ \nu \frac{\partial y}{\partial \mathbf{n}}(t,\mathbf{s},\nu) = \sum_{i=1}^m u_i(t) b_i(\mathbf{s}), & \text{a.e. on } \Sigma \\ y(0,\mathbf{x},\nu) = y_\circ(\mathbf{x}), & \text{a.e. in } \Omega \end{cases} \tag{1}$$

where $Q = (0,\infty) \times \Omega$, $\Sigma = (0,\infty) \times \Gamma$, $\Gamma = \partial\Omega$, $b_i : \Gamma \to \mathbb{R}$, given shape functions, $\nu \in \mathbb{R}$ fixed parameter.

Cost Functional

$$\min_{u \in \mathcal{U}} J(u) = \min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \int_0^\infty e^{-\lambda t} \left\| y(t) - \bar{y} \right\|^2_{L^2(\Omega)} dt + \frac{1}{2} \sum_{i=1}^m \sigma_i \left\| u_i \right\|^2_{L^2(0,\infty)} \right\}$$

where $\mathcal{U} := \{u \in L^2(0,+\infty;\mathbb{R}^m) | u_a \le u(t) \le u_b \ \forall t)\}$

Goal: find a control which steers the trajectories to a desired state.

Finite Element Formulation

Considering the weak formulation and using a FE scheme of piecewise linear basis functions $\varphi_i, i = 1, \ldots, \mathcal{N}$, we lead to:

$$\mathrm{M}\mathrm{y}'(t) = \nu \mathrm{A}\mathrm{y}(t) + \mathrm{B}u(t) \text{ for } t \in (0, T], \quad \mathrm{y}(0) = \mathrm{y}_\circ. \tag{2}$$

where

$$\mathrm{A} = \big(\big(a(\varphi_j, \varphi_i)\big)\big)_{1 \le i,j \le \mathcal{N}}, \quad \mathrm{B} = \big(\big(\langle b_j, \varphi_i \rangle_{L^2(\Gamma)}\big)\big)_{1 \le i \le \mathcal{N}, 1 \le j \le M},$$

$$\mathrm{y}(t) = \big(y_i^{\mathcal{N}}(t)\big)_{1 \le i \le \mathcal{N}}, \qquad y_\circ = \big(y_{\circ i}^{\mathcal{N}}\big)_{1 \le i \le \mathcal{N}}.$$

Proper Ortoghonal Decomposition (Kunisch, Hinze, Volkwein,...)

We have to solve an optimal control problem for a large system of ODEs $\mapsto$ POD decomposition allows to reduce the number of variables to approximate partial differential equations.

Proper Orthogonal Decomposition and SVD

Given snapshots $Y = [y_1, \ldots, y_n] \in \mathbb{R}^{\mathcal{N} \times n}$ with $y_i = y(t; \cdot)$.

We look for an orthonormal basis $\{\psi_i\}_{i=1}^{\ell}$ in $\mathbb{R}^m$ with $\ell \leq \min\{n, m\}$ s.t.

$$J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{m} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle \psi_i \right\|_{L^2(\Omega)}^2 = \sum_{i=\ell+1}^{d} \sigma_i^2$$

reaches a minimum where $\{\alpha_j\}_{j=1}^{n} \in \mathbb{R}^+$.

$$\min J(\psi_1, \ldots, \psi_\ell) \quad s.t. \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

Singular Value Decomposition: $Y = \Psi \Sigma V^T$.

For $\ell \in \{1, \ldots, d = rank(Y)\}$, $\{\psi_i\}_{i=1}^{\ell}$ are called *POD basis* of rank ℓ.
Ansatz:

$$y(x, t) \approx \sum_{i=1}^{\ell} y_i^{\ell}(t) \psi_i(x).$$

Reduced dynamics

The system can be expressed as:

$$\dot{\mathrm{y}}(t) = F\big(\mathrm{y}(t), u(t)\big) \text{ for } t > 0, \quad \mathrm{y}(0) = \mathrm{y}_0,$$

with $F(\mathrm{y}) = \mathrm{M}^{-1}(\mathrm{A}\mathrm{y} + \mathrm{B}u + f) \in \mathbb{R}^\ell$, for $\mathrm{y} \in \mathbb{R}^\ell$, $u \in \mathbb{R}^m$.

Reduced Cost Functional

$$J^\ell(u) = \int_0^\infty e^{-\lambda t} L\big(\mathrm{y}(t), u(t)\big)\, dt$$

with:

$$L(\mathrm{y}, u) = \frac{1}{2}\Big((\mathrm{y} - \bar{\mathrm{y}})^\top \mathrm{M}(\mathrm{y} - \bar{\mathrm{y}}) + \sigma u^\top u\Big) \quad \text{for } (\mathrm{y}, u) \in \mathbb{R}^\ell \times \mathbb{R}^m.$$

Test 1

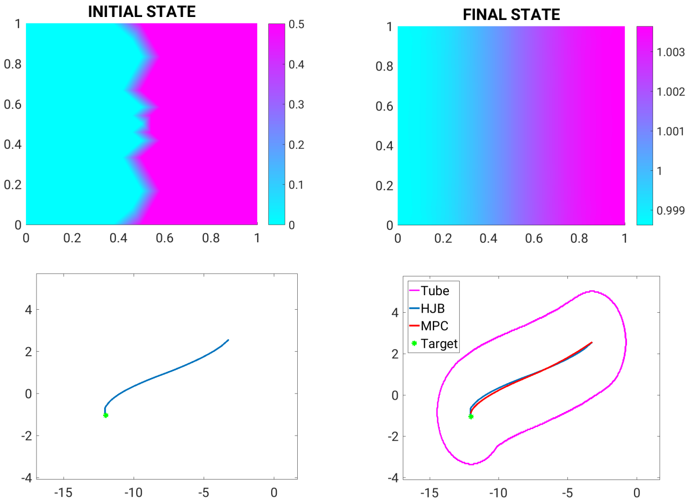
Dynamics and Cost Functional

$$\begin{cases} y_t(t,\mathbf{x}) - \nu\Delta y(t,\mathbf{x}) = 0, & \text{a.e. in } Q, \\ \nu \frac{\partial y}{\partial \mathbf{n}}(t,\mathbf{s},\nu) = \sum_{i=1}^m u_i(t) b_i(\mathbf{s}), & \text{a.e. on } \Sigma \\ y(0,\mathbf{x},\nu) = y_\circ(\mathbf{x}), & \text{a.e. in } \Omega \end{cases}$$

$$J(u) = \frac{1}{2}\int_0^\infty e^{-\lambda t} \|y(t) - \bar{y}\|^2_{L^2(\Omega)}\, dt + \frac{1}{2}\sum_{i=1}^m \sigma_i \|u_i\|^2_{L^2(0,\infty)}$$

Parameters:

Snapshots: $Q = [0,1]\times\Omega$, $\Omega = [0,1]^2$,$\nu = 1$, $\mu = 0$, $m = 1$, $y_\circ(\mathbf{x}) = 0\chi_{[0,0.5]} + 0.5\chi_{[0.5,1]}$ $\Delta x = 0.5$ and $\Delta t = 0.02$, $u(t) \equiv 1$, $\ell = 2$ (number of POD basis).
MPC: $\Delta t_{MPC} = 0.02, N = 5, \lambda = 1, u(t) \in [-4,4], \bar{y} \equiv 1.$
HJB: $\Delta t_{HJB} = 0.05, \Delta t_{traj} = 0.02$, $\#$ contr value$= 21$, $\#$ contr traj$= 81$


INITIAL STATE
FINAL STATE
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
0.5
0.4
0.3
0.2
0.1
0
1.003
1.002
1.001
1
0.999
4
2
0
-2
-4
-15
-10
-5
0
Tube
HJB
MPC
Target

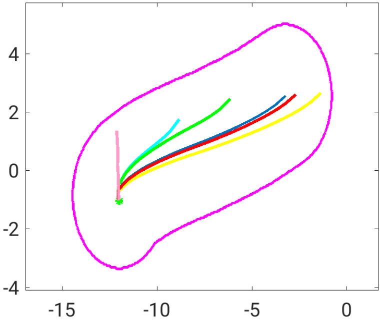
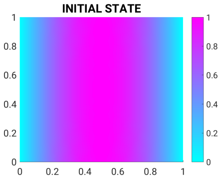


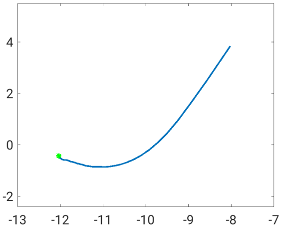
Figure: optimal trajectories for different intial profiles.

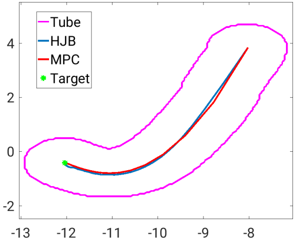
$\lambda = 1$	**MPC N=5**	**HJB in Ω_ρ**	**HJB in Ω**
CPU time [s]	37	72	198
$J(u)$	0.05	0.05	0.05
$\|y(T) - \bar{y}\|_{L^2}$	0.1	0.02	0.01

Test 2

Parameters: same as Test 1, different initial profile: $y_\circ(\mathbf{x}) = \sin(\pi \mathbf{x})$.







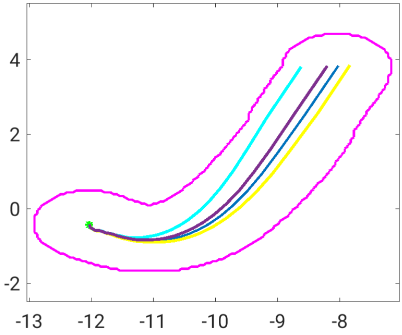


Figure: optimal trajectories for different intial profiles.

$\lambda = 1$	**MPC N=5**	**HJB in Ω_ρ**	**HJB in Ω**
CPU time [s]	72	88	205
$J(u)$	0.013	0.012	0.012
$\Vert y(T) - \bar{y}\Vert_{L^2}$	0.02	0.003	0.003

CONCLUSIONS

- Local version of dynamic programming approach for infinite horizon optimal control problems
- The coupling between MPC and DP methods can produce rather accurate results
- Computational speed-up

FUTURE DIRECTIONS

- Advection term (using more POD basis)
- A posteriori-error estimate including the advection term in the equation.

References

- A. Alla, G. Fabrini, M. Falcone, *Coupling MPC and DP methods for an efficient solution of optimal control problems*, Conference Proceedings of IFIP 2015.
- M. Bardi, I. Capuzzo Dolcetta,*Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, 1997.
- G. Fabrini, M. Falcone, S. Volkwein, *Coupling MPC and HJB for the computation of POD-basis feedback laws*, to appear in ENUMATH proceedings, 2017.
- M. Falcone, R. Ferretti, *Semi-Lagrangian Approximation schemes for linear and Hamilton-Jacobi equations*, SIAM 2013.
- L. Grüne, J. Pannek, *Nonlinear Model Predictive Control,* Springer London, 2011.
- V. Gaitsgory, L. Grüne, N. Thatcher, *Stabilization with discounted optimal control*, 2014.
- S. Volkwein, F. Trölzsch *POD a-posteriori error estimates for linear-quadratic optimal control problems*, Computational Optimization and Applications, 44:83-115, 2009.
- S. Volkwein. *Model Reduction using Proper Orthogonal Decomposition*, Lecture Notes, University of Konstanz, 2013.

THANK YOU FOR YOUR ATTENTION