Elliptic Operators with Unbounded Coefficients

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8th June 2018

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Motivation

Consider the Stochastic Differential Equation

$$\begin{cases} dX(t,x) = F(X(t,x))dt + Q(X(t,x))dB(t), & t > 0, \\ X(0) = x \in \mathbb{R}^N \ , \end{cases} \tag{SDE}$$

$B(t)$ is a standard Brownian motion and $Q(x) \in \mathcal{L}(\mathbb{R}^N)$.

- Probabilistic model of the physical process of diffusion
- Model in mathematical finance
- Model in biology

Motivation

Consider the Stochastic Differential Equation

$$\begin{cases} dX(t,x) = F(X(t,x))dt + Q(X(t,x))dB(t), & t > 0, \\ X(0) = x \in \mathbb{R}^N \, , \end{cases}$$

$$u(t,x) := \mathbb{E}(\varphi(X(t,x)), \quad \varphi \in C_b(\mathbb{R}^N)$$

Then $\Rightarrow$ u solves Kolomogorov/Fokker-Planck equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u(t,x) + F(x) \cdot \nabla u(t,x), & t > 0, \\ u(0,x) = \varphi(x), \qquad x \in \mathbb{R}^N, \end{cases} \tag{FP}$$

$(a_{ij}(x)) := \frac{1}{2} Q(x) Q(x)^*$ can be unbounded.

Examples

- The Black-Scholes equation:

$$
\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\sigma^2}{2} x^2 \dfrac{\partial^2 v}{\partial x^2} - rx \dfrac{\partial v}{\partial x} + rv = 0, & 0 < t \leq T, \\ v(0, x) = (x - K)^+, & x > 0. \end{cases} \tag{BS}
$$

  - $v(t, x) = u(T - t, x)$ is the value of *european* type option on the asset price x at time t;
  - σ is the stock volatility;
  - r is the risk-free rate;
  - $0 < x$ is the underlying asset;
  - K is the prescribed price.

Set $y = \log x$. Then,

$$v(t,x) = x^{-\frac{\alpha-1}{2}} e^{-\frac{(\alpha+1)^2}{8}\sigma^2 t} w\left(\frac{\sigma^2}{2}t, \log x\right),$$

where $\alpha = \frac{2r}{\sigma^2}$ and w solves the heat equation

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial y^2} & t > 0,\ y \in \mathbb{R}, \\ w(0,y) = e^{\frac{\alpha-1}{2}y}(e^y - K)^+ & y \in \mathbb{R}. \end{cases}$$

Examples

- Ornstein-Uhlenbeck operator:

$$\begin{cases} dX(t,x) = MX(t,x)dt + dB(t), & t > 0, \\ X(0,x) = x \in \mathbb{R}^N, \end{cases} \tag{OU}$$

$M = (m_{ij}) \in \mathcal{L}(\mathbb{R}^N)$. Set $M_t := \int_0^t e^{sM} e^{sM*}\, ds$. Then

$$\begin{aligned} u(t,x) &: = \mathbb{E}(f(X(t,x))) \\ &= (2\pi)^{-\frac{N}{2}} (\det M_t)^{-\frac{1}{2}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}|M_t^{-1/2}(e^{tM}x - y)|^2} f(y)\, dy \end{aligned}$$

solves the Ornstein-Ulhenbeck equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + Mx \cdot \nabla u(t,x), & t > 0, \\ u(0,x) = f(x) \qquad x \in \mathbb{R}^N. \end{cases} \tag{OU}$$

Semigroups in short

Let X be a Banach space.
A family $(T(t))_{t\geq 0}$ of bounded operators on X is called a *strongly continuous* or C_0*-semigroup* if

- $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$;
- $t \mapsto T(t)x \in X$ is continuous for every $x \in X$.

Semigroups in short

The generator of a strongly continuous semigroup A is defined as

$$D(A) := \{x \in X : t \mapsto T(t)x \text{ is differentiable on } [0, \infty)\}$$

$$Ax := \frac{d}{dt} T(t)x_{|t=0} = \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x).$$

$$T(t) \rightsquigarrow e^{tA}$$

Semigroup approach to initial-boundary value problems

Given X Banach space, $A : D(A) \subset X \to X$ with boundary conditions in $D(A)$

$$(ACP_1) \begin{cases} \frac{\partial u}{\partial t}(t) = Au(t) \\ u(0) = u_0 \end{cases}$$

- $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $X \Leftrightarrow (ACP_1)$ well posed with solution

$$u(t) = T(t)u_0.$$

- Study qualitative properties: positivity, stability, regularity, ...

The resolvent

The **Resolvent Operator** is the operator

$$R(\lambda, A) := (\lambda - A)^{-1} \ , \quad \lambda \in \rho(A) := \{\lambda \in \mathbb{C} \, : \, \lambda - A \to X \text{ is bijective}\}$$

The semigroup is related to the Resolvent Operator
$T(t)$ is a C_0-semigroup, $\|T(t)\| \leq e^{\omega t}$ then $\lambda \in \rho(A)$ for $\mathrm{Re}\lambda > \omega$

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda s} T(s)f \, ds, \quad \|R(\lambda, A)\| \leq \frac{1}{\mathrm{Re}\lambda - \omega} \ .$$

Generation Theorems

- Hille, Yosida, (1948). Let $(A, D(A))$ closed, densely defined and $\lambda \in \rho(A)$ for every $\lambda > \omega$

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega} \Longrightarrow A \text{ generates } \|T(t)\| \leq e^{\omega t}$$

- Lumer, Phillips, (1961).

$$\overline{Rg(\lambda - A)} = X, \; \|(\lambda - A)f\| \geq \lambda\|f\|, \text{ for all } \lambda > 0 \Longrightarrow$$
$$\overline{A} \text{ generates } \|T(t)\| \leq 1$$

Some trivial example

Let $\omega \in \mathbb{R}$ consider the operator $A : \mathbb{R} \to \mathbb{R},\ Ax = \omega x$

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) = \omega x(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

$R(\lambda, A) = (\lambda - A)^{-1}$ solves the equation $\lambda x - \omega x = y \Longleftrightarrow x = \frac{y}{\lambda - \omega}$

$$R(\lambda, A)y = \frac{y}{\lambda - \omega} \text{ and } \rho(A) = \mathbb{C} \setminus \{\omega\}$$

$$\left.\begin{array}{l} A \text{ closed, dense} \\ \lambda \in \rho(A) \text{ if } \lambda > \omega \\ \|R(\lambda, A)\| = \frac{1}{\lambda - \omega} \end{array}\right\} \overset{\text{H. Y. Theorem}}{\Longrightarrow} A \text{ generates } \|T(t)\| \leq e^{\omega t}$$

For $t > 0,\ T(t) : \mathbb{R} \to \mathbb{R}$
Fixing x_0, $T(t)x_0 = x(t)$ is a function in t and solves (1)

Some trivial example

$$T(t)x = e^{\omega t}x$$

Semigroup laws

$$\begin{aligned}&i)\ T(t+s)x = e^{\omega(t+s)}x = e^{\omega t}e^{\omega s}x = T(t)T(s)x\\ &ii)\ T(0)x = e^{\mu 0}x = x\\ &iii)\ \lim_{t\to 0} T(t)x = x\end{aligned}$$

Generator

$$Ax = \lim_{t\downarrow 0}\frac{T(t)x - x}{t} = \lim_{t\downarrow 0}\frac{e^{\omega t}x - x}{t} = \omega x$$

Resolvent

$$R(\lambda, A)y = \frac{y}{\lambda - \omega} = \int_0^\infty e^{-\lambda s}T(s)yds = \int_0^\infty e^{-\lambda s}e^{\omega s}yds$$

Kernel Representation

If the coefficients of the differential operator have suitable regularity, the semigroup has a kernel representation.

$$T(t)f(x) = \int_{\mathbb{R}^N} k(t,x,y)f(y)dy \; .$$

Consider the heat equation $\partial_t u(t,x) = \Delta u(t,x)$, $u(0,x) = f(x)$

$$u(t,x) = T(t)f(x) = \frac{1}{(4\pi t)^{N/2}} \int e^{-\frac{|x-y|^2}{4t}} f(y)dy$$

In this case $k(t,x,y) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-y|^2}{4t}}$

- The behavior of the semigroup depends on the behavior of the kernel
- The kernel is related to the eigenvalues and the ground state of the problem

Elliptic operators with unbounded coefficients

We consider elliptic operators with unbounded coefficients of the form

$$\mathcal{A}u(x) = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij}u(x) + \sum_{i=1}^{N} b_i(x) D_i u(x) + V(x)u(x).$$

The realisation A of $\mathcal{A}$ in $C_b(\mathbb{R}^N)$ with *maximal domain*

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap \bigcap_{1 \le p < \infty} W^{2,p}_{loc}(\mathbb{R}^N) : \mathcal{A}u \in C_b(\mathbb{R}^N)\}$$

$$Au = \mathcal{A}u.$$

$$\begin{cases} \partial_t u(t,x) = Au(t,x) & t > 0, \quad x \in \mathbb{R}^N \\ u(0,x) = f(x) & \qquad\quad\; x \in \mathbb{R}^N, \end{cases} \tag{2}$$

with $f \in C_b(\mathbb{R}^N)$.
To have solution we assume that for some $\alpha \in (0,1)$,

(1) $a_{ij},\ b_i,\ V \in C^{\alpha}_{loc}(\mathbb{R}^N),\ \forall\, i,j = 1,...,N$;

(2) $a_{ij} = a_{ji}$ and

$$\langle a(x)\xi, \xi\rangle = \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq k(x)|\xi|^2$$

$x, \xi \in \mathbb{R}^N$, $k(x) > 0$;

(3) $\exists c_0 \in \mathbb{R}$ s.t.

$$V(x) \leq c_0, \qquad x \in \mathbb{R}^N.$$

Consider the problem on bounded domains

$$\begin{cases} \partial_t u_R(t,x) = A u_R(t,x) & t > 0, \quad x \in B_R \\ u_R(t,x) = 0 & t > 0, \quad x \in \partial B_R \\ u_R(0,x) = f(x) & \qquad\quad\; x \in B_R, \end{cases} \tag{3}$$

with $f \in C_b(\mathbb{R}^N)$.
Then A is uniformly elliptic on compacts of $\mathbb{R}^N$ and (3) admits unique classical solution

$$u_R(t,x) = T_R(t)f(x), \quad t \geq 0,\, x \in \overline{B}_R$$

with $T_R(t)$ analytic semigroup in $C(\overline{B}_R)$.
The infinitesimal generator of $(T_R(t))$ is $(A, D_R(A))$,

$$D_R(A) = \{u \in C_0(\overline{B}_R) \cap \bigcap_{1<p<\infty} W^{2,p}(B_R), : Au \in C(\overline{B}_R)\}.$$

Theorem 1

(i) $(T_R(t))$ *has the integral representation*

$$T_R(t)f(x) = \int_{B_R} p_R(t,x,y)f(y)dy, \quad f \in C(\overline{B}_R),\ t > 0,\ x \in \overline{B}_R$$

with strictly positive kernel $p_R \in C((0,+\infty) \times B_R \times B_R)$.

(ii) $T_R(t) \in \mathcal{L}(L^p(B_R))$ *per ogni* $t \geq 0$ *e per ogni* $1 < p < +\infty$;

(iii) $T_R(t)$ *is contractive in* $C(\overline{B}_R)$;

(iv) *for all fixed* $y \in \overline{B}_R$, $p_R(\cdot,\cdot,y) \in C^{1+\frac{\alpha}{2},2+\alpha}([s,t_0] \times \overline{B}_R)$ *for all* $0 < s < t_0$ *and*

$$\partial_t p_R(t,x,y) = Ap_R(t,x,y), \quad \forall\,(t,x) \in (0,+\infty) \times \overline{B}_R.$$

$f \in C(\overline{B}_R) \; + \; (iv)$, gives

$$u_R \in C^{1+\frac{\alpha}{2},2+\alpha}([s,t_0] \times \overline{B}_R).$$

Proposition 1

Let $f \in C_b(\mathbb{R}^N)$ and $t \geq 0$; then it exists

$$T(t)f(x) = \lim_{R \to +\infty} T_R(t)f(x), \quad \forall\, x \in \mathbb{R}^N \tag{4}$$

and $(T(t))$ is a positive semigroup in $C_b(\mathbb{R}^N)$.

Schauder interior estimates

$$\|u_R\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{B}_{R-1})} \leq C\|u_R\|_{L^\infty((0,T)\times B_R)} \leq Ce^{\lambda_0 T}\|f\|_\infty$$

Theorem 2

Let $f \in C_b(\mathbb{R}^N)$*, then the function*

$$u(t,x) = T(t)f(x)$$

belongs to $C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,+\infty)\times\mathbb{R}^N)$ *and it solves*

$$\begin{cases} u_t(t,x) = Au(t,x) & t>0, x\in\mathbb{R}^N, \\ u(0,x) = f(x) & x\in\mathbb{R}^N. \end{cases}$$

The operator

For $c > 0$, $b \in \mathbb{R}$, $\alpha > 2$ and $\beta > \alpha - 2$ we consider on $L^p(\mathbb{R}^N)$

$$A := (1 + |x|^\alpha)\Delta + b|x|^{\alpha-2}x \cdot \nabla - c|x|^\beta. \tag{5}$$

Aim.

- Solvability of $\lambda u - Au = f$
- Properties of the maximal domains
- Generation of positive analytic semigroup

Related results

- $b = c = 0$: **Unbounded Diffusion**: $A = (1 + |x|^\alpha)\Delta$

  [*G. Metafune, C. Spina,'10*] $\alpha > 2$, $p > \frac{N}{N-2}$

- $b = 0$: **Schrödinger-Type Operator:** $A = (1 + |x|^\alpha)\Delta - c|x|^\beta$

  [*L. Lorenzi, A. Rhandi,*'15] $0 \le \alpha \le 2, \beta \ge 0$

  [*A. Canale, A. Rhandi, C.Tacelli,*'16] $\alpha > 2, \beta > \alpha - 2$

- $c = 0$: **Unbounded Diffusion** & **Drift**: $A = (1 + |x|^\alpha)\Delta + b|x|^{\alpha-2}x \cdot \nabla$

  [*S. Fornaro, L. Lorenzi '07*]: $0 \le \alpha \le 2.$

  [*Metafune, Spina, Tacelli,'14*] $\alpha > 2, b > 2 - N \to p > \frac{N}{N-2+b}$

- **Complete** : $A = |x|^\alpha\Delta + b|x|^{\alpha-2}x \cdot \nabla_x - c|x|^{\alpha-2}$

  [G. Metafune, N. Okazawa, M. Sobajima, C. Spina,'16]

$N/p \in (s_1 + \min\{0, 2 - \alpha\}, s_2 + \max\{0, 2 - \alpha\})$, $c + s(N - 2 + b - s) = 0$

$$A := (1 + |x|^{\alpha})\Delta + b|x|^{\alpha-2} x \cdot \nabla - c|x|^{\beta}.$$

Remark

- $(1 + |x|^{\alpha})\Delta$ and $b|x|^{\alpha-1}\frac{x}{|x|} \cdot \nabla$ are homogeneous at infinity. They have the same "influence " on the behaviour of A. (possible dependence on coefficient b);
- $|x|^{\beta}$ with $\beta > \alpha - 2$ is super homogeneous.
  No critical exponent, but strong unboundedness with respect to diffusion and drift. A different approach is required;
- $b|x|^{\alpha-1}\frac{x}{|x|} \cdot \nabla$ is not a small perturbation of $(1 + |x|^{\alpha})\Delta - c|x|^{\beta}$.

Solvability in $C_0(\mathbb{R}^N)$

First consider the operator $(A, D_{max}(A))$ on $C_b(\mathbb{R}^N)$ where

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap \bigcap_{1 \le p < \infty} W^{2,p}_{loc}(\mathbb{R}^N) : Au \in C_b(\mathbb{R}^N)\}.$$

It is known that we can associate to the parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N,\ t > 0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N, f \in C_b(\mathbb{R}^N) \end{cases} \tag{6}$$

a semigroup of bounded operators $(T_{min}(t))_{t \ge 0}$ in $C_b(\mathbb{R}^N)$ generated by $A_{min} = (A, \hat{D})$, where $\hat{D} \subset D_{max}$.

Solvability in $C_0(\mathbb{R}^N)$

The uniqueness relies on the existence of suitable Lyapunov function for A, i.e.

$$\exists 0 \le \phi \in C^2(\mathbb{R}^N) \ : \ \lim_{|x|\to\infty} \phi(x) = +\infty, \ A\phi - \lambda\phi \le 0, \quad \lambda > 0.$$

Proposition 2

Assume that $\alpha > 2, \beta > \alpha - 2$. Then $\phi = 1 + |x|^\gamma \ : \gamma > 2$ is Lypunov function for A.

Proposition 3

$T_{\min}(t)$ is generated by $(A, D_{max}(A)) \cap C_0(\mathbb{R}^N)$, is compact, preserves $C_0(\mathbb{R}^N)$.

Solvability of $\lambda u - Au = f$ in L^p

The transformation $v = u\sqrt{\phi}$ where $\phi = (1+|x|^\alpha)^{\frac{b}{\alpha}}$ gives

$$\lambda u - Au = f \quad \Leftrightarrow \quad -\underbrace{(\Delta - U)}_{H} v = \tilde{f} := \frac{f\sqrt{\phi}}{1+|x|^\alpha}.$$

$$U = -\frac{1}{4}\left|\frac{\nabla\phi}{\phi}\right|^2 + \frac{1}{2}\frac{\Delta\phi}{\phi} + \frac{\lambda + c|x|^\beta}{1+|x|^\alpha} \backsim \frac{c|x|^\beta}{1+|x|^\alpha}$$

For $\lambda \geq \lambda_0$ we have $0 \leq U \in L^1_{loc}$ then there exists $G(x,y)$ such that

$$v(x) = \int_{\mathbb{R}^N} G(x,y)\tilde{f}(y)dy \quad \textit{solves} \quad -Hv = \tilde{f}$$

$$u(x) = Lf(x) := \int_{\mathbb{R}^N} \frac{G(x,y)}{\sqrt{\phi(x)}}\frac{\sqrt{\phi(y)}}{1+|y|^\alpha} f(y)dy \quad \textit{solves} \quad \lambda u - Au = f$$

We study the L^p-boundedness of the operator L by estimates of G.

Green function estimates of $\Delta - U$

Since $U(0) = \lambda > 0$ and U behaves like $|x|^{\beta-\alpha}$ as $|x| \to \infty$ we have the following estimates

$$C_1(1 + |x|^{\beta-\alpha}) \leq U \leq C_2(1 + |x|^{\beta-\alpha}) \quad \text{if } \beta \geq \alpha, \tag{7}$$

$$C_3 \frac{1}{1 + |x|^{\alpha-\beta}} \leq U \leq C_4 \frac{1}{1 + |x|^{\alpha-\beta}} \quad \text{if } \alpha - 2 < \beta < \alpha$$

for some positive constants C_1, C_2, C_3, C_4.

Z. Shen '95, gives estimate of $G(x, y)$ if the potential belong to the reverse Holder class B_q if $q \geq \frac{N}{2}$.

Green function estimates of $\Delta - U$

$f \geq 0$ is said to be in B_q if

$$\exists C > 0 : \left(\frac{1}{|B|} \int_B f^q dx \right)^{1/q} \leq C \left(\frac{1}{|B|} \int_B f dx \right) \quad \forall B \in \mathbb{R}^N.$$

If $\beta - \alpha \geq 0$ then $|x|^{\beta-\alpha} \in B_\infty$. If $-\frac{N}{q} < (\beta - \alpha) < 0$ then $|x|^{\beta-\alpha} \in B_q$

- $\beta > \alpha - 2 \Rightarrow U \in B_{\frac{N}{2}}$
- $\beta \leq \alpha - 2 \Rightarrow U \notin B_{\frac{N}{2}}$

Green function estimates of $\Delta - U$

For every $k > 0$ there is some constant $C(k) > 0$ such that

$$|G(x,y)| \leq \frac{C_k}{(1+m(x)|x-y|)^k} \cdot \frac{1}{|x-y|^{N-2}}$$

$$\frac{1}{m(x)} := \sup_{r>0} \left\{ r \,:\, \frac{1}{r^{N-2}} \int_{B(x,r)} U(y)dy \leq 1 \right\}, \quad x \in \mathbb{R}^N.$$

Proposition 4

$$m(x) \geq C(1+|x|)^{\frac{\beta-\alpha}{2}}, \quad \beta > \alpha - 2, C = C(\alpha, \beta, N)$$

Sketch of Proof. Observe that $U \geq C\tilde{V}$.

Green function estimates of $\Delta - U$

Lemma 3

if $\beta > \alpha - 2$

$$G(x,y) \leq C_k \frac{1}{1+|x-y|^k\,(1+|x|)^{\frac{\beta-\alpha}{2}k}}\,\frac{1}{|x-y|^{N-2}}$$

Finally we can prove the boundedness of L in $L^p(\mathbb{R}^N)$.

Theorem 4

$\exists C = C(\lambda)$*:* $\quad \forall \gamma \in [0,\beta]$ *and* $f \in L^p(\mathbb{R}^N)$

$$\||x|^\gamma Lf\|_p \leq C\|f\|_p.$$

For every $f \in C_c^\infty(\mathbb{R}^N)$ the function u solves $\lambda u - Au = f$

Closedness & Invertibility of $\lambda - A_p$ *in* $D_{p,max}(A)$

Theorem 5

$$D_{p,max}(A) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \ : \ Au \in L^p(\mathbb{R}^N)\}.$$

- *Assume that* $N > 2$, $\alpha > 2$ *and* $\beta > \alpha - 2$. *For* $p \in (1, \infty)$ *the following holds*

$$D_{p,max}(A) = \{u \in W^{2,p}(\mathbb{R}^N) \ : \ Au \in L^p(\mathbb{R}^N)\}.$$

- *The operator* $\lambda - A_p$ *is closed and invertible. Moreover* $\exists C = C(\lambda) > 0$: $\forall \gamma \in [0, \beta]$ *and* $\lambda \geq \lambda_0$, *we have*

$$\| |\cdot|^\gamma u\|_p \leq C\|\lambda u - A_p u\|_p, \quad \forall u \in D_{p,max}(A) \ .$$

- *The inverse of* $\lambda - A_p$ *is a positive operator* $\forall \lambda \geq \lambda_0$. *Moreover, if* $f \in L^p \cap C_0$ *then* $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$.

Weighted gradient and second derivative estimates

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N) \ : \ Vu,\ (1+|x|^{\alpha-1})\nabla u,\ (1+|x|^{\alpha})D^2 u \in L^p(\mathbb{R}^N)\}$$

Lemma 6

$\exists C > 0$: $\forall u \in D_p(A)$ we have

$$\|(1+|x|^{\alpha-1})\nabla u\|_p \leq C(\|A_p u\|_p + \|u\|_p)\ ,$$

$$\|(1+|x|^{\alpha})D^2 u\|_p \leq C(\|A_p u\|_p + \|u\|_p)\ .$$

The space $C_c^\infty(\mathbb{R}^N)$ is dense in $D_p(A)$ endowed with the norm

$$||u||_{D_p(A)} := ||u||_p + ||Vu||_p + ||(1+|x|^{\alpha-1})|\nabla u|||_p + ||(1+|x|^{\alpha})|D^2 u|||_p.$$

Generation of Analytic Semigroup

Theorem 7

$(A, D_p(A))$ generates a analytic semigroup in $L^p(\mathbb{R}^N)$.

Some References

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