Automorphism groups of Hahn groups and fields

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Introduction

Hahn fields are fields of generalised power series. In some particular cases, their automorphism groups have been studied successfully, e.g., Schilling described the (internal) automorphism group of the field of Laurent series, using methods from valuation theory. The construction of a Hahn field can be generalised to that of a Hahn group. Many aspects of the theory of Hahn groups parallel those of Hahn fields. We use these analogies in order to transfer Schilling's ideas to the study of automorphisms of more general Hahn fields and Hahn groups.

This is joint work with S. Kuhlmann.

Definitions and notation

Let $(\Gamma, <)$ be a totally ordered set (a chain) and $\{A_\gamma : \gamma \in \Gamma\}$ a family of archimedean groups (i.e., subgroups of $(\mathbb{R}, +)$). For an element $a = (a_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma A_\gamma$ the **support** of a is the set $\{\gamma \in \Gamma : a_\gamma \neq 0\}$.

Definition 1 (Hahn group). The group of elements $(a_\gamma)_{\gamma\in\Gamma}$ with **well ordered support** is the Hahn product. Denoted by $\mathbf{H}_\Gamma A_\gamma$.

The group of elements with **finite support** is called **Hahn sum** and denoted by $\coprod_\Gamma A_\gamma$.

A **Hahn group** is any subgroup G of a Hahn product. On a Hahn group we can define a **valuation** v, with value set Γ, by setting, for $a \neq 0$,

$$v(a) = \min \operatorname{Supp} a \tag{1}$$

and we can order it lexicographically by setting

$$a >_{\text{lex}} 0 \iff a_{v(a)} > 0 \tag{2}$$

Definition 2 (Hahn field). Let H be a totally ordered abelian group and k an archimedean field. The field of formal generalised power series $K = k((H))$ is called a **(maximal) Hahn field**. Its elements are formal expressions of the form

$$a = \sum_{h\in H} a_h t^h,\ a_h \in k,\ h \in H$$

where the support $\{h \in H : a_h \neq 0\}$ is well ordered. A **Hahn field** is a subfield of a maximal Hahn field. Again, for all $a \in K$, (1) defines a valuation on K with value group H and (2) defines the ordering.

Remark 3. A maximal Hahn field $k((H))$ is nothing but the Hahn product $\mathbf{H}_H\, k$ endowed with an additional field multiplication. The adjective *maximal* comes from this analogy, as Hahn products are maximally valued.

Relevant subgroups (resp. subfields) Let κ be an infinite regular cardinal. The set of elements of a Hahn group (resp. fied) with support of cardinality smaller than κ is called the **κ-bounded subgroup (resp. subfield)**.

Automorphism groups

We want to study the group of order-preserving automorphisms of Hahn groups and Hahn fields. In particular, for a Hahn group $G \leq \mathbf{H}_\Gamma A_\gamma$ and a Hahn field $K = k((H))$:

- The automorphism group $\operatorname{Aut}(G, <)$. We want to describe it with respect to the automorphisms of its **skeleton**, that is, of its associated chain $(\Gamma, <)$ and the family $\{A_\gamma : \gamma \in \Gamma\}$.
- The automorphism group $\operatorname{Aut}(K, <)$ with respect to the automorphisms of its value group $\operatorname{Aut}(H, <)$ and of its residue field $\operatorname{Aut} k$.

  Note that, if k is archimedean (which we assume here) $\operatorname{Aut} k$ is trivial.

To the skeleton and back

An automorphism of a Hahn group $\sigma \in \operatorname{Aut}(G, <)$ canonically induces an automorphism on its skeleton $S(G) = [\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ given by $[\tilde\sigma; \{\sigma_\gamma \text{ s.t. } \gamma \in \Gamma\}]$ where

$$\tilde\sigma \in \operatorname{Aut}(\Gamma, <), v(a) \mapsto v(\sigma(a)) \text{ and}$$
$$\sigma_\gamma : A_\gamma \to A_{\tilde\sigma(\gamma)}, a_\gamma \mapsto (\sigma(a))_{\tilde\sigma(\gamma)} \text{ is an isomorphism.}$$

Similarly, an automorphism σ of $K = k((H))$ induces an automorphism $\tilde\sigma$ of H by $\tilde\sigma(v(a)) = v(\sigma(a))$

Hence, we have two group homomorphisms, both denoted by Φ:

$$\Phi\colon \operatorname{Aut} G \longrightarrow \operatorname{Aut} S(G);$$
$$\Phi\colon \operatorname{Aut} K \longrightarrow \operatorname{Aut} H.$$

Definition 4. The kernel of Φ is the group of **internal automorphisms** and it is denoted by $\operatorname{Int}\operatorname{Aut} G$ (resp. $\operatorname{Int}\operatorname{Aut} K$).

We want to study Φ in more detail: understand its image and whether or under what conditions it is surjective.

The lifting property

Definition 5. Let σ be an automorphism of $S(G)$ (resp. of H). Then σ **lifts** if there exists $\tau \in \operatorname{Aut} G$ (resp. $\tau \in \operatorname{Aut} K$) such that $[\tilde\tau; \tau_\gamma] = \sigma$ (resp. $\tilde\tau = \sigma$), i.e., $\sigma \in \operatorname{im}\Phi$.

If all automorphisms of the skeleton (resp. the value group) lift, that is, if Φ is surjective, we say that G (resp. K) has the **lifting property**.

Goal 1. Characterise Hahn groups and Hahn fields with the lifting property.

Proposition 6. *The κ-bounded groups and fields have the lifting property and the lift is unique up to composition with an internal automorphism. Moreover, an automorphism of the skeleton (resp. the rank) induces an automorphism on the group (resp. field), thereby giving embeddings*

$$\Psi\colon \operatorname{Aut} S(G) \longrightarrow \operatorname{Aut} G$$
$$\Psi\colon \operatorname{Aut} H \longrightarrow \operatorname{Aut} K.$$

We give here the formula for the field case: if $\sigma \in \operatorname{Aut} H$ then $\Psi(\sigma)$ is defined by $\Psi(\sigma)(\sum a_h t^h) = \sum a_h t^{\sigma(h)}$. The group one is notationally heavier.

Definition 7. The image of Ψ is called the group of **external automorphisms** and denoted by $\operatorname{Ext}\operatorname{Aut} G$ (resp. $\operatorname{Ext}\operatorname{Aut} K$).

Structure of the automorphism group

From now on we fix a κ-bounded Hahn group G and a κ-bounded Hahn field K.

Goal 2. Describe the structure of the automorphism group.

Proposition 8. *We can decompose the automorphism group into the following (inner) semidirect product:*

$$\operatorname{Aut} G = \operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Ext}\operatorname{Aut} G \simeq \operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Aut} S(G) \tag{3a}$$
$$\operatorname{Aut} K = \operatorname{Int}\operatorname{Aut} K \rtimes \operatorname{Ext}\operatorname{Aut} K \simeq \operatorname{Int}\operatorname{Aut} K \rtimes \operatorname{Aut} H. \tag{3b}$$

The result of (3b) appeared in [Hof91].

Balanced case

We consider the problem just proposed in the case where G has all isomorphic components, namely $G \leq \mathbf{H}_\Gamma A$, for a group A. In this case G is called **balanced**. Then we immediately describe the external automorphisms in terms of the automorphisms of the chain and those of A. Every such automorphism is determined by an automorphism of the chain Γ and, for each $\gamma \in \Gamma$, an automorphism of A. So

$$\operatorname{Ext}\operatorname{Aut} G \simeq \operatorname{Aut}(\Gamma, <) \times \prod_{\gamma\in\Gamma} \operatorname{Aut}(A).$$

Therefore, using (3a), the study of the structure of the automorphism group reduces to

Goal 3. Describe the group of internal automorphisms.

Example: $\mathbf{H}_\mathbb{Z}\,\mathbb{R}$

We consider the case of $G = \mathbf{H}_\mathbb{Z}\,\mathbb{R}$, as an ordered additive abelian group and, in parallel, that of $K = \mathbb{R}((\mathbb{Z}))$, the field of Laurent series with real coefficients, as a field. Notice that the underlying additive group of K is nothing but G.

External automorphisms From the above we have

$$\operatorname{Ext}\operatorname{Aut} G = \operatorname{Aut}(\mathbb{Z}, <) \times \prod_{n\in\mathbb{Z}} \operatorname{Aut}(\mathbb{R}, +, <) \tag{4}$$

where we regard $\mathbb{Z}$ only as a chain and $\mathbb{R}$ only as an additive group and

$$\operatorname{Ext}\operatorname{Aut} K = \operatorname{Aut}(\mathbb{Z}, +, <) \times \prod_{n\in\mathbb{Z}} \operatorname{Aut}(\mathbb{R}, +, \cdot, <) \tag{5}$$

where $\mathbb{Z}$ is seen as an ordered additive group and $\mathbb{R}$ as an ordered field.

Since both $(\mathbb{Z}, +, <)$ and $(\mathbb{R}, +, \cdot, <)$ only admit the identity as an automorphism, then $\operatorname{Ext}\operatorname{Aut} K = \{\mathrm{id}\}$. On the other hand, in the case of G we have

$$\operatorname{Aut}(\mathbb{Z}, <) \simeq (\mathbb{Z}, +) \text{ and } \operatorname{Aut}(\mathbb{R}, +, <) \simeq (\mathbb{R}_{>0}, \cdot)$$

automorphisms of $(\mathbb{Z}, +)$ are given by shifting by an integer number and automorphisms of $(\mathbb{R}, +, <)$ by multiplication by a positive real number. So the external automorphisms are fully described.

Internal automorphisms In the field case they were described by Schilling (see [Sch44]). For $K = \mathbb{R}((\mathbb{Z})) = \mathbb{R}((t))$ we have, as just seen, $\operatorname{Aut} K = \operatorname{Int}\operatorname{Aut} K$.

- An automorphism $\sigma \in \operatorname{Aut} K$ is uniquely determined by its action on t. Indeed, for all $d \in \mathbb{Z}$ we have

$$\sigma\left(\sum_{n=m}^{\infty} a_n t^n\right)_d = \sigma\left(\sum_{n=m}^{d} a_n t^n + \sum_{n=d+1}^{\infty} a_n t^n\right)_d = \left(\sum_{n=m}^{d} a_n \sigma(t)^n\right)_d$$

- Since $v(t) = 1$ then $\sigma(t)$ must have valuation 1, so be of the form tu for a unit u in the valuation ring of K.
- The correspondence between $\operatorname{Aut} K$ and U, the group of units in the valuation ring, is one to one. It becomes an group isomorphism if U is equipped with the appropriate group operation:

$$\operatorname{Aut} K \simeq (U, \times),\ u_1 \times u_2 := u_1\sigma_{u_1}(u_2)$$

  where σ_{u_1} is the automorphism of K mapping t onto tu_1.
- The **order preserving** automorphisms are those corresponding to **positive units**.

Conclusions and forthcoming work

- For Goal 1, we know the κ-bounded groups have the lifting property. We don't know whether, up to isomorphism, they are the only ones. We have a characterisation of groups with the lifting property in terms of their principal convex subgroups. It is probably challenging to relate this to the κ-boundedness.
- Goal 2 is partly achieved, thanks to Proposition 8, as describing the external automorphisms is done through the automorphisms of the skeleton or, for the field case, of the value group and residue field.
- Goal 3 is the missing piece to complete Goal 2. An idea would be to transfer Schilling's methods to the general case, by finding elements that uniquely determine an automorphism, like the variable t does in the Laurent series case.

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