Projective Limits Techniques for the Infinite Dimensional Moment Problem

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Abstract We study the following general version of the classical moment problem: when can a linear functional on a unital commutative $\mathbb{R}$–algebra A be represented as an integral w.r.t. a Radon measure on the character space $X(A)$ of A equipped with the Borel σ–algebra generated by the weak topology τ_A? Constructing $X(A)$ as a projective limit of a certain family of Borel measurable spaces and using the classical Prokhorov theorem allows us to generalize to infinitely (even uncountably) generated algebras some of the classical theorems for the moment problem, e.g. the ones by Nussbaum and Putinar, in a uniform fashion. Our results apply in particular to the polynomial algebra in an arbitrary number of variables.

Projective Limit of Measurable Spaces

Let $(I,<)$ be a directed partially ordered set and $\mathcal{P} := \{(X_i,\Sigma_i),\pi_{i,j},I\}$ be a *projective system of measurable spaces* (X_i,Σ_i), i.e. $\pi_{i,j}: X_j \to X_i$ is defined and measurable if $i \leq j$ in I and $\pi_{i,j} = \pi_{i,k}\circ\pi_{k,j}$ for all $i \leq k \leq j$.

Definition. *A projective limit of $\mathcal{P}$ is a measurable space (X_I,Σ_I) together with a family of maps $\pi_i : X_I \to X_i$ $(i\in I)$ s.th. $\pi_i = \pi_{i,j}\circ\pi_j$ if $i \leq j$ in I, Σ_I is the smallest σ–algebra s.th. all π_i's are measurable, and the universal property holds.*

Projective systems and projective limits of topological spaces are defined analogously.

Prokhorov Theorem

Definition. *An exact projective system of measures on $\mathcal{P}$ is a family $\mathcal{M} := \{\mu_i : i \in I\}$ s.th. each μ_i is a measure on (X_i,Σ_i) and $\mu_i = \pi_{i,j\#}\mu_j$ if $i \leq j$ in I, i.e. $\mu_i(E_i) = \mu_j(\pi_{i,j}^{-1}(E_i))$ for all $E_i \in \Sigma_i$.*

If $\Sigma_i = \mathcal{B}_i$ is the Borel σ–algebra w.r.t. some Hausdorff topology τ_i on X_i and μ_i is a Radon measure s.th. $\mu_i(X_i) = 1$, then $\mathcal{M}$ is called an exact projective system of Radon probabilities. Recall that a Radon measure on (X_i,τ_i) is a measure μ_i defined on $\mathcal{B}_i$ s.th. μ_i is locally finite and inner regular.

Definition. *A cylindrical measure μ w.r.t. $\mathcal{P}$ is a measure μ on (X_I,Σ_I) s.th. $\pi_{i\#}\mu$ is a measure on Σ_i for all $i \in I$.*

Let $\mathcal{T} := \{(X_i,\tau_i),\pi_{i,j},I\}$ be a projective system of Hausdorff topological spaces, $\mathcal{B}_i$ the Borel σ–algebra w.r.t. τ_i. Further, let $\{(X_I,\tau_I),\pi_i,I\}$ be a projective limit of $\mathcal{T}$ and $\{(X_I,\Sigma_I),\pi_i,I\}$ be a projective limit of $\{(X_i,\mathcal{B}_i),\pi_{i,j},I\}$.

Theorem (Prokhorov, cf. [1, Theorem 21-b, p. 75])**.** *If $\{\mu_i : i \in I\}$ is an exact projective system of Radon probabilities w.r.t. $\mathcal{T}$, then there exists a unique cylindrical measure μ on (X_I,Σ_I) s.th. $\pi_{i\#}\mu = \mu_i$ for all $i \in I$ and $\mu(X_I) = 1$. Moreover, the measure μ (uniquely) extends to a Radon probability ν if and only if*

$$\forall \varepsilon > 0\ \exists\, K \subset X_I \text{ compact } : i \in I \Rightarrow \mu_i(\pi_i(K)) \geq 1-\varepsilon. \tag{1}$$

Character Space as Projective Limit

Let A be a unital commutative $\mathbb{R}$–algebra. Its character space $X(A)$, i.e. the set of all homomorphisms $\alpha : A \to \mathbb{R}$, is equipped with the initial topology τ_A w.r.t the maps $\hat{a} : X(A) \to \mathbb{R}, \alpha \mapsto \alpha(a)$, where $a \in A$. The inclusion is a directed partial order on

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

For $S \in J$ set $X_S := X(S)$, τ_S the initial topology on X_S w.r.t. $\{\hat{a}\restriction_S : a \in S\}$ and $\mathcal{B}_S$ the Borel σ–algebra on X_S w.r.t. τ_S. Then $\pi_{S,T} : X_T \to X_S, \alpha \mapsto \alpha\restriction_S$ is continuous if $S \subseteq T$ in J so that both $\{(X_S,\mathcal{B}_S),\pi_{S,T},J\}$ and $\{(X_S,\tau_S),\pi_{S,T},J\}$ are projective systems of measurable resp. topological spaces. Define $\pi_S : X(A) \to X_S, \alpha \mapsto \alpha\restriction_S$ and let Σ_J be the smallest σ–algebra on $X(A)$ s.th. all π_S's are measurable resp. τ_J the initial topology on $X(A)$ w.r.t. $\{\pi_S : S \in J\}$. We always assume $X(A) \neq \emptyset$.

Then $\{(X(A),\Sigma_J),\pi_S,J\}$ is the projective limit of $\{(X_S,\mathcal{B}_S),\pi_{S,T},J\}$, respectively $\{(X(A),\tau_J),\pi_S,J\}$ is the projective limit of $\{(X_S,\tau_S),\pi_{S,T},J\}$. Moreover, $\tau_A = \tau_J$.

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Moment Problem for $\mathbb{R}$–Algebras

Let A be a unital commutative $\mathbb{R}$–algebra and consider $(X(A),\tau_A)$. Given a closed subset $K \subseteq X(A)$ and a linear functional $L : A \to \mathbb{R}$ s.th. $L(1) = 1$, does there exist a Radon measure ν defined on the Borel σ–algebra w.r.t. τ_A whose support is contained in K s.th.

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\, d\nu(\alpha) \quad \text{for all } a \in A? \tag{2}$$

If such a measure ν exists, it is called a K–representing Radon probability for L.

Results

Let $L : A \to \mathbb{R}$ be a linear functional on A s.th. $L(1) = 1$.

Theorem. *Assume for all $S \in J$ there exists a unique K_S–representing measure μ_S for $L\restriction_S$, where K_S is a closed subset of X_S s.th. $\pi_{S,T}(K_T) \subseteq K_S$ if $S \subseteq T$ in J. Then there exists a unique cylindrical measure μ on $(X(A),\Sigma_J)$ s.th. (2) holds.*

If in addition (1) holds, then there exists a unique K–representing Radon probability ν for L, where $K := \bigcap_{S\in J}\pi_S^{-1}(K_S)$.

The proof of this theorem utilizes Prokhorov's theorem. Further, we obtain generalized versions of the theorems by Nussbaum and Putinar as corollaries:

Corollary (Nussbaum, cf. [2, Theorem 10])**.** *Assume that*
i) $L(Q) \subseteq [0,+\infty)$ for some quadratic module Q in A.
ii) For all $a \in A$, the class $\mathcal{C}\{\sqrt{L(a^{2n})}\}$ is quasi-analytic.
Then there exists a unique cylindrical measure μ on $(X(A),\Sigma_J)$ s.th. (2) holds.

If in addition (1) holds, where $\mu_S := \pi_{S\#}\mu$, then there exists a unique K_Q–representing Radon probability ν for L, where $K_Q := \{\alpha \in X(A) : \alpha(q) \geq 0 \text{ for all } q \in Q\}$.

Note that if A is countably generated, then (1) is always satisfied.

Corollary (Putinar, cf. [3, Lemma 3.2])**.** *Assume $L(Q) \subseteq [0,+\infty)$ for some Archimedean quadratic module Q of A, then there exists a unique K_Q–representing Radon probability for L.*

Applications

Our results apply when $A = \mathbb{R}[X_i : i \in \Omega]$ is the polynomial algebra (where Ω is an arbitrary index set) which allows us to retrieve some results in [4]. In this case $X(A) = \mathbb{R}^\Omega$.

Corollary ([4, Corollary 4.8])**.** *If $L(\sum \mathbb{R}[X_i : i \in \Omega]^2) \subseteq [0,+\infty)$ and Carleman's condition holds for each $i \in \Omega$, i.e. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$, then there exists a unique cylindrical measure μ on $\mathbb{R}^\Omega$ s.th. (2) holds.*

Combining our version of Nussbaum's resp. Putinar's theorem, we are able to generalize [4, Theorem 5.4] in the following way:

Corollary. *Assume that $L(Q) \subseteq [0,+\infty)$ for some quadratic module Q in A and that there exist subalgebras B_a, B_c of A s.th. B_c is countably generated and*
i) $B_a \cup B_c$ generates A as an $\mathbb{R}$–algebra.
ii) $Q \cap B_a$ is Archimedean in B_a.
iii) For all $a \in B_c$ the class $\mathcal{C}\{\sqrt{L(a^{2n})}\}$ is quasi-analytic.
Then there exists a unique K_Q–representing Radon probability for L.

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