Dynamical zeta functions, Lefschetz formulae and applications

Polyxeni Spilioti

December 6th-7th, 2018

KWIM-Festtage

Konstanz Women in Mathematics (KWIM)

Spectral geometry

→ What is it?

✓ It is the field of mathematics that concerns the connections between the geometry of manifolds and the spectrum of differential operators.

→ Marc Kac, mid-60's:

Can one hear the shape of a drum?

✓ The answer is not always positive, in particular when we deal with manifolds with singularities.

→ How can one obtain information about the geometry of a manifold, such as the volume, the curvature, or the length of the closed geodesics, provided that we can study the spectrum of certain differential operators?

Tools from spectral geometry

Harmonic analysis on locally symmetric spaces is a powerful machinery in studying problems related to spectral invariants, such as the analytic torsion, the eta function, and the dynamical zeta functions of Ruelle and Selberg.

→tools from spectral geometry

- trace formula
- Lefschetz formula

Why are these topics important?

✓ There are numerous applications in

- mathematical physics
- dynamics, ergodic theory
- graph theory

✓ More...ongoing area of research: spectral geometry and computational disciplines

- spectral shape analysis, medical imaging
  Using spectral theory of the Laplace-Beltrami operator, heat kernels etc...
  ✓ see for example the work of Michael M. Bronstein, Alex M. Bronstein, L. Guibas, M. Reuter
- data analysis, spectral graph theory, clustering

“Beautiful” articles-books

- Ruelle: Dynamical Zeta Functions and Transfer Operators ([Rue02])
- Pollicott: Dynamical Zeta functions ([Pol])
- Terras: Zeta functions of Graphs, A Stroll through the garden ([Ter10])

The dynamical zeta functions are represented by **Euler-type products**.

- Analogy to the **Riemann zeta function**

$$\left.\begin{array}{r}\zeta(s)=\displaystyle\prod_{p=\text{prime}}(1-p^{-s})^{-1}\\ \mathrm{Re}(s)>1\end{array}\right\}\leftrightarrow\left.\begin{array}{r}R(s)=\displaystyle\prod_{\gamma=\text{prime}}(1-e^{-sl(\gamma)})^{-1}\\ \mathrm{Re}(s)>1\end{array}\right\}$$

Compact hyperbolic manifolds

obtained as follows:

- $G = SO^0(d,1)$, $K = SO(d)$, $d = 2n+1, n \in \mathbb{N}_>$
- $\widetilde{X} := G/K \cong \mathbb{H}^d$ using the Killing form
- Γ discrete, cocompact, torsion-free subgroup of G
- $X = \Gamma\backslash\widetilde{X}$ is a d-dimensional locally symmetric compact hyperbolic manifold

Fix notation: $\mathfrak{g}$ =Lie algebra of G, $\mathfrak{k}$ =Lie algebra of K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{a}$ = a maximal abelian subalgebra of $\mathfrak{p}$, $M := \text{Centr}_K(A)$

Twisted Selberg and Ruelle zeta functions

✓ The dynamical zeta functions are associated with the geodesic flow on $S(X) \cong \Gamma \backslash G/M$.

✓ Representation theory is involved: modern consideration of the dynamical zeta functions. Let $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$.

Definition (Twisted Selberg zeta function)

$$Z(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\text{ prime}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)l(\gamma)} \right),$$

for $\mathrm{Re}(s) > r$, r positive constant.

Definition (twisted Ruelle zeta function)

$$R(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right)^{(-1)^{d-1}},$$

for $\mathrm{Re}(s) > c$, c positive constant.

- The product runs over the prime conjugacy classes $[\gamma]$ of Γ, which correspond to the prime closed geodesics on X of length $l(\gamma)$.

Arbitrary representations of Γ on a f.d. vector space

$$\chi : \Gamma \to \mathrm{GL}(V_\chi)$$

- **Problem:** there is no Hermitian metric h^χ, which is compatible with the flat connection on the flat vector bundle ⇢ we deal with non self-adjoint Laplace and Dirac operators
- **Solution:** Twisted Bochner-Laplace operator $\Delta^\sharp_{\tau,\chi}$ acting on smooth sections of twisted vector bundles $E_\tau \otimes E_\chi$.
  - it is an elliptic operator ⇢ nice spectral properties ⇢ its spectrum is discrete and contained in a translate of a positive cone in $\mathbb{C}$.
  - Consider the corresponding heat semi-group $e^{-t\Delta^\sharp_{\tau,\chi}}$. It is an integral operator with smooth kernel.
  - Consider the trace of the operator $e^{-t\Delta^\sharp_{\tau,\chi}}$ and derive a trace formula.

Trace formula

([Spi18])

Theorem

For every $\sigma \in \widehat{M}$ we have

$$\mathrm{Tr}(e^{-tA^{\sharp}_{\chi}(\sigma)}) = \dim(V_\chi)\,\mathrm{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)d\lambda + \sum_{[\gamma]\neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L(\gamma;\sigma,\chi) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}};$$

where

$$L(\gamma;\sigma,\chi) = \frac{\mathrm{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{n}})}.$$

Meromorphic continuation for arbitrary representations of Γ

([Spi18])

Theorem

Let $\sigma \in \widehat{M}$ and $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ. The twisted Selberg zeta function $Z(s;\sigma,\chi)$ associated with σ and χ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. Its singularities are described in terms of the discrete eigenvalues of the twisted operators $A^\sharp_\chi(\sigma)$ and $D^\sharp_\chi(\sigma)$ and their orders are described by the corresponding algebraic multiplicities.

Theorem

For every $\sigma \in \widehat{M}$ and for every finite dimensional representation χ of Γ, the twisted Ruelle zeta function $R(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$.

Functional equations

([Spi15])

Theorem

The Selberg zeta function $Z(s;\sigma,\chi)$ satisfies the functional equation

$$\frac{Z(s;\sigma,\chi)}{Z(-s;\sigma,\chi)} = \exp\left(-4\pi \dim(V_\chi) \operatorname{Vol}(X) \int_0^s P_\sigma(r)dr \right),$$

where P_σ denotes the Plancherel polynomial associated with $\sigma \in \widehat{M}$.

Theorem

The super Ruelle zeta function associated with a non-Weyl invariant representation $\sigma \in \widehat{M}$ satisfies the functional equation

$$R^s(s;\sigma,\chi)R^s(-s;\sigma,\chi) = e^{2i\pi\eta(D^\sharp_{p,\chi}(\sigma))},$$

where $\eta(D^\sharp_{p,\chi}(\sigma))$ denotes the eta invariant of the twisted Dirac

Determinant formula

([Spi15])

Proposition

The Ruelle zeta function has the representation

$$R(s;\sigma,\chi)=\prod_{p=0}^{d-1}\det\left(A^{\sharp}_{\chi}(\sigma_p\otimes\sigma)+(s+\rho-\lambda)^2\right)^{(-1)^p}$$
$$\exp\left(-2\pi(d+1)\dim(V_\chi)\dim(V_\sigma)\operatorname{Vol}(X)s\right),$$

where σ_p denotes the p-th exterior power of the standard representation of M and $\det(A^{\sharp}_{\chi}(\sigma_p\otimes\sigma)+(s+\rho-\lambda)^2)$ the regularized determinant of the operator $A^{\sharp}_{\chi}(\sigma_p\otimes\sigma)+(s+\rho-\lambda)^2$.

The Lefschetz formula (1)

✓ The Lefschetz formula together with the Selberg trace formula are often used to prove and extend results such as the prime geodesic theorem.

✓ Lefschetz formula expresses the number of fixed points of an endomorphism of a topological space in terms of the traces of the corresponding endomorphisms in the (co)homology groups.

✓ Setting:

- X differentiable compact orientable manifold
- $f : X \to X$ differentiable mapping
- $x \in X$ isolated fixed point for f, with $\det(df_x - \mathrm{Id}) \neq 0$ (non-singular point)
- for such an x, define the index $i(x) = \operatorname{sign} \det(df_x - \mathrm{Id})$

The Lefschetz formula (1)

Lefschetz number $\Lambda(f, X)$

$$\Lambda(f, X) = \sum_{i=1}^{\infty} (-1)^i \operatorname{Tr}(f_* | H_i(X, \mathbb{Q})) =$$

$$\sharp\{\text{fixed points with index } \quad 1\} - \sharp\{\text{fixed points with index } -1\}$$

$\rightsquigarrow$ consequences: Hopf formula, Hairy ball theorem.

Euler charecteristic $\chi(X)$

$$\chi(X) = \sum_k i(x_k, v)$$

The Lefschetz formula (2)

✓Locally symmetric spaces: rank 1

Theorem

([Juh01, Thm 3.9]) For all test functions $\phi \in \mathbb{C}_0^\infty(\mathbb{R}^+)$ the identity

$$\sum_{\pi \in \widehat{G}} N_\Gamma(\pi)\Big(\sum_p (-1)^p \int_0^\infty \theta_A((H^p(\mathfrak{n}^-,(V_\pi)_K) \otimes V_\sigma)^M)(\exp tH_0)\phi(t)dt\Big)$$

$$= (-1)^{d-1} \sum_c |c_0| \operatorname{tr}(\sigma(m_c)) \det(\mathrm{Id} - P_c^-)^{-1})\phi(|c|)$$

holds, where the sum on the right-hand side runs over the periodic orbits c of the action of A^+ on $\Gamma \backslash G/M$.

The Lefschetz formula (2)

✓Locally symmetric spaces: higher rank

Theorem

([Dei04, Theorem 1.2]) There exists $\mu \in \mathfrak{a}^*$ such that for any suitable function ϕ on A, we have

$$\sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_\lambda^\sigma(\pi) \int_{A^-} \phi(a) a^{\lambda+\rho} da = \sum_{\gamma \in E(\Gamma)} \frac{\mathrm{Vol}(\Gamma_\gamma \backslash G_\gamma)}{\det(\mathrm{Id} - a_\gamma m_\gamma | \mathfrak{n})} \operatorname{tr} \sigma(m_\gamma) \phi(a_\gamma),$$

where the sum in the right-hand side runs over the conjugacy classes in Γ consisting of split regular elements and

$$m_\lambda^\sigma(\pi) = \sum_{q=0}^{\dim \mathfrak{n}} (-1)^{q+\dim \mathfrak{n}} \dim(H^q(\mathfrak{n}, \pi_K)_\lambda \otimes \check{\sigma})^M.$$

The Lefschetz formula (2)

In the right-hand side of equation (geometrical side), the sum runs over the conjugacy classes of split regular elements. These elements corresponds to the elements a_γ that lie in the negative Weyl chamber A^-. For every regular element in Γ, there exists a closed geodesic c in X, which gets closed by γ. The closed geodesic lies in a unique maximal flat submanifold of X.
$\rightsquigarrow$ consequence: Prime geodesic theroem ([Dei04]) For $T_1, \ldots, T_r > 0$, let

$$\psi(T_1, \ldots, T_r) = \sum_{c:\ a_{c,j} \leq T_j} \lambda_c,$$

where λ_c is the volume of the unique maximal flat submanifold that c lies in. Then, as $T_j \to \infty$ for every j,

$$\psi(T_1, \ldots, T_r) \sim T_1 \ldots T_r.$$

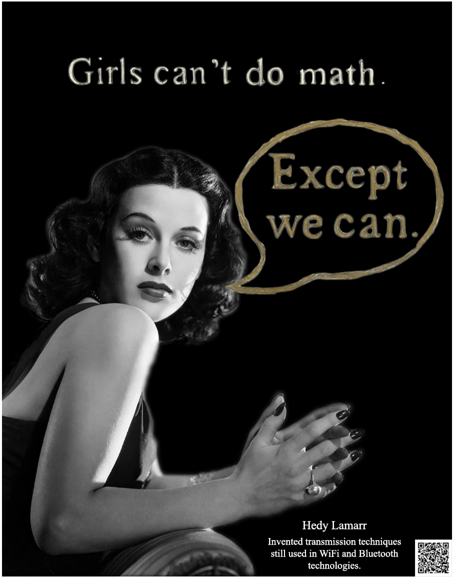
Further research

✓ Derive the Lefschetz formula in a non-compact setting: non-cocompact arithmetic subgroups

✓ prime geodesic theorem This task demands advanced analysis and representation theory, Arthur trace formula ([DGS17])

✓ Conclusion The theory of trace formulas involve objects from different fields (spectral theory, representation theory, number theory) such as: zeta functions, Eisenstein series, Dirichlet series, automorphic forms in relation with Langlands program.

Thank you!!


Girls can't do math.
Except we can.
Hedy Lamarr
Invented transmission techniques still used in WiFi and Bluetooth technologies.

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